HOLOMORPHIC PARTIAL DIFFERENTIAL EQUATIONS AND CLASSICAL POTENTIAL THEORY

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Preface

These notes represent a series of lectures I have given at the Universidad de La Laguna in the Spring semester of 1995. They are directed mainly at the graduate students and a wide audience of analysts that are not assumed to be experts in the theory of holomorphic partial differential equations. The major purpose of such a course was simply to give the audience a good first taste of a subject which is already sufficiently rich with deep results and also provides exciting grounds for a nice interplay between some parts of modern analysis and a number of attractive themes in classical "physical" mathematics of the last century rather than trying to produce a some sort of encyclopedic treatise. Thus, let me stress again, the reader I had in mind was most certainly not a working expert in the field but a curious analyst, or a graduate student. Accordingly, whenever a choice between clarity and simplicity vs. generality appeared I have chosen the first while trying to supply enough references to special literature to satisfy a more requiring reader. Also, wherever possible I tried to preserve an informal way of communicating the material characteristic for the lecture hall rather than resorting to a more formal style of an "academic" textbook.

My understanding of topics covered in the lectures has developed during a close, decade long collaboration with Professor Harold S. Shapiro from the Royal Institute of Technology in Stockholm. Thus, his influence on these notes goes far beyond what one may see from the references to his and our joint works. (Needless to say though that he bears no responsibility whatsoever for any possible errors).

I have also benefited greatly from numerous stimulating discussions I have had over the years on a number of related topics with Professors B. Gustafsson, G. Johnsson, L. Karp and H. Shahgholian of the Royal Institute and with Professor P. Ebenfelt of the University of California at La Jolla.

It is my pleasure to thank Professor Fernando Pérez-González of the Univesidad de La Laguna who conceived the idea of such a course and

initiated my invitation to La Laguna. Professors A. Bonilla, D. I. Cruz Baez, U. Fraga, J. García Melián, F. Pérez-González, J. Sabina and R. Trujillo-González all attended the lectures and contributed a great many suggestions and improvements into the text of these Notes. Also I want to thank them all for their selfless efforts and the time they put into typesetting the manuscript.

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La Laguna, June 1995.

Chapter 1
Introduction

A general unifying theme of these lectures is inspired by a number of simple questions in Classical Potential Theory. So, let us start with discussing some of those problems.

A) In 1914 G. Herglotz ([Her]) studied the continuation of potentials inside the region occupied by masses. More precisely, suppose we are in $\mathbb{R}^3$ and consider a bounded domain Ω whose boundary is real analytic. Let $p(x)$ be a polynomial in (x_1, x_2, x_3) which is the density of the potential

$$U_{\Omega,p}(x) = -\frac{1}{4\pi}\int_\Omega \frac{p(y)}{|x-y|}dy.$$

Of course, $U_{\Omega,p}(x)$ is well defined and harmonic for $x \notin \overline{\Omega}$, and the question is - keeping in mind that $p(y)$ and $\partial\Omega$ are extremely smooth - how far $U_{\Omega,p}(x)$ can be continued across $\partial\Omega$ as a harmonic function inside Ω. In particular, if $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$ and $p \equiv 1$, then according to the mean value property

$$U_{\Omega,p}(x) = -\frac{1}{4\pi}\int_\Omega \frac{1}{|x-y|}dy = \frac{C}{|x|},$$

C being a constant, so $U_{\Omega,p}$ extends into $\Omega\setminus\{0\}$. The following examples are even more intriguing (cf. [Jo], [KS1]).

Example 1.1 Let Ω be the ellipsoid

$$\Omega = \{x : \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} \le 1, a > b\}$$

and suppose also that $p \equiv 1$. Then $U_{\Omega,p}$ extends to $\Omega \setminus \{(x_1, x_2, 0) : x_1^2 + x_2^2 \leq a^2 - b^2\}$. In fact, $U_{\Omega,p}$ extends as a multi-valued function across all points inside the circle $x_1^2 + x_2^2 = a^2 - b^2$ in the plane $\{x_3 = 0\}$ and has a singularity of a square root type on that circle.

Example 1.2 Let $\Omega = \{x : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{b^2} \leq 1, a > b\}$. Then $U_{\Omega,p}$ extends as an analytic multi-valued function to $\Omega \setminus \{(x_1, 0, 0) : |x_1| \leq \sqrt{a^2 - b^2}\}$. However the singularities are in this case of a logarithmic type.

B) The following rather simple question posed by N. Nadirashvili and communicated to us by H. S. Shapiro is the following. Consider the spherical shell $\Omega = \{x : 1 < |x| < 2\} \subset \mathbb{R}^3$, and let u be a harmonic function in Ω. Suppose we know that $u(x_1, 0, 0)_{-2 < x_1 < -1} = 0$. Does $u(x_1, 0, 0)_{1 < x_1 < 2} = 0$?. The two-dimensional case is easy and is recommended as an exercise.

C) SCHWARZ' REFLECTION PRINCIPLE. Recall that if γ is a real analytic curve in the complex plane then for any point A sufficiently close to γ there exists another point B on the other side of γ such that

$$u(A) + u(B) = 0 \tag{1.1}$$

for all functions u harmonic near γ and vanishing on γ (cf. [Sh1]). The simplest cases are when $\gamma = \mathbb{R}$, and when $\gamma =$ unit circle. Now, what happens in higher dimensions? If γ is the hyperplane $\gamma = \{x : x_n = 0\}$ the reflection principle (1.1) holds with $A = (x_1, ..., x_{n-1}, x_n)$ and $B = (x_1, ..., x_{n-1}, -x_n)$. When $\gamma = \{x : |x| = 1\}$, the unit sphere, (1.1) still holds although in a more complicated form

$$u(x) + |x|^{2-n} u\left(\frac{x}{|x|^2}\right) = 0 \tag{1.2}$$

for all u harmonic near the sphere γ and vanishing on it. This is due to Kelvin (cf. [Ke], [Sh1]). However an answer to the question whether

an analogue of (1.1) or (1.2) perhaps with a different constant holds for analytic hypersurfaces in $\mathbb{R}^n$, other than planes or spheres, has remained unknown until very recently (cf. the discussion in [Sh1] and the most recent paper [EK]).

As it turns out, all these questions have a common denominator: the Cauchy problem for the Laplace operator, and more generally, the Cauchy problem for linear PDE (partial differential equations) with holomorphic coefficients.

Recall that if we consider a linear ODE (ordinary differential equation)

$$w^{(n)}(z) + a_{n-1}(z)w^{(n-1)}(z) + ... + a_1(z)w'(z) + a_0(z)w(z) = f(z) , \tag{1.3}$$

where the coefficients a_j's and f are holomorphic in some domain Ω, and pose the Cauchy problem looking for solutions of (1.3) such that

$$w(0) = w_0,\ w'(0) = w_1,\ ....,\ w^{(n-1)}(0) = w_{n-1} \tag{1.4}$$

(we tacitly assume here, of course, that $0 \in \Omega$), then, as is well-known (cf. [I]), there exists a unique solution w of the Cauchy problem (1.3), (1.4) that extends as an analytic function to the whole domain Ω. Unfortunately the situation with PDE's is much more complicated. The following simple example is a good illustration of how things can go wrong in the multi-variate case.

Example 1.3 Consider in $\mathbb{C}^2$ the function

$$w(z_1, z_2) = \frac{z_1}{1 - z_1 z_2} .$$

Then,

$$\frac{\partial w}{\partial z_2} = \frac{z_1^2}{(1 - z_1 z_2)^2}$$

and

$$\frac{\partial w}{\partial z_1} = \frac{1}{(1 - z_1 z_2)^2} ,$$

so w is the (unique) solution of the Cauchy problem

$$\frac{\partial w}{\partial z_2} - z_1^2 \frac{\partial w}{\partial z_1} = 0, \quad w(z_1, 0) = z_1. \tag{1.5}$$

We have a good equation, a good initial surface (a hyperplane), excellent data, but nevertheless the solution blows up arbitrarily close to the initial surface.

With those remarks we close this introductory section and proceed with the study of the holomorphic Cauchy problem for pde's. Let us say a few words concerning the rather standard multi-variate notation (cf. [Hör1, Hör2]) that we use.

Notations. Let $z \in \mathbb{C}^n$, $z = (z_1, z_2, ..., z_n) = x + iy$ where $x, y \in \mathbb{R}^n$. Then,

$$\begin{aligned} < z, \xi > &= \sum_1^n z_j \overline{\xi_j}, \\ z \cdot \xi &= \sum_1^n z_j \xi_j, \\ \|z\| &= < z, z >^{\frac{1}{2}}. \end{aligned}$$

A multi-index α is a vector $(\alpha_1, \alpha_2, ..., \alpha_n), \alpha_j \in \mathbb{Z}_+$, and we set

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + ... + \alpha_n; \\ \alpha! &= \alpha_1! \cdot \alpha_2! \cdot \cdots \alpha_n!; \\ z^\alpha &= z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}; \\ \partial_j &= \frac{\partial}{\partial z_j}; \\ \partial^\alpha &= (\frac{\partial}{\partial z_1})^{\alpha_1} (\frac{\partial}{\partial z_2})^{\alpha_2} \cdots (\frac{\partial}{\partial z_n})^{\alpha_n}. \end{aligned}$$

$f(z)$ being holomorphic near z^0 means $f(z) = \sum_{|\alpha|=0}^{\infty} \frac{\partial^\alpha f(z^0)}{\alpha!}(z - z^0)^\alpha$ and the series converges absolutely and uniformly in some neighborhood of z^0.

A useful abbreviation is the following: $z' = (z_1, ..., z_{n-1})$ denotes the projection of $z = (z_1, z_2, ..., z_n)$ onto the plane $z_n = 0$. The polydisk $D(z^0, R)$ centered at z^0 of radius $R > 0$ is, as usual, the set

$$D(z^0, R) = \{z : |z_j - z_j^0| < R, j = 1, 2, ..., n\}.$$

Finally, it is sometimes convenient in $\mathbb{C}^n$ to use "the polydisk norm"

$$|||z||| = \max\{|z_1|, |z_2|, ..., |z_n|\}.$$

Notes

A problem discussed in A) is stated in the Herglotz memoir [Her], although there are earlier works of C. Neumann, E. Schmidt and others. A fairly detailed survey of the literature can be found in [KS1] and even more so in [Sh1]. Herglotz has completely solved problem A) in two dimensions; in higher dimensions it remains unsolved even today (cf. [KS1], [Sh1]). Examples (1.1) and (1.2) are studied in great detail in G. Johnsson's thesis [Jo]. The question B) was posed and answered in the affirmative by N. Nadirashvili [N] by making use of symmetrization techniques. At the end of Chapter 8 we shall outline a different approach to it. [Sh1] contains a detailed discussion of various topics associated with the Schwarz Reflection Principle and gives detailed references. A complete solution of the problem posed in C) has been obtained only recently in [EK].

Chapter 2

The Cauchy-Kovalevskaya Theorem with Estimates

Theorem 2.1 *Let $\beta = (0, \ldots, m)$ and suppose that for each $\alpha \in \mathbb{Z}^n_+$, $|\alpha| \leq m$, $\alpha \neq \beta$ the functions $a_\alpha(z)$ and $f(z)$ are holomorphic in $\overline{D(0,R)}$. Then, given $t < 1$, there exists a unique holomorphic solution $w = w(z)$ to the problem*

$$\begin{cases} \partial^\beta w = \displaystyle\sum_{|\alpha| \leq m, \alpha \neq \beta} a_\alpha(z) \partial^\alpha w + f(z) \\ \dfrac{\partial^j w}{\partial z_n^j} = 0, \quad 0 \leq j \leq m-1, \end{cases} \tag{2.1}$$

defined in the polydisk $\{z : |z'| < tR\, |z_n| < \delta t R\}$. Most importantly, the quantity $\delta > 0$ only depends on t and the coefficients $a_\alpha(z)$ but does not depend on $f(z)$.

PROOF: *Uniqueness.* If $w_1(z), w_2(z)$ were two solutions then $w = w_1 - w_2$ would satisfy

$$\begin{cases} \partial^\beta w = \displaystyle\sum_{|\alpha| \leq m, \alpha \neq \beta} a_\alpha(z) \partial^\alpha w \\ \dfrac{\partial^j w}{\partial z_n^j} = 0, \quad 0 \leq j \leq m-1. \end{cases}$$

The initial conditions give $\partial^\alpha w(z', 0) = 0$ for $|\alpha| \leq m-1$ while

$$\partial_n^m w(z', 0) = \sum_{|\alpha| \leq m, \alpha \neq \beta} a_\alpha \partial^\alpha w(z', 0) = 0, \quad |z'| < tR. \tag{2.2}$$

Thus $\partial_{z'}^\gamma \partial_n^m w(z', 0) = 0$ for any multi-index γ. In particular, $\partial^\alpha w(\mathbf{0}, 0) = 0$ for each α, $|\alpha| \leq m$. Differentiating (2.2) with respect to z_n we obtain $\partial_{z_n}^{m+1} w(z', 0) = 0$, and so $\partial_{z'}^\gamma \partial_{z_n}^{m+1} w(z', 0) = 0$ for every γ. Hence

$\partial^\alpha w(z', 0) = 0$ for all $|\alpha| \leq m+1$. Proceeding in the same way leads to $\partial^\alpha w(\mathbf{0}, 0) = 0$ for all α. Since $w(z)$ is holomorphic in $\{z : |z'| < tR, |z_n| < \delta t R\}$, $w(z) \equiv 0$. Thus $w_1(z) = w_2(z)$.

Existence. The proof of existence shall rely upon the following proposition.

Proposition 2.1 *Let $A_\alpha(z), F(z)$ be holomorphic in the polydisk $\overline{D(0,1)}$. Then, given $t : 0 < t < 1$ there exists a unique holomorphic solution $w(z)$ in $D(0,t) = \{z : |||z||| < t\}$ to the problem*

$$\begin{cases} \partial^\beta w = \displaystyle\sum_{|\alpha| \leq m, \alpha \neq \beta} A_\alpha(z) \partial^\alpha w + F(z), \quad \beta = (0, \ldots, m) \\ \partial^\alpha w(z', 0) = 0, \quad |\alpha| \leq m-1, \end{cases} \tag{2.3}$$

provided that

$$\sup_{z \in D(0,1)} \sum_{|\alpha| \leq m, \alpha \neq \beta} |A_\alpha(z)| \leq C,$$

where $C = C(t)$ is a sufficiently small constant that depends only on t.

Let us continue now with the proof of the C-K theorem. So, let $0 < t < 1$ and $\delta > 0$ such that $\{|z'| < R, |z_n| < \delta R\} \subset D(0,R)$. By changing the variables $z = (z', z_n) \to \xi = (\xi', \xi_n)$, $\xi' = \frac{z'}{R}$, $\xi_n = \frac{z_n}{\delta R}$ we arrive at the equation:

$$\partial_{\xi_n}^m w = \sum_{|\alpha| \leq m, \alpha \neq \beta} \delta^{m-\alpha_n} R^{m-|\alpha|} a_\alpha \partial_\xi^\alpha w(z(\xi)) + (\delta R)^m f(z(\xi))$$

or,

$$\partial_{\xi_n}^m w = \sum_{|\alpha| \leq m, \alpha \neq \beta} A_\alpha(\xi) \partial_\xi^\alpha w + F(\xi), \quad \xi \in D(0,1).$$

Since $A_\alpha(\xi) := \delta^{m-\alpha_n} R^{m-|\alpha|} a_\alpha(z(\xi))$, $\alpha \neq \beta$, we have

$$\sup_{D(0,1)} \sum_{|\alpha| \leq m, \alpha \neq \beta} |A_\alpha| = O(\delta).$$

Therefore, for a fixed t: $0 < t < 1$, this supremum can be made smaller than $C(t)$ in Proposition 2.1 provided that δ is sufficiently small and the theorem follows.

PROOF OF THE PROPOSITION: For each k define w_{k+1} to be the solution of the Cauchy problem

$$\begin{cases} \partial^\beta w_{k+1} = \sum_{|\alpha|\le m, \alpha\ne\beta} A_\alpha(z)\partial^\alpha w_k + F(z); \\ \dfrac{\partial^j w_{k+1}}{\partial z_n^j} = 0, \quad 0 \le j \le m-1, \end{cases} \tag{2.4}$$

and set $w_o(z) = 0$. Existence of solutions w_k is proven below in Lemma 2.3. Now, set $v_k = w_k - w_{k-1}$, $k \ge 1$. Then, for each k we have

$$\begin{cases} \partial^\beta v_{k+1} = \sum_{|\alpha|\le m, \alpha\ne\beta} A_\alpha(z)\partial^\alpha v_k; \\ \dfrac{\partial^j v_{k+1}}{\partial z_n^j} = 0, \quad 0 \le j \le m-1. \end{cases}$$

Our immediate objective is to show convergence of the telescopic series $\sum v_k$. This requires the following lemmas.

Lemma 2.1 *If $f(z)$ is analytic in $\mathbb{D} = \{z : |z| < 1\} \subset \mathbb{C}$, $f(0) = 0$ and*

$$|f'(z)| \le \frac{1}{(1-|z|)^p} \quad \text{for some } p > 0,$$

then

$$|f(z)| \le \frac{1}{p(1-|z|)^p}$$

Lemma 2.2 *If $f(z)$ is analytic in $\mathbb{D}$ and*

$$|f(z)| \le \frac{1}{(1-|z|)^p} \ ,$$

then

$$|f'(z)| \le \frac{(p+1)e}{(1-|z|)^{p+1}} \ .$$

As a consequence of Lemmas 2.1 and 2.2 we claim that the following estimate holds for $v_{k+1}(z)$

$$|v_{k+1}(z)| \le \frac{N(Ce^m)^k}{[(1-|z_1|)\ldots(1-|z_n|)]^{mk}} \ , \tag{2.5}$$

where $N = N(F)$ is a constant that only depends on F and

$$C = \sup_{z\in D(0,1)} \sum_{|\alpha|\le m, \alpha\ne\beta} |A_\alpha(z)|.$$

Assuming the claim, we have for each $z \in D(0,t)$

$$\frac{Ce^m}{[(1-|z_1|)\ldots(1-|z_n|)]^m} \le \frac{Ce^m}{(1-t)^{nm}}$$

and provided that

$$C \ < \ C(t) =: \frac{(1-t)^{nm}}{e^m} \ ,$$

the series $\sum |v_k(z)|$ is majorized by a geometric series. Hence, the series $\sum v_k$ converges uniformly inside $D(0,t)$ to a holomorphic function $w(z)$. So, its partial sums w_k converge to w and, in view of (2.4), w is a solution of (2.3).

Let us now prove the estimate (2.5). We proceed by induction. Since $w_o = 0$ we have $v_1 = w_1$, and the estimate of $|w_1|$ by a constant $N = N(F)$, i.e., the validity of (2.5) for $k = 0$, follows from the following lemma.

Lemma 2.3 *Suppose that $F = F(z)$ is holomorphic in the polydisk $D(0,1)$. Then, the problem,*

$$\begin{cases} \partial^{\beta} w = F, & \beta = (0,\ldots,m) \\ \partial^{j}_{z_n} w(z',0) = 0, & 0 \le j \le m-1, \end{cases} \tag{2.6}$$

admits a unique holomorphic solution $w = w(z)$ defined in $D(0,1)$ and satisfying $|w(z)| \le N$ on $D(0,1)$ for a positive constant N that only depends on F.

Proof of Lemma 2.3: Set

$$w(z', z_n) = \int_0^{z_n} \frac{(z_n - \lambda)^{m-1}}{(m-1)!} F(z', \lambda)\, d\lambda.$$

A routine check shows that w is a solution of (2.6) and the lemma follows. Assume now that (2.5) holds for $k+1$, i.e.

$$|v_{k+1}(z)| \le \frac{N(Ce^m)^k}{[(1-|z_1|)\ldots(1-|z_n|)]^{mk}}\ .$$

For $z \in D(0,1)$ we have

$$|\partial^{\beta} v_{k+2}(z)| \le C \sup_{|\alpha| \le m, \alpha \ne \beta} |\partial^{\alpha} v_{k+1}(z)|\ .$$

A repeated application of Lemma 2.2 and the inductive hypothesis then yield

$$|\partial^{\alpha} v_{k+1}(z)| \le \frac{\gamma(\alpha)\ N\ C\ e^m (Ce^m)^k}{[(1-|z_1|)\ldots(1-|z_n|)]^{mk}(1-|z_1|)^{\alpha_1}\ldots(1-|z_n|)^{\alpha_n}}\ ,$$

where

$$\gamma(\alpha) := ((mk+\alpha_1)\ldots(mk+1))\ldots((mk+\alpha_n)\ldots(mk+1)).$$

Thus, taking into account that for each $z \in D(0,1)$, and $\alpha \in \mathbb{Z}^n_+$, $|\alpha| \le m$

$$(1-|z_1|)^{\alpha_1}\ldots(1-|z_n|)^{\alpha_n} \ge (1-|z_1|)^m \ldots (1-|z_n|)^m$$

and applying repeatedly Lemma 2.1 we obtain

$$|v_{k+2}(z)| \le \frac{\gamma(\alpha)}{[m(k+1)]^m} \frac{N(Ce^m)^{k+1}}{[(1-|z_1|)\ldots(1-|z_n|)]^{m(k+1)}}$$

and (2.5) follows for obviously

$$\frac{\gamma(\alpha)}{[m(k+1)]^m} \le 1\ .$$

Finally, let us supply the proofs of Lemmas 2.1 and 2.2.

Proof of Lemma 2.1: For each $z \in \mathbb{D}$ we have

$$f(z) = \int_0^z f'(\xi)\, d\xi.$$

Thus,

$$|f(z)| \le \int_0^{|z|} \frac{dt}{(1-t)^p} \le \frac{1}{p(1-|z|)^p},$$

for p: $0 < p \le 1$. If $p > 1$ we must check that

$$\frac{1}{p-1}\left((1-|z|)^{1-p} - 1\right) \le \frac{1}{p(1-|z|)^p},$$

or, setting $\zeta = 1 - |z|$,

$$\zeta(1-\zeta^{p-1}) \le \frac{p-1}{p}, \quad 0 \le \zeta \le 1.$$

But the maximum value of the expression in the left hand side is attained at $\zeta_o = p^{-\frac{1}{p-1}}$, that value being $p^{-\frac{1}{p-1}}(\frac{p-1}{p})$. That number is obviously less than $\frac{p-1}{p}$ provided $p > 1$.

Proof of Lemma 2.2: Take $z \in \mathbb{D}$. For any s: $0 < s < 1$ the circle $\gamma := \{\xi : |\xi - z| = s(1 - |z|)\}$ is contained in $\mathbb{D}$. Using Cauchy's formula we obtain

$$|f'(z)| \le \frac{1}{2\pi s(1-|z|)} \left\{ \int_0^{2\pi} |f(z + s(1-|z|)e^{i\theta})|\, d\theta \right\}.$$

Since

$$(1 - |\xi|)^p \ge (1 - |z|)^p (1 - s)^p$$

for each $\xi \in \gamma$,

$$|f'(z)| \le \frac{1}{(1-|z|)^{p+1} s(1-s)^p} .$$

For $\varphi(s) = s(1-s)^p$ the maximum value for s: $0 < s < 1$ is $\varphi_m = \frac{1}{p+1}\left(1 - \frac{1}{p+1}\right)^p$. To get the desired estimate it suffices to show that

$$\frac{1}{p+1}\left(1 - \frac{1}{p+1}\right)^p \ge \frac{1}{e(p+1)} \tag{2.7}$$

for each $p > 0$. If we put $\zeta = \frac{1}{p+1}$, then (2.7) is equivalent to showing that $\log(1-\zeta) + \frac{\zeta}{1-\zeta} > 0$ for $0 < \zeta < 1$. The latter follows, e. g., from the identity

$$\log(1-\zeta) + \frac{\zeta}{1-\zeta} = \sum_{n \ge 2} \frac{n-1}{n} \zeta^n, \quad 0 < \zeta < 1.$$

Therefore,

$$|f'(z)| \le \frac{(p+1)e}{(1-|z|)^{p+1}}$$

as desired.

Notes

Almost every textbook on PDE's contains a statement and a proof of the C-K theorem, the only truly general theorem in the theory of

PDE, although usually coached in a slightly different language of real-analytic functions (cf. e.g. [G3], [J2], [Ha]). The statement presented here, and especifically the proof based on Picard's method of iterations, often used in ODE's, are from Hörmander's book [Hör1]. The main feature of Hörmander's version that makes it so crucial for applications is a rather precise estimate on the size of a domain of existence for the solution to Cauchy's problem that is *independent* of how large the data is, or as presented in this section how large the non-homogeneous term is, provided that they are holomorphic in a sufficiently large domain.

Chapter 3

Remarks on the Cauchy-Kovalevskaya Theorem

3.1. One can replace the zero data in the C-K theorem by any data holomorphic in $D(0', R)$, i.e., in (2.1) we can ask for a solution w satisfying

$$\begin{aligned} w|_{\{z_n=0\}} &= g_0(z'), \\ \partial_n w|_{\{z_n=0\}} &= g_1(z'), \\ \dots &\quad \dots \\ \partial_n^{m-1} w|_{\{z_n=0\}} &= g_{m-1}(z'), \end{aligned}$$

where $g_0, \dots, g_{m-1}$ are arbitrary functions of $(n-1)$ variables holomorphic in $\{|||z'||| < R\}$.

Indeed, consider $G(z) := g_0(z') + \frac{z_n}{1!} g_1(z') + \cdots + \frac{z_n^{m-1}}{(m-1)!} g_{m-1}(z')$. Then, $(\partial_n)^k (G - w)|_{\{z_n=0\}} = 0$ for all $k \le m-1$.

Thus, we have

$$\begin{aligned} \partial_n^m w &= (\partial_n)^m (w - G) \\ &= \sum_{\substack{|\alpha| \le m \\ \alpha \ne (0,..,m)}} a_\alpha(z) \partial^\alpha (w - G) + f + \sum_{\substack{|\alpha| \le m \\ \alpha \ne (0,..,m)}} a_\alpha(z) \partial^\alpha G. \end{aligned}$$

Since G is holomorphic in $D(0', R) \times \mathbb{C}$,

$$f_1 := f(z) + \sum_{\substack{|\alpha| \le m \\ \alpha \ne (0,..,m)}} a_\alpha(z) \partial^\alpha G$$

is holomorphic in $D(0, R)$.

3.2. The multi-index β in the C-K theorem need not be of the form $(0, .., m)$, it can be an arbitrary multi-index $\beta = (\beta_1, .., \beta_n)$ with $|\beta| = m$.

However, in that case, the data must be given in the form

$$\begin{aligned} w|_{\{z_1=0\}} &= \cdots = \partial_1^{\beta_1 - 1} w|_{\{z_1=0\}} = 0 \\ &\quad \vdots \\ w|_{\{z_n=0\}} &= \cdots = \partial_n^{\beta_n - 1} w|_{\{z_n=0\}} = 0. \end{aligned}$$

This type of problem is known as *Goursat Problem.* The statement (and the proof) of the C-K theorem for that situation remain essentially the same. The obvious changes needed in Lemma 2.3 are left as an

4 exercise (cf. [Hörl, §9.~~1~~]).

3.3. An important example of the Goursat problem used in the sequel is the following.

Let $a(z, w)$, $b(z, w)$, $c(z, w)$ be holomorphic in a neighborhood of a point $(\xi, \eta) \in \mathbb{C}^2$ and seek a solution R of the Cauchy problem

$$\begin{cases} \frac{\partial^2 R}{\partial z \partial w} = \frac{\partial (aR)}{\partial z} + \frac{\partial (bR)}{\partial w} - cR; \\ R|_{\{z=\xi\}} = exp\left(\int_\eta^w a(\xi, \tau)\, d\tau\right); \\ R|_{\{w=\eta\}} = exp\left(\int_\xi^z b(\tau, \eta)\, d\tau\right). \end{cases}$$

The (unique) holomorphic solution $R := R(z, w; \xi, \eta)$ whose existence follows from the general C-K theorem is called the Riemann function for the operator $\mathcal{L} := \frac{\partial^2}{\partial z \partial w} + a\frac{\partial}{\partial z} + b\frac{\partial}{\partial w} + c$ at (ξ, η) and is the key to the Riemann method of integration of the hyperbolic equations in two variables.

3.4. In case when the order of derivatives in the right hand side in the Cauchy problem is larger than m the theorem may fail altogether.

The following example which is due to Kovalevskaya herself [Ko] illustrates the above comment rather well. Consider the "heat equation"

$$\begin{cases} \dfrac{\partial w}{\partial z_2} = \dfrac{\partial^2 w}{\partial z_1^2} \\ w(z_1, 0) = f(z_1) \end{cases}$$

with data f being a holomorphic function near the origin.

Suppose $w(z_1, z_2)$ is a solution holomorphic in a bi-disk centered at the origin. Then we can write

$$w(z_1, z_2) = \sum_{n=0}^{\infty} a_n(z_1) z_2^n,$$

where $a_0(z_1) = f(z_1)$ and all a_n's are holomorphic in a disk $\{\|z_1\| < R\}$.

Applying the equation we find

$$\sum_{n=1}^{\infty} n a_n(z_1) z_2^{n-1} = \sum_{n=1}^{\infty} a''_{n-1} z_2^{n-1}.$$

Hence $n a_n(z_1) = a''_{n-1}$ for all $n \geq 1$. Since $a_0(z_1) = f(z_1)$ we obtain that $a_n(z_1) = \frac{f^{(2n)}(z_1)}{n!}$.

In particular, $w(0, z_2) = \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{n!} z_2^n$. So, if we take, e.g., $f(z_1) := \sum z_1^n$, we have $f^{(2n)}(0) = (2n)!$ and that gives us the expansion $w(0, z_2) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} z_2^n$ that converges nowhere!

Moreover, one can even show that a necessary condition for existence of a holomorphic solution of this problem is for $f(z_1)$ to be an entire function of order at most 2, i.e., $|f(z_1)| \leq const \cdot exp(C|z_1|^2)$ (cf. [KS5]).

3.5. Suppose Γ is a non-singular analytic hypersurface in a neighborhood U of $z^o \in \Gamma$, i.e.,

$$\Gamma = \{z \in \mathbb{C}^n \ : \ \varphi(z) = 0, \varphi \text{ is holomorphic in } U, \\ \nabla_z \varphi := (\partial_1 \varphi, \cdot\cdot, \partial_n \varphi) \neq (0, \cdot\cdot, 0) \text{ in } U\}.$$

We want to solve the following Cauchy problem

$$\begin{cases} \sum_{|\alpha| \leq m} a_\alpha(z) \partial^\alpha w = f(z); \\ \partial^\alpha (w - g) = 0 \text{ on } \Gamma \bigcap U, \ |\alpha| \leq m - 1, \end{cases}$$

where a_α, f and g are, say, holomorphic in a slightly larger neighborhood U' of z^o.

Without loss of generality we can assume $z^o = 0$ and $\frac{\partial \varphi}{\partial z_n} \neq 0$ in perhaps slightly smaller neighborhood $V \subset U$. Then, the change of variables $z = (z', z_n) \to \xi := T(z) = (\xi', \xi_n)$ given by $\xi' = z'$ and $\xi_n = \varphi(z)$ is bi-holomorphic near z^o. Changing the coordinates to ξ by setting $a_\alpha(z) = a_\alpha(T^{-1}(\xi)) =: A_\alpha(\xi)$, $f(z) = F(\xi)$, $g(z) = G(\xi)$ and $w(z) = W(\xi) = W(z', \varphi(z))$ and performing straightforward but tedious calculations, we find that our differential equation is transformed into

$$C(\xi) \frac{\partial^m}{\partial \xi_n^m} W + \sum_{\substack{|\alpha| \leq m \\ \alpha \neq (0, \cdot\cdot, m)}} B_\alpha(\xi) \partial^\alpha W = F(\xi),$$

where the coefficients $B_\alpha(\xi)$ are composed from $A'_\alpha s$ and the derivatives of φ, while the coefficient $C(\xi)$ is calculated by

$$C(\xi) = C(T(z)) =: c(z) = \sum_{|\alpha| = m} a_\alpha(z) (\nabla_z \varphi)^\alpha.$$

Thus, in order to be able to write down our Cauchy problem in ξ variables in a canonical form (cf. Chapter 2)

$$\begin{cases} \frac{\partial^m}{\partial \xi_n^m} W = \sum_{\substack{|\alpha| \leq m \\ \alpha \neq (0, \cdot\cdot, m)}} \mathcal{D}_\alpha(\xi) \partial^\alpha W + F_1(\xi); \\ \partial^\alpha (W - G)|_{\{\xi_n = 0\}} = 0, \text{ for all } \alpha : \ |\alpha| \leq m - 1 \end{cases}$$

with coefficients $\mathcal{D}_\alpha$, F_1 holomorphic near the origin, it is necessary and sufficient that $c(z) \neq 0$ near z^o, or, by continuity, that

$$\sum_{|\alpha| = m} a_\alpha(z) (\nabla_z \varphi)^\alpha |_{z^o} \neq 0.$$

Thus we have the following extension of the C-K theorem.

Theorem 3.1 *The holomorphic solution w of the Cauchy problem*

$$\begin{cases} \sum_{|\alpha|\le m} a_\alpha(z)\partial^\alpha w = f(z) \\ \partial^\alpha(w-g) = 0 \text{ on } \Gamma\bigcap U, \quad \text{for all } \alpha : |\alpha| \le m-1 \end{cases}$$

exists and is unique in a neighborhood V of z^o, whose size depends only on a_α, Γ, and f but not on g, provided that $c(z^o) = \sum_{|\alpha|=m} a_\alpha(z)(\nabla\varphi)^\alpha|_{z^o} \neq 0$.

$c(z)$ is called a *characteristic form* of the differential operator

$$\mathcal{L} := \sum_{|\alpha|\le m} a_\alpha(z)\partial^\alpha.$$

If $c(z^o) \neq 0$, $z^o \in \Gamma$, Γ is called *non-characteristic* (w.r.t. the operator $\mathcal{L}$) at z^o.

If Γ is characteristic at z^o, i.e., $c(z^o) = 0$, both uniqueness and existence in the C-K theorem may fail.

Indeed, recall the Cauchy problem for the heat equation

$$\begin{cases} \dfrac{\partial^2 w}{\partial z_1^2} - \dfrac{\partial w}{\partial z_2} = 0; \\ w(z_1, 0) = f(z_1) \end{cases}$$

where as we saw a solution need not exist. Since in this case $\Gamma = \{z \in \mathbb{C}^2 : \varphi(z) := z_2 = 0\}$, $c(z) = (\frac{\partial\varphi}{\partial z_1})^2 \equiv 0$, i.e., Γ is everywhere characteristic for the heat operator.

To see the failure of uniqueness consider the Laplace operator $\triangle := \sum_{j=1}^n \frac{\partial^2}{\partial z_j^2}$ and $\Gamma = \{z : \sum_{j=1}^n a_j z_j = 0\}$ with $\sum_{j=1}^n a_j^2 = 0$. Then Γ is everywhere characteristic w.r.t. $\triangle$. On the other hand, for any $p \ge 2$, the function $w(z) = (\sum_{j=1}^n a_j z_j)^p$ is harmonic in $\mathbb{C}^n$ and satisfies $w = \nabla_z w = 0$ on Γ.

Notes

In presenting the C-K theorem in the most general form (Theorem 3.1) we still follow Hörmander's book [Hör1]. The notion of a characteristic point together with the relevant examples associated with the heat equation appear in S. Kovalevskaya's paper [Ko]. (See [KS5] for some related historical background). B. Riemann introduced the Riemann function in order to calculate explicitly solutions of Cauchy problems for hyperbolic equations in two variables. More on properties of the Riemann function together with other proofs of its existence and uniqueness can be found e.g. in [G3], [J2], [Ha], [Hen], [V] and references cited therein.

Chapter 4

Holmgren's Uniqueness Theorem

Let us start out by recalling a simple uniqueness theorem for harmonic functions.

Theorem 4.1 *Let Ω be a C^1-bounded domain in $\mathbb{R}^n$, and Γ be a C^1-hypersurface inside Ω that divides it into two parts: Ω^+ and Ω^-. Let $u \in C^2(\Omega^+ \cup \Omega^-) \cap C^1(\overline{\Omega})$. Suppose $\Delta u = 0$ in $\Omega^+ \cup \Omega^- = \Omega \setminus \Gamma$. Then u extends as a harmonic function to all of Ω.*

Corollary 4.1 *If u is harmonic in Ω^+, $u \in C^1(\overline{\Omega^+})$ and $u|_\Gamma = \nabla u|_\Gamma = 0$, then $u \equiv 0$.*

Proof of the Corollary: Set $u|_{\Omega^-} \equiv 0$. By Theorem 4.1, u is harmonic in Ω and thus $u \equiv 0$.

Remark 4.1 This is an analogue for harmonic functions of a well-known result on removable singularities of analytic functions based on Morera's theorem: let Ω, $\Gamma \subset \mathbb{R}^2$ be as in Theorem 4.1 and let f be analytic in $\Omega^+ \cup \Omega^-$ and continuous in $\overline{\Omega}$. Then f extends as an analytic function to all of Ω.

Proof of the Theorem 4.1: For the sake of simplicity take $n = 3$.

Fix $x \in \Omega^+$, we have

$$u(x) = \frac{1}{4\pi} \int_{\partial\Omega^+} [u(y) \frac{\partial}{\partial n_y}(|x-y|^{-1}) - \frac{\partial u}{\partial n_y}(|x-y|^{-1})] dS_y, \qquad (4.1)$$

where n is the outer normal and dS_y is Lebesgue measure on $\partial\Omega^+$.

Also, (by Green's formula)

$$0 = \frac{1}{4\pi} \int_{\partial\Omega^-} [u(y) \frac{\partial}{\partial n_y}(|x-y|^{-1}) - \frac{\partial u}{\partial n_y}(|x-y|^{-1})] dS_y.$$

Hence, in (4.1) the part of integral over Γ can be replaced by that over $\partial\Omega^- \setminus \Gamma$. Substituting this into (4.1) gives the extension of u into all of Ω.

Holmgren's uniqueness theorem is a far reaching extension of Corollary 4.1 to solutions of arbitrary equations with real-analytic coefficients. Here, for the sake of simplicity, we shall present it in a slightly reduced form.

Theorem 4.2 *Let Ω be a domain in $\mathbb{R}^n$ with C^1 boundary, $\Omega \subset \{x_n > 0\}$, and $\{x' : \|x'\| < R\} \subset \partial\Omega \cap \{x_n = 0\}$. Let $w \in C^m(\Omega) \cap C^{m-1}(\overline{\Omega})$ be a solution of*

$$\partial_n^m w = \sum_{\substack{|\alpha| \le m \\ \alpha \ne (0,\cdots,0,m)}} a_\alpha(x) \partial^\alpha w$$

in Ω and $w|_{\{x_n=0\}} = \cdots = \partial_n^{m-1} w|_{\{x_n=0\}} = 0$. Then , if all a_α's are real-analytic in $\overline{\Omega}$, there exists $\delta > 0$ such that $w \equiv 0$ in $\{x : \|x\| \le \delta\} \cap \Omega$.

The idea of proof comes from the ODE's .

Indeed, let us discuss the following example. Consider a solution u of the o.d.e. $u'' + pu' + qu = 0$ on $[0,1]$, with the coefficients $p, q \in C^1[0,1]$ satisfying $u(0) = u'(0) = 0$. Then , $u \equiv 0$ on $[0,1]$.

Take $0 < a < 1$. We want to show $u|_{[0,a]} \equiv 0$. Take any $v \in C^1[0,a]$ such that $v(a) = v'(a) = 0$ Integration by parts yields:

$$\int_0^a u''v dx = u'v|_0^a - \int_0^a u'v' dx = \int_0^a uv'' dx;$$

$$\int_0^a pu'v dx = -\int_0^a u(pv)' dx.$$

So denoting by L the ordinary differential operator $L : \dfrac{d^2}{dx^2} + p\dfrac{d}{dx} + q$, we have

$$\int_0^a (Lu)v\,dx = 0 = \int_0^a u(v'' - (pv)' + q)dx = \int_0^a uL^\sharp v\,dx$$

where $L^\sharp := \dfrac{d^2}{dx^2} - p\dfrac{d}{dx} + (q - p')$ denotes the adjoint operator.

By the Main Existence Theorem for ODE's we can solve the initial value problem

$$\begin{cases} L^\sharp v = f; \\ v(a) = v'(a) = 0 \end{cases}$$

on $[0, 1]$ for any reasonable function f, say a polynomial.

Therefore, $\int_0^a uf\,dx = 0$ for all polynomials f and hence $u \equiv 0$.

Proof of the Theorem: For the sake of simplicity, we take up the case $m = n = 2$. We have

$$\frac{\partial^2 w}{\partial y^2} = -a(x,y)\frac{\partial^2 w}{\partial x^2} - b(x,y)\frac{\partial^2 w}{\partial x \partial y} - c(x,y)\frac{\partial w}{\partial x} - d(x,y)\frac{\partial w}{\partial y} - e(x,y)w,$$

where a, b, c, d, e are analytic in Ω.

Consider $C_\varepsilon := \{(x,y) : y + x^2 = \varepsilon\}$ and let $\Omega_\varepsilon \subset \Omega$ be a domain bounded by C_ε and $\{y = 0\}$. Take $v \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega_\varepsilon})$ such that v, ∇v vanish on C_ε.

Claim.

$$\int\int_{\Omega_\varepsilon} (Lu)v\,dxdy = \int\int_{\Omega_\varepsilon} u(L^\sharp v)dxdy = 0,$$

where

$$L = \frac{\partial^2}{\partial y^2} + a(x,y)\frac{\partial^2}{\partial x^2} + b(x,y)\frac{\partial^2}{\partial x \partial y} + c(x,y)\frac{\partial}{\partial x} + d(x,y)\frac{\partial}{\partial y} + e(x,y)$$

and $L^\sharp$ is the adjoint operator

$$L^\sharp w = \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2(aw)}{\partial x^2} + \frac{\partial^2(bw)}{\partial x \partial y} - \frac{\partial(cw)}{\partial x} - \frac{\partial(dw)}{\partial y} + ew$$

Proof of the claim: Note that all the products uv, $u_x v, \cdots, u_y v_y$ vanish on $\partial\Omega_\varepsilon$.

The proof of the claim is obtained by a straightforward computation involving integration by parts. For the reader's convenience we shall perform it for two of the terms involved.

Integrating by parts twice, we obtain

$$\begin{aligned}
\int\int_{\Omega_\varepsilon} bv\frac{\partial^2 u}{\partial x \partial y}dxdy &= \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left\{ \int_0^{\varepsilon - x^2} bv\frac{\partial^2 u}{\partial x \partial y}dy \right\} dx \\
&= -\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left\{ \int_0^{\varepsilon - x^2} \frac{\partial(bv)}{\partial y}\frac{\partial u}{\partial x}dy \right\} dx \\
&= -\int_0^{\varepsilon} \left\{ \int_{-\sqrt{\varepsilon - y}}^{\sqrt{\varepsilon - y}} \frac{\partial(bv)}{\partial y}\frac{\partial u}{\partial x}dx \right\} dy \\
&= \int_0^{\varepsilon} \left\{ \int_{-\sqrt{\varepsilon - y}}^{\sqrt{\varepsilon - y}} \frac{\partial^2(bv)}{\partial x \partial y}u\,dx \right\} dy \\
&= \int\int_{\Omega_\varepsilon} u\frac{\partial^2(bv)}{\partial x \partial y}dxdy.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int\int_{\Omega_\varepsilon} cv\frac{\partial u}{\partial x}dxdy &= \int_0^{\varepsilon} \left\{ \int_{-\sqrt{\varepsilon - y}}^{\sqrt{\varepsilon - y}} cv\frac{\partial u}{\partial x}dx \right\} dy \\
&= -\int_0^{\varepsilon} \left\{ \int_{-\sqrt{\varepsilon - y}}^{\sqrt{\varepsilon - y}} \frac{\partial(cv)}{\partial x}u\,dx \right\} dy
\end{aligned}$$

$$= -\int\int_{\Omega_\varepsilon} u\frac{\partial(cv)}{\partial x}dxdy.$$

Provided that C_ε is not characteristic with respect to $L^\sharp$ for small ε, we can solve the Cauchy problem

$$\begin{cases} L^\sharp v = p(x,y) \\ v = \nabla v = 0 \text{ on } C_\varepsilon \end{cases}$$

with $p(x,y)$ being an arbitrary polynomial.

By the C-K theorem (cf. Chapter 3), the solution v exists in a domain D of size $const\sqrt{\varepsilon}$ which therefore covers the origin. Now the claim implies that for all sufficiently small ε, $\int\int_{\Omega_\varepsilon} up(x,y)dxdy = 0$ for any polynomial p and therefore $u \equiv 0$ in Ω_ε.

To check that C_ε is non-characteristic for $L^\sharp$ we note that C_ε is given by the equation $C_\varepsilon : \varphi(x,y) := x^2 + y - \varepsilon = 0$ and the characteristic (with respect to $L^\sharp$) form for the function φ can be written as

$$Char(L^\sharp, C_\varepsilon) = 1 + (\text{terms containing } \frac{\partial\varphi}{\partial x}).$$

Since $\dfrac{\partial\varphi}{\partial x} = 0$ at $(0,0)$, C_ε is non-characteristic for small ε.

The following exercise (for an ambitious reader) establishes Holmgren's theorem in full generality.

Exercise 4.1 Extend Holmgren's theorem to general operators $L = \sum a_\alpha(x)\partial^\alpha$ with holomorphic coefficients and the "data" surface $\{x_n = 0\}$ replaced by an arbitrary C^1-surface Γ that is non-characteristic with respect to L.

Notes

The Holmgren uniqueness theorem (Theorem 4.2) was proved by Holmgren [Hol] in a special case and by John [J1] in full generality (Exercise 4.1). Also, see [Hör1], [J2].

The discussion of a rather special case presented here follows Petrovsky [P]. Theorem 4.1 is from Kellogg's classic [Ke]. He attributes the result to Kovalevskaya.

Chapter 5

The Continuity Method of F. John

Sometimes one can extend Holmgren's uniqueness theorem, which is local in nature, globally. The following example serves as a good illustration of the methods one may use.

Theorem 5.1 *Let a C^2-smooth function u satisfy the following equation in $\mathbb{R}^{n+1}$:*

$$\frac{\partial^2 u}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{n} a_j \frac{\partial u}{\partial x_j} + a_{n+1} \frac{\partial u}{\partial t} + a_0 u$$

in the double cone $C = \{(x,t) : \| x \| + | t | < R\}$ and have zero Cauchy data on $\{(x,0) : \| x \| < R\}$ (the base of the cone). Then $u \equiv 0$ provided that all the coefficients are real analytic in a neighbourhood of C.

For the sake of simplicity, we shall prove the theorem assuming in addition that $n = 1$ and $R = 1$. The following lemma is elementary and its proof is left as an exercise.

Lemma 5.1 *Fix a point (x_0, t_0) in C with $t_0 > 0$. There exists a function f on $[-1, 1]$ satisfying the following properties:*

(i) $f(-1) = f(1) = 0$;

(ii) $f \in C^1[-1, 1]$;

(iii) $f(x_0) > t_0$;

(iv) $| f'(x) | < 1$ *on* $[-1, 1]$.

Proof of the theorem 5.1: For $s \in [0, 1]$ define

$$\mathcal{D}_s = \{(x,t) : t - sf(x) < 0\}$$

where f is as in Lemma 5.1. Clearly $(x_0, t_0) \in \mathcal{D}_1$.

Let $E = \{s \in [0,1] : u \equiv 0 \text{ in } \mathcal{D}_s\}$. We shall show that $E = [0,1]$. Indeed,

(i) E is nonempty in view of Holmgren's theorem;

(ii) E is obviously closed. For if $s_n \to s_0$, $\{s_n\}_{n=1}^{\infty} \subset E$ then

$$\mathcal{D}_{s_0} \subset \cup_{n=1}^{\infty} \mathcal{D}_{s_n}$$

and so $u \equiv 0$ in $\mathcal{D}_{s_0}$, implying that $s_0 \in E$.

(iii) E is open. It is again a direct consequence of Holmgren's uniqueness theorem provided that we check that the part of the boundary of $\mathcal{D}_s$ given by the curve $t = sf(x)$ is non-characteristic. But this is obvious since setting $\varphi(x,t) := t - sf(x)$ we have

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 - \left(\frac{\partial \varphi}{\partial t}\right)^2 = s^2 \mid f'(x) \mid^2 - 1 < 0$$

because of property (iv) in Lemma 5.1 satisfied by the function f.

So we have $E = [0,1]$ and $u \equiv 0$ in the upper-half of the double cone. The argument for the lower half of C is the same.

Notes

The method of proof of Theorem 5.1 is due to F. John [J2], also cf. [Hör1]. In the following sections we shall see how similar ideas can be applied to expand the domain of existence of solutions of analytic PDE's.

Chapter 6

Zerner's Theorem

Consider differential operator with holomorphic coefficients in $\mathbb{C}^n$

$$\mathcal{L} = \sum_{|\alpha| \leq m} a_\alpha(z) \partial^\alpha$$

and a real hyperplane H in $\mathbb{C}^n$ given by

$$H = \{z : \sum_{j=1}^{n} a_j x_j + b_j y_j = t\},$$

where $x_j = Re\ z_j$, $y_j = Im\ z_j$ and a_j, b_j, t are real numbers. Writing $x_j = \frac{z_j + \overline{z_j}}{2}$, $y_j = \frac{z_j - \overline{z_j}}{2i}$, we can express this as

$$H: \quad \sum_{j=1}^{n} \frac{a_j - ib_j}{2} z_j + \frac{a_j + ib_j}{2} \overline{z_j} = t$$

or, setting $\lambda_j = \frac{1}{2}(a_j - ib_j)$, we obtain

$$H: \quad Re\Big(\sum_{j=1}^{n} \lambda_j z_j\Big) = t.$$

Note that H contains a unique complex hyperplane

$$\Pi: \quad \sum_{j=1}^{n} \lambda_j z_j = t,' \qquad Re\, t' = t\ , \quad \Pi \ni z^0$$

Definition 6.1 *A real hyperplane H passing through a point $z_0 \in \mathbb{C}^n$ is called Zerner characteristic with respect to the operator $\mathcal{L}$ at z_0 if Π is characteristic with respect to $\mathcal{L}$ at z_0, i.e.*

$$\sum_{|\alpha| = m} a_\alpha(z_0) \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n} = 0.$$

Definition 6.2 *A C^1-real hypersurface Γ given by $\{z : \varphi(z) = 0\}$, where φ is a C^1 real-valued function, $\nabla\varphi \neq 0$ near a point $z_0 \in \Gamma$, is called Zerner characteristic with respect to the operator $\mathcal{L}$ at z_0 if the real hyperplane H tangent to Γ at z_0 is Zerner characteristic with respect to $\mathcal{L}$ at z_0.*

Because H is the tangent plane to Γ at z_0, we can write its equation in the form

$$H: \quad \sum_{j=1}^{n} \frac{\partial\varphi}{\partial x_j}\Big|_{z_0} x_j + \frac{\partial\varphi}{\partial y_j}\Big|_{z_0} y_j = t,$$

or,

$$H: \quad Re \sum_{j=1}^{n} \frac{1}{2}\Big(\frac{\partial\varphi}{\partial x_j} - i\frac{\partial\varphi}{\partial y_j}\Big)\Big|_{z_0} z_j = t.$$

But then Π (the complex hyperplane tangent to Γ at z_0) is given by

$$\Pi: \quad \sum_{j=1}^{n} \frac{\partial\varphi}{\partial z_j}\Big|_{z_0} z_j = t$$

where as usual

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big),$$

and so Π is Zerner characteristic with respect to $\mathcal{L}$ at z_0 if and only if

$$\sum_{|\alpha| = m} a_\alpha(z_0)(\nabla_z\varphi)^\alpha\Big|_{z_0} = 0.$$

Now we can state *Zerner's theorem.*

Theorem 6.1 *Let u be a holomorphic solution of the equation $\mathcal{L}u = f$ in a domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary, and assume that the coefficients a_α, $|\alpha| \leq m$ and f are holomorphic in $\overline{\Omega}$. Let $z_0 \in \partial\Omega$. If $\partial\Omega$ is non-characteristic at z_0 with respect to $\mathcal{L}$ then u extends holomorphically into a neighbourhood of z_0.*

Example 6.1 Let $\Omega = B_1(0) =: \{z \in \mathbb{C}^n : \| z \| < 1\}$ and $\mathcal{L} = \sum_{j=1}^n \frac{\partial^2}{\partial z_j^2}$. In this case, Ω is a domain of holomorphy, i.e. there exist functions which are holomorphic in Ω and do not extend to a larger set. Let us find Zerner characteristic points with respect to $\mathcal{L}$ in $\partial\Omega$. The equation of $\partial\Omega$ can be written in the form

$$\sum_{j=1}^n z_j \overline{z_j} - 1 = 0,$$

and so $z \in \partial\Omega$ is characteristic if and only if

$$\sum_{j=1}^n z_j^2 = 0.$$

Thus, except for the points that lie on the cone $\{z : \sum_{j=1}^n z_j^2 = 0\}$, called an *isotropic cone*, any solution of $\mathcal{L}u = 0$ can be continued holomorphically across all other points on the unit sphere.

Proof of the theorem 6.1: Without loss of generality we may assume $z_0 = 0$, and that the normal vector $(0, 0, \dots, 1)$ is pointing inside Ω. Thus the real hyperplane H tangent to $\partial\Omega$ at z_0 is $\{Re\, z_n = 0\}$. For $\varepsilon > 0$ consider the hyperplanes

$$H_\varepsilon : \qquad Re\, z_n = \varepsilon.$$

By continuity, H_ε is still non-characteristic with respect to $\mathcal{L}$ if ε is sufficiently small. Let Π_ε be the unique complex hyperplane inside H_ε.

Claim. $\Pi_\varepsilon \cap \Omega$ contains an $(n-1)$- dimensional polydisk centered at the point $(0, \dots, 0, \varepsilon)$ of radius $R(\varepsilon)$ such that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{R(\varepsilon)} = 0.$$

Assuming the claim, we have u holomorphic in the polydisk of radius $R(\varepsilon)$ inside Π_ε. Also Π_ε is non-characteristic with respect to $\mathcal{L}$. So, we can solve the Cauchy problem

$$\begin{cases} \mathcal{L}v = f & \text{near} \quad \Pi_\varepsilon \cap \Omega \\ \partial^\alpha (v - u) = 0, \ | \alpha | \le m - 1 & \text{on} \quad \Pi_\varepsilon. \end{cases}$$

By the uniqueness for solutions of Cauchy problems, $v = u$, and hence applying the Cauchy-Kovalevskaya theorem we conclude that u is holomorphic in a polydisk centered at $(0, \dots, 0, \varepsilon)$ of radius $C \cdot R(\varepsilon)$, where C is a constant depending on the coefficients of $\mathcal{L}$ *but not on u*.

In view of the claim, this polydisk covers the origin for small ε and so the conclusion of the theorem follows.

Proof of the claim. According to our normalization, the boundary of Ω near the origin is given by

$$Re\, z_n = \psi(z', Im\, z_n)$$

where $\psi \in C^1$, $\psi(0) = 0$ and $d\psi(0) = 0$. Thus, the claim simply states that $\psi(z', Im\, z_n) = o(| z |)$ near the origin.

Remark. Zerner's theorem is a $\mathbb{C}^n$-*theorem only*. One can easily find a function harmonic inside the unit disk in $\mathbb{C}$ that does not extend across any point on the unit circle, although the surface $\varphi(x, y) := Re\,(x^2 + y^2 - 1) = 0$ in $\mathbb{C}^2$ does not have any characteristic points with respect to the Laplacian. Indeed,

$$Re \left\{ \left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right\} = Re \left(4(x^2 + y^2) \right) = 4.$$

Corollary 6.1 *(Delassus-Le Roux). If u is holomorphic in a domain Ω in $\mathbb{C}^n$ except for points on a holomorphic hypersurface $\Gamma = \{z : \varphi(z) = 0\}$, and satisfies a PDE $\mathcal{L}u = f$ in $\Omega \setminus \Gamma$ with holomorphic coefficients in Ω then u extends holomorphically to Ω provided Γ is non-characteristic with respect to $\mathcal{L}$. Thus, Γ can only be a non-removable singularity set for u if Γ is characteristic with respect to $\mathcal{L}$.*

PROOF: Imbed Γ into a real hypersurface S given by the equation $Re\,\varphi(z) = 0$. Near Γ S is nowhere Zerner characteristic with respect to $\mathcal{L}$. Indeed, the characteristicity condition is in this case

$$\sum_{|\alpha|=m} a_\alpha(z)(\nabla_z\varphi)^\alpha\Big|_\Gamma = 0$$

because $\partial_z\overline{\varphi} = 0$. So all points on $\Gamma \subset S$ satisfy the hypothesis of Zerner's theorem and the corollary follows.

Example 6.2. If the rational function f/g satisfies the Laplace equation, with f and g relatively prime polynomials, then

$$\sum_{j=1}^{n}\left(\frac{\partial g}{\partial z_j}\right)^2 = 0$$

on the variety $\{g = 0\}$.

Method of globalizing families

Suppose that we want to continue a solution u of the equation $\mathcal{L}u = f$ holomorphic in a domain $\Omega \subset \mathbb{C}^n$ to a larger domain $\widetilde{\Omega}$ containing Ω. Of course, we shall always assume that all the coefficients are holomorphic in $\widetilde{\Omega}$.

A simple idea is then to construct a family of bounded domains Ω_t, $t \in [0,1]$ (we shall call it a *globalizing family*-GF) satisfying the following properties

(i) $\Omega_t \subset \Omega_s$ if $t \le s$;

(ii) $\Omega_1 = \widetilde{\Omega}$;

(iii) $\Omega_t \subset \Omega$ for t sufficiently small;

(iv) All points on the boundaries of all $\partial\Omega_t \setminus \Omega$ satisfy the hypothesis of Zerner's theorem (with respect to the operator $\mathcal{L}$);

(v) (*Continuity property*) $\bigcup_{s<t} \Omega_s = \Omega_t$, $\bigcap_{s>t} \Omega_s = \overline{\Omega}_t$.

Globalizing Principle. *If for given $\mathcal{L}, \Omega, \widetilde{\Omega}$ we can find a globalizing family satisfying (i)-(v), then any solution u of $\mathcal{L}u = f$ holomorphic in Ω extends to $\widetilde{\Omega}$.*

Indeed let $E = \{t \in [0,1] : u \text{ is holomorphic in } \Omega_t\}$. Then

a) E is nonempty because of (iii).

b) E is obviously closed (follows at once from (i) and (v)).

c) E is open in view of (iv), Zerner's theorem, and compactness of $\partial\Omega_t$.

Hence $E = [0,1]$ and so u extends to $\widetilde{\Omega}$ by (ii).

The following simple examples hopefully illustrate well the power of the method.

Corollary 6.2 *Let u be holomorphic in a $\mathbb{C}^n$-neighbourhood of the isotropic cone*

$$\{z \in \mathbb{C}^n : \sum_{j=1}^{n} z_j^2 = 0\}$$

and satisfy the Laplace equation there. Then u extends as an entire function to all of $\mathbb{C}^n$.

PROOF: The corollary follows at once from the Globalizing Principle if we choose the following globalizing family:

$$\Omega_t = \{z : \sum_{j=1}^{n} |z_j|^2 \le t^2\},\ t \in [0,\infty).$$

Remark. In Zerner's theorem the hypothesis of Ω being C^1 can be relaxed somewhat. It suffices to require that $\partial\Omega$ has a tangent everywhere and when that tangent plane is moved parallel to itself inside Ω

by the distance ε in normal direction the intersection with Ω contains a polydisk of radius $R(\varepsilon)$ so that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{R(\varepsilon)}=0.$$

Corollary 6.3 *The solution of the Cauchy-Goursat problem*

$$\begin{cases} \partial^{\beta}w &= \sum_{|\alpha|<m} a_\alpha(z)\partial^\alpha w + f;\\ w &= \cdots = \partial^{\beta_1-1}w = 0 \quad on\ \{z_1=0\};\\ \cdot & \cdots \quad \cdots \\ w &= \cdots = \partial^{\beta_n-1}w = 0 \quad on\ \{z_n=0\}; \end{cases}$$

is an entire function provided all the coefficients a_α, f are entire.

PROOF: We are going to prove this with a particular choice of $\beta = (0,0,\ldots,m)$, leaving the general case as an exercise.

Let us define the globalizing family to be the family of cocentric balls

$$\Omega_t = \{z : \| z \| < t, 0\le t<\infty\}.$$

It is easily seen that a point on $\partial\Omega$ is characteristic if and only if $\overline{z_n}^m = 0$, and so characteristic points are all located in the initial hyperplane.

Near those points the solution is holomorphic by the Cauchy-Kovalevskaya theorem. For all other points on $\partial\Omega_t$, w extends across by Zerner's theorem. Thus, again letting $E=\{t\in[0,\infty) : w$ is holomorphic in $\Omega_t\}$, it follows as before that $E=[0,\infty)$ and so w is an entire function.

Corollary 6.4 *The Riemann function $R(z,w;\xi,\eta)$ (cf. §3.3) for the operator*

$$\mathcal{L} := \frac{\partial^2}{\partial z\partial w} + a(z,w)\frac{\partial}{\partial z} + b(z,w)\frac{\partial}{\partial w} + c(z,w)$$

is entire for any point (ξ,η) provided that all the coefficients are entire functions.

Example 6.3. Let $\mathcal{L} = \frac{\partial^2}{\partial z\partial w} + \lambda^2$ be the Helmholtz operator. Then, we have (cf. [Ha])

$$R(z,w;\xi,\eta) = J_0(\lambda\sqrt{(z-\xi)(w-\eta)}),$$

where J_0 is the Bessel function of order zero given by

$$J_0(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}.$$

Corollary 6.5 *(The generalized heat equation). If u is a holomorphic solution of the equation*

$$\frac{\partial^2 u}{\partial z^2} + a(z,w)\frac{\partial u}{\partial w} + b(z,w)\frac{\partial u}{\partial z} + c(z,w) = f(z,w)$$

in a bidisk $\mathcal{D}(0,R)$ and a,b,c,f are entire functions then u extends as a holomorphic function to the "cylinder"

$$\{(z,w) : z\in\mathbb{C}, | w | < R\}.$$

In particular, the "data" $u(z,0)$ must be entire.

PROOF: Consider the family of ellipsoids

$$\Omega_t = \{(z,w) : \frac{|z|^2}{t^2} + \frac{|w|^2}{(R')^2}\le 1\},$$

where $R'<R$ is fixed.

To check that $\{\Omega_t\}_{t\ge 0}$ is a globalizing family we only have to verify the condition (iv) in the definition of (GF). The defining function for the boundary $\partial\Omega_t$ is

$$\varphi(z,w) = \frac{z\bar{z}}{t^2} + \frac{w\bar{w}}{(R')^2} - 1 = 0.$$

So the characteristic points are precisely those for which $z = 0$. But these points are in $D(0, R)$ where u is holomorphic. The rest of the argument follows *mutatis mutandis.*

Notes

Zerner's theorem is proved in [Ze]. Here, we have followed the exposition in Hörmander's book [Hör1, Theorem 9.4.7]. Theorem of Delassus and Le Roux can be found, e.g., in Hadamard's book [Ha].The method of globalizing families in various editions based on a continuous deformation of the boundary of a domain of regularity has appeared in works of many authors: F. John [J2] and L. Hörmander [Hör1] used it for proving various uniqueness theorems in $\mathbb{R}^n$; J. Bony and P. Schapira [BS] applied this idea to extend the domain of regularity of the solution of a PDE in $\mathbb{C}^n$ (also cf. [Hör1, Theorem 9.4.8]). The method has been developed further and used extensively by Johnsson [Jo]. Here, we followed the outline given in [KS2]. Corollary 6.2 is due to Johnsson (private communication). A proof of corollaries 6.3 and 6.4 (but not the facts) is due to the author. For a "classical" proof of Corollary 6.4 see, e.g., Vekua's book [V]. Corollary 6.5 for the heat equation was observed by Kovalevskaya in her Habilitationsschrift [Ko]. An extended version and the proof given here follow [KS5].

Chapter 7

The Bony-Schapira Theorem

Theorem 7.1 *Let $\Omega_1 \subset \Omega_2$ be convex, bounded domains in $\mathbb{C}^n$. Let*

$$\mathcal{L} = \sum_{|\alpha| \le m} a_\alpha(z)\partial^\alpha$$

be a differential operator whose coefficients a_α are holomorphic in $\overline{\Omega}_2$. Every w holomorphic in $\overline{\Omega}_1$ and satisfying $\mathcal{L}\, w = f$, with f holomorphic in Ω_2 extends holomorphically to Ω_2 provided the following condition holds: for each $z \in \Omega_2$ and each real hyperplane H passing through z and characteristic with respect to $\mathcal{L}$ at z, $H \cap \Omega_1 \neq \emptyset$.

Corollary 7.1 *Assume that all leading coefficients are constants, i.e. a_α=const, $|\alpha| = m$. If Ω_1, Ω_3 are convex, open sets in $\mathbb{C}^n$ such that $\Omega_1 \cap \Omega_3 \neq \emptyset$ and every characteristic real hyperplane H that meets Ω_3 also meets Ω_1, then every solution w of $\mathcal{L}\, w = f$ holomorphic in $\overline{\Omega}_1$ extends to $\Omega_2 := co(\Omega_1 \cup \Omega_3) := \{convex\ hull\ of\ \Omega_1 \cup \Omega_3\}$, provided that all coefficients are holomorphic in $\overline{\Omega}_2$.*

Proof of the corollary 7.1: $\Omega_2 \supset \Omega_1$ and convex so it remains to check the hypothesis of the theorem. Let $z^0 \in \Omega_2$ and $H \ni z^0$ be a real hyperplane so that H is characteristic with respect to $\mathcal{L}$. We have three posibilities:

(i) $H \cap \Omega_3 \neq \emptyset$. H is characteristic in $H \cap \Omega_3$ and hence by the hypothesis H meets Ω_1;

(ii) If $H \cap \Omega_1 \neq \emptyset$ there is nothing to prove;

(iii) If $H \cap \Omega_1 = \emptyset$, $H \cap \Omega_3 = \emptyset$ then because $\Omega_1 \cup \Omega_3$ is connected, Ω_1 , Ω_3 both lie in the same half-space with respect to H. Therefore $\Omega_2 =$

$co(\Omega_1 \cup \Omega_2)$ lies in the same half-space contradictory to our assumption that $\Omega_2 \cap H \neq \emptyset$.

Corollary 7.2 *Let u be harmonic in the ball of radius R in $\mathbb{R}^n$, $n \geq 2$ centered at the origin. Then it extends as a holomorphic function to the ball $B\left(0, \frac{R}{\sqrt{2}}\right)$ in $\mathbb{C}^n$ and the constant $\frac{R}{\sqrt{2}}$ is the best possible, i.e. u need not extend to any ball $B(0, R')$ in $\mathbb{C}^n$ with $R' > \frac{R}{\sqrt{2}}$.*

Proof of the corollary 7.2: Without lost of generality we can assume $R = 1$ and u to be harmonic in the closed unit ball $\overline{B_{\mathbb{R}^n}(0;1)}$. Since u is harmonic in $\overline{B_{\mathbb{R}^n}(0;1)}$ it is holomorphic in a closed convex $\mathbb{C}^n$ neighborhood $\overline{\Omega}_1$ of $B_{\mathbb{R}^n}(0,1)$. Let $\Omega_3 = B_{\mathbb{C}^n}\left(0, \frac{1}{\sqrt{2}}\right)$. By Corollary 7.1 it suffices to check that $\forall z^0 : |z^0| < \frac{1}{\sqrt{2}}$ and any charararacteristic plane $H : \sum_1^n a_j x_j + b_j y_j = t$ passing through z^0, $H \cap B_{\mathbb{R}^n}(0,1) \neq \emptyset$. The characteristicity of H (with respect to the Laplacian) is equivalent to

$$\sum_1^n a_j^2 = \sum_1^n b_j^2 \text{ and } \sum_1^n a_j b_j = 0.$$

Hence, without loss of generality, we can rescale the equation of H so that $\|a\| = \|b\| = 1$. If $H \cap B_{\mathbb{R}^n}(0,1) = \emptyset$, then for any $x \in \mathbb{R}^n$: $< a, x >= t$ it follows that $\|x\| \geq 1$. Hence, $|t| \geq 1$, otherwise we could simply take $x = t\,a$. Now since H passes through z^0, we have $(z^0 = x^0 + i\,y^0)$:

$$\begin{aligned} 1 \leq |t| &= |< a, x^0 > + < b, y^0 >| \\ &\leq \|x^0\| + \|y^0\| \\ &\leq \sqrt{2\left(\|x^0\|^2 + \|y^0\|^2\right)} \\ &= \sqrt{2}\|z^0\| \end{aligned}$$

and then $|z^0| \geq \frac{1}{\sqrt{2}}$. The proof that the constant $\frac{1}{\sqrt{2}}$ is the best possible is easy and left as an exercise.

Remark 7.1 The same conclusion holds for any operator

$$\mathcal{L} = \sum_1^n \frac{\partial^2}{\partial z_j^2} + \textit{(lower order terms)},$$

provided that the coefficients are holomorphic in $\overline{B_{\mathbb{C}^n}(0,R)}$.

In particular, we obtain the following

Corollary 7.3 *If u satisfies the equation*

$$\sum_1^n \frac{\partial^2 u}{\partial z_j^2} + \sum_{|\alpha| \leq 1} a_\alpha(z) \partial^\alpha u = f$$

in a $\mathbb{C}^n$-neighborhood of $\mathbb{R}^n$ and all $a_\alpha' s$ and f are entire functions, u extends as an entire function to all of $\mathbb{C}^n$.

Let us now turn to the proof of the theorem.

Proof of the theorem 7.1: Let $z'' \in \Omega_2 \setminus \overline{\Omega}_1$. Take $z' \in \Omega_1$, consider $[z', z''] := \{z_t := tz' + (1-t)z'' : 0 \leq t \leq 1\} \subset \Omega_2$. Take a convex neighborhood $U \supset \Omega_1$, so that u is holomorphic in $\overline{U}$. Without ambiguity we shall denote $U := \Omega_2 \cap U$ and can assume that $\partial U \cap \Omega_2$ is smooth. Choose $\delta > 0$ so that $B_t := \overline{B_{\mathbb{C}^n}(z_t, \delta)} \subset \Omega_2$, for all $z_t \in [z', z'']$. Let D_t denote a convex hull of $\overline{U}$ and B_t. For small t, $D_t = \overline{U}$. Note that $\{D_t\}_{0 \leq t \leq 1}$ satisfy properties (i)-(v) of a Globalizing Family (cf. Chapter 6). Now the argument proceeds as in similar situations in chapters 5 and 6. We want to show that u extends to all $(D_t)'s$, $0 \leq t \leq 1$ by showing that the set $\{t : 0 \leq t \leq 1 : u$ extends holomorphically to $D_t\}$ is nonempty, open and closed. Indeed, as usual, the only point that really needs checking is that the set $\{t : 0 \leq t \leq 1 : u$ extends to $D_t\}$ is open. For any t^0, all points on $\partial D_{t^0} \setminus \overline{U}$ are non-characteristic with respect to $\mathcal{L}$ since a real tangent plane at any such boundary point does not meet $U \cap \Omega_2$ and hence cannot be characteristic by the hypothesis. Although perhaps not everywhere C^1-smooth ∂D_{t^0} nevertheless satisfies the hypothesis of an extended version of Zerner's theorem (see the remark following the proof of Corollary 6.2). Hence, u extends across all points on $\partial D_{t^0} \setminus \overline{U}$. Also, u is holomorphic at all points on $\partial D_{t^0} \cap \overline{U}$. Hence u extends to a neighborhood of $\overline{D}_{t^0}$, i.e. to D_s, for $s > t^0$ sufficiently close to t^0.

Remark 7.2 If all a_α, $|\alpha| = m$ are constants one can relax the hypothesis a bit assuming only that u is holomorphic in Ω_1. The argument is proceeding along the following lines: Consider $\mathcal{O} \supset [z', z'']$, $\overline{\mathcal{O}} \subset \Omega_2$, where for $\mathcal{O}$ we can take a convex "tube" around $[z', z'']$.

Claim. There exists a compact set $K \subset \Omega_1$ such that all real hyperplanes H that meet $\mathcal{O}$ and are characteristic with respect to $\mathcal{L}$ also meet K.

PROOF OF THE CLAIM: Suppose there exists a sequence of points $z_j \in \overline{\mathcal{O}}$ and characteristic planes $H_j \ni z_j$ so that $H_j \cap K_j = \emptyset$ where $\{K_j\}$ is an increasing sequence of compact sets in Ω_1 satisfying $\bigcup_j K_j = \Omega_1$. By compactness we can assume that $z_j \to z^0 \in \overline{\mathcal{O}}$, $H_j \to H_0 \ni z^0$, H_0 being a characteristic plane. But then, $H_0 \cap K_j = \emptyset$ for all j, and so $H_0 \cap \Omega_1 = \emptyset$, a contradiction.

Now the remark follows from Corollary 7.1 if we take a convex, smoothly bounded, open set U: $\overline{U} \subset \Omega_1$, $U \supset K \cup z'$ and $\mathcal{O}$ for Ω_1 and Ω_3 respectively.

The following "Corollary" follows rather from the method of proof than the statement of Theorem 7.1. (cf. Chapter 5!).

Corollary 7.4 *Let $\mathcal{L}$ be a differential operator with constant coefficients in the principal part. Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ be bounded convex domains and all coefficients are real-analytic in $\overline{\Omega}_2$. Assume that for any $z^0 \in \Omega_2$ and any hyperplane $H \ni z^0$ characteristic with respect to $\mathcal{L}$, $H \cap \Omega_1 \neq \emptyset$. If u satisfies $\mathcal{L} u = 0$ in Ω_2 and $u \equiv 0$ in Ω_1, then $u \equiv 0$ in Ω_2.*

PROOF: Repeat the construction in the proof of the Bony-Schapira theorem (modified as in the latter remark) and use Holmgren's theorem instead of Zerner's theorem.

In particular, Corollary 7.4 implies the following

Corollary 7.5 (cf. Theorem 5.1).*Let*

$$\mathcal{L} = \sum_1^n \frac{\partial^2}{\partial x_j^2} - \frac{\partial^2}{\partial x_{n+1}^2} + (\textit{lower order terms}),$$

where the coefficients in the lower order terms are real analytic in $\mathbb{R}^{n+1}$. If $u \in C^2(\mathbb{R}^{n+1})$ satisfies $\mathcal{L} u = 0$ and $u \equiv 0$ in a neighborhood of $\mathbb{R}^n \times \{0\}$, then $u \equiv 0$.

Notes

Theorem 7.1 is due to J.Bony and P.Schapira [BS]. Also, cf. [Hör1]. Kiselman [Ki] proved a special version of the result for equations with constant coefficients by Fourier analysis methods. Corollary 7.2 is due to W.Hayman [Hay], whose proof is based on the power series expansion. More on this and other closely related topics can be found in the books by N.Aronszajn, Th. Creese and L.Lipkin [ACL] and V.Avanissian [Av]. Corollary 7.4 is due to Hörmander [Hör1, Theorem 8.6.5].

Chapter 8

Applications of the Bony-Schapira Theorem. Vekua's Theory.

Example 8.1 *The Lie ball in $\mathbb{C}^2$.*

Let us find the maximal domain in $\mathbb{C}^2$ to which all functions harmonic in the unit disk $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 : |x|^2+|y|^2 < 1\}$ extend as holomorphic functions of complex variables X,Y.

Proposition 8.1 *Any function u harmonic in $\mathbb{D}$ extends holomorphically to the Lie ball*

$$\widehat{\mathbb{D}} := \{(X,Y) \in \mathbb{C}^2 : |X|^2+|Y|^2+2\sqrt{x_2^2y_1^2+x_1^2y_2^2-2x_1x_2y_1y_2} < 1\}$$

where $X = x_1 + ix_2$, $Y = y_1 + iy_2$, x_i, $y_i \in \mathbb{R}$, i=1,2.

PROOF: Change variables to $z = x+iy$, $\overline{z} = x-iy$, so $\Delta := \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}}$. Since solutions of the equation $\frac{\partial f}{\partial \overline{z}} = 0$ in $\mathbb{D}$ are analytic functions, while those of $\frac{\partial f}{\partial z} = 0$ are anti-analytic functions, any harmonic function u can be written in the form $u = f(z)+g(\overline{z})$, where f,g are analytic in $\mathbb{D}$. Clearly, $f(z)$ extends as a holomorphic function in $\mathbb{C}^2$ to the cylinder $\{(X,Y) : X+iY \in \mathbb{D}\}$ by simply setting $f(X,Y) = f(X+iY)$. Similarly $g(\overline{z})$ extends as a holomorphic function to $\{(X,Y) : X-iY \in \mathbb{D}\}$,by $g(X,Y) = g(X-iY)$.

Thus u extends holomorphically to the domain in $\mathbb{C}^2$ defined by $\{(X,Y) : |X+iY| < 1 \text{ and } |X-iY| < 1\}$. Now

$$\begin{aligned} |X+iY|^2 < 1 \quad &\Leftrightarrow \quad (X+iY)(\overline{X}-i\overline{Y}) < 1 \\ &\Leftrightarrow \quad |X|^2+|Y|^2+i(\overline{X}Y-X\overline{Y}) < 1 \\ &\Leftrightarrow \quad |X|^2+|Y|^2-2Im(X\overline{Y}) < 1. \end{aligned}$$

Similary,

$$|X-iY|^2 < 1 \Leftrightarrow |X|^2+|Y|^2+2Im(X\overline{Y}) < 1 .$$

Thus, u extends to $\{(X,Y) : |X|^2+|Y|^2+2|Im(X\overline{Y})| < 1\}$ which is the Lie ball $\widehat{\mathbb{D}}$.

The following exercises may help better familiarize the reader with the geometry of $\widehat{\mathbb{D}}$

Exercise 8.1 (i) Show that there exists a function u harmonic in $\mathbb{D}$ that does not extend holomorphically to any domain containing $\widehat{\mathbb{D}}$.

(ii) Show that the n-dimensional analogue of the Lie ball is defined by

$$\{z \in \mathbb{C}^n : (||z||^4 - |\sum_{j=1}^n z_j^2|^2)^{1/2} + ||z||^2 < 1\}.$$

(iii) Let $m \geq 1$. Show that any function u satisfying $\Delta^m u = 0$ in $\mathbb{D}$ extends holomorphically to $\widehat{\mathbb{D}}$.

(The exercises (i), (ii) are not exactly routine and intended for an ambitious reader. See Notes at the end of this chapter).

Now let us adapt slightly the concept of Globalizing Families (GF) from Chapter 6 to the situation studied in this section.

Let $V_1 \subset V_2 \subset \mathbb{C}^n$ be bounded open domains and

$$\mathfrak{L} = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$$

be a differential operator with coefficients a_α holomorphic in V_2.

We say that a family of domains $\{\Omega_t\}_{0 \leq t \leq 1}$ is a Globalizing Family (GF) for $(V_1, V_2, \mathfrak{L})$ if the following properties hold:

(i) All Ω_t are bounded and $\Omega_{t_1} \subset \Omega_{t_2}$, $t_1 < t_2$.

(ii) There exists $\delta > 0$ so that $\Omega_t \subset V_1, t < \delta$.

(iii) The family Ω_t is continuous, i.e

$$\Omega_{t_0} = \cup_{t<t_0} \Omega_t;$$

$$\overline{\Omega_{t_0}} = \cap_{t>t_0}\Omega_t.$$

(iv) $\partial\Omega_t \backslash V_1$ are sufficiently smooth to satisfy the hypothesis of Zerner's theorem (see Remark following Corollary 6.2) with respect to the operator $\mathfrak{L}$.

(v) $V_2 = (\cup_t \Omega_t) \cup V_1$.

Then, we have the following "Meta-theorem" that we shall call the Globalizing Principle (GP).

GLOBALIZING PRINCIPLE (GP). If u is a holomorphic solution of $\mathfrak{L}u = f$ in V_1 with f holomorphic in $\overline{V_2}$, then u extends holomorphically to V_2.

(The proof of (GP) is immediate -cf. Chapter 6). The following Corollary follows from the proof of the Bony-Schapira Theorem.

Corollary 8.1 *Let $\mathfrak{D}_1 \subset V \subset \mathbb{C}^n$ be bounded, open, convex domains and the coefficients of $\mathfrak{L}$ are holomorphic in V. Then for any $z^0 \in V$, there exist a convex domain $\mathfrak{D}_2 \subset V$ so that $z^0 \in \mathfrak{D}_2$ and a (GF) that extends any solution of $\mathfrak{L}u = f$ holomorphic in $\overline{\mathfrak{D}_1}$ holomorphically into $\mathfrak{D}_2$ (provided that f is holomorphic in V).*

Remark 8.1 If the principal part of $\mathfrak{L}$ has constant coefficients, the same is true only if $\mathfrak{L}u = f$ in $\mathfrak{D}_1$.

We assume that from now on $n = 2$ and

$$\mathfrak{L} = \triangle^m + \sum_{|\alpha| \le 2m-1} a_\alpha(X,Y)\partial^\alpha.$$

For the sake of simplicity all the coefficients a_α and non-homogeneous terms are assumed to be entire functions in $\mathbb{C}^2$.

Definition 8.1 *Let $\Omega \subset \mathbb{R}^2$ be a domain. A domain $\widehat{\Omega} \subset \mathbb{C}^2$ is called the Vekua hull (or, the harmonicity hull) of Ω iff for any $(X,Y) \in \widehat{\Omega}$, $X + iY \in \Omega$, $X - iY \in \Omega^* := \{\overline{z} : z \in \Omega\}$.*

Example 8.2 If we let $\Omega = \mathbb{D}$, then $\widehat{\Omega}$ is the Lie ball.

Proposition 8.2 *If Ω is convex, then $\widehat{\Omega}$ is also convex.*

PROOF: Let $Z_1 = (X_1, Y_1)$, $Z_2 = (X_2, Y_2) \in \widehat{\Omega}$. We want to show that $tZ_1 + (1-t)Z_2 \in \widehat{\Omega}$, $0 \le t \le 1$. We have

$$tX_1 + (1-t)X_2 + i(tY_1 + (1-t)Y_2 = t(X_1 + iY_1) + (1-t)(X_2 + iY_2) \in \Omega$$

because $X_j + iY_j \in \Omega$, $j = 1,2$ and Ω is convex. Similarly,

$$tX_1 + (1-t)X_2 - i(tY_1 + (1-t)Y_2) = t(X_1 - iY_1) + (1-t)(X_2 - iY_2) \in \Omega^*$$

because Ω^* is also convex.

Corollary 8.2 *Let $\Omega \subset \mathbb{C} \cong \mathbb{R}^2$ be a convex domain, $\widehat{\Omega}$ its Vekua hull. Then any solution of $\mathfrak{L}u = f$ in Ω extends holomorphically to $\widehat{\Omega}$. Moreover for any $z^0 \in \widehat{\Omega}$, there exists a convex domain $\mathfrak{D}_2$ containing z^0, $\mathfrak{D}_2 \subset \widehat{\Omega}$ and a (GF) extending solution u from $\mathfrak{D}_1$, a convex neighbohood of Ω in $\mathbb{C}^2$, to $\mathfrak{D}_2$.*

PROOF: In view of Definition 8.1 and Proposition 8.2 the corollary follows at once from Corollary 8.1.

Proposition 8.3 *Let Ω, $\Omega' \subset \mathbb{C}$ be simply connected domains. Let f map Ω conformally onto Ω'. Then, there exists a biholomorphic map F of $\widehat{\Omega}$ onto $\widehat{\Omega'}$ so that $F|_\Omega = f$ and F preserves all (GF) with respect to the operator*

$$\mathfrak{L} = \triangle^m + \sum_{|\alpha| \le 2m-1} a_\alpha(z)\partial^\alpha.$$

PROOF: Changing the coordinates to $z = X + iY$, $w = X - iY$ we see that $\widehat{\Omega} \cong \Omega \times \Omega^*$, $\widehat{\Omega'} \cong \Omega' \times (\Omega')^*$ ($\cong$ means the domains being biholomorphically equivalent). Set $f^*(w) = \overline{f(\overline{w})}$, f^* is a conformal mapping of Ω^* onto $(\Omega')^*$. Set $F(z,w) := (f(z), f^*(w))$. F maps $\Omega \times \Omega^*$

onto $\Omega' \times (\Omega')^*$ biholomorphically. For the second part, we only have to check that F preserves non-characteristic points (with respect to $\mathfrak{L}$) on smooth real hypersurfaces.

Let $\Gamma' := \{(z', w') : g(z', w') = 0\}$ be a C^1-real hypersurface in $\widehat{\Omega}'$ and let $\Gamma := \{(z, w) : G(z, w) = g(F(z, w)) = 0\}$ be its pull-back to $\widehat{\Omega}$. The characteristic points on Γ' are those satisfying

$$\frac{\partial g}{\partial z'} \frac{\partial g}{\partial w'} = 0$$

and on Γ:

$$\frac{\partial G}{\partial z} \frac{\partial G}{\partial w} = 0.$$

But $(F := (F_1, F_2))$,

$$\frac{\partial G}{\partial z} \frac{\partial G}{\partial w} = \left(\frac{\partial g}{\partial z'} \frac{\partial F_1}{\partial z} \right) \left(\frac{\partial g}{\partial w'} \frac{\partial F_2}{\partial w} \right)$$

$$= \frac{\partial g}{\partial z'} \frac{\partial g}{\partial w'} f'(z)(f^*)'(w)$$

and the proposition follows since f', $(f^*)'$ do not vanish in Ω, Ω^* respectively.

Corollary 8.3 *Let u be a solution of $\mathfrak{L}u = f$ in a simply connected domain $\Omega \subset \mathbb{C}$. Then, u extends holomorphically to $\widehat{\Omega}$.*

Proof: Follows at once from Corollary 8.2, Proposition 8.3 and (GP) since by the Riemann mapping theorem Ω is conformally equivalent to a convex bounded domain Ω', e.g., the unit disk $\mathbb{D}$.

Corollary 8.4 *Let as above $\mathfrak{L} = (\frac{\partial^2}{\partial z \partial w})^m +$ (lower order terms), and let $\Gamma = \{w = S(z)\}$ be a holomorphic hypersurface in $\mathbb{C}^2$. Suppose S is univalent in $U_1 \subset \mathbb{C}_z$, where U_1 is a simply connected and bounded domain. Let $U_2 = S(U_1)$. Then any solution of the Cauchy Problem*

$$\begin{cases} \mathfrak{L}u = f; \\ \partial^\alpha (u - g) = 0, \ |\alpha| \le 2m - 1 \ on \ \Gamma, \end{cases}$$

with f, g holomorphic in $\overline{\Omega} := \overline{U_1} \times \overline{U_2}$, extends holomorphically to Ω.

Proof: Let ψ be a conformal mapping of U_2 onto the unit disk $\mathbb{D}$. Define $F(z, w) : \Omega \longrightarrow \mathbb{D}^2$ by $F(z, w) = (\psi(S(z)), \psi(w))$. F is a biholomorphic isomorphism of Ω onto $\mathbb{D}^2$. Clearly,

$$F(\Gamma) = \{(z', w') \in \mathbb{D}^2 : z' = w'\} =: \Gamma'.$$

Γ' is convex and each real hyperplane through a point $Z' = ((z^0)', (w^0)') \in \mathbb{D}^2$ that is characteristic with respect to the operator

$$(\frac{\partial^2}{\partial z' \partial w'})^m + \text{(lower order terms)}$$

i.e., $Re(z') = Re(z^0)'$ or $Re(w') = Re(w^0)'$, intersects Γ'. Thus, for any convex neighborhood V' of Γ' and any point $Z' \in \mathbb{D}^2$ there is a (GF) with respect to $(V', \mathbb{D}^2, \mathfrak{L}' := (\frac{\partial^2}{\partial z' \partial w'})^m + ...)$ as in Corollary 8.1. The pre-image of this (GF) is then a (GF) with respect to a neighborhood V of Γ, Ω, $\mathfrak{L}$ and the point $F^{-1}(Z') \in \Omega$. The corollary now follows from the (GP).

Remark 8.2 Let us pause for a moment to come back to question B posed in the Introduction but set in the dimension 2 rather than 3.

The function u harmonic in the annulus $\Omega = \{z \in \mathbb{C} : 1 < |z| < 2\}$ extends holomorphically to its Vekua hull $\widehat{\Omega}$ (perhaps, as a multivalued function, since Ω is not simply connected). It is easy to see that $\widehat{\Omega} \cap \{Y = 0\} \supseteq \{(X, 0) : 1 < |X| < 2\}$. Indeed, for any $(X^0, 0)$, $1 < |X^0| < 2$, both complex lines $X \pm iY = X^0$ intersect $\mathbb{R}^2$ at points $x^0 \pm iy^0 = X^0, \overline{X^0} \in \Omega$. Thus, the line segment $[-2, -1]$ and $[1, 2]$ that are disconnected in Ω are connected through $\widehat{\Omega}$. Hence, since u is holomorphic on $\widehat{\Omega} \cap \{Y = 0\}$ and vanishes on $\{(x, 0) : -2 < x < -1\}$

it must vanish on the whole intersection $\widehat{\Omega} \cap \{Y = 0\}$, in particular, on $\{(x, 0) : 1 < x < 2\}$. Although there is a simple direct proof of this fact for $n = 2$ (the reader is encouraged to find it!), the advantage of this viewpoint is that with no extra work it allows the same conclusion for solutions of other equations, e.g. $\Delta^m u = 0$, or exhibiting a little bit more care, even $\mathfrak{L}u = 0$ with $\mathfrak{L}$ as above. Still with a bit more technicalities but essentially no new ideas, one can extend the result to higher dimensions to answer the question B completely.

Notes

The notion of the Vekua hull (of course, not the name) appears in Vekua's work [V]. Also, cf.[Gil], [Hen] and the references therein. The extension to higher dimensions under the name of "hull of harmonicity" is due independently to P. Lelong and N. Aronszajn-cf. [Av], [ACL]. The Lie ball is discussed in [Av], [ACL] in great detail. Corollaries 8.3 and 8.4, the main achievements of Vekua's theory, are due to Vekua [V], [Hen]. His methods are based on explicit representations of solutions by Riemann's formulae for solutions of hyperbolic equations in two variables and restricted to second order operators or constant coefficient operators for the higher order case. Also, Vekua states Corollary 8.4 for a domain $\Omega \subset \mathbb{R}^2$ with the Cauchy problem posed on an analytic curve $\gamma \subset \Omega$. In that context, one has to assume that Ω is conformally symmetric with respect to γ (cf. [V], [Da], [Hen]). That means that γ is given by the equation $\overline{z} = S(z)$, S being an analytic function in Ω so that $\overline{S}$ maps Ω bijectively onto itself. (More on this is contained in Chapter 9). The presentation here, based on a powerful though geometrically simple method of Globalizing Families allowing to bypass entirely the explicit representation formulae, and essentially ignore the lower order terms of the operator, is from [KS2].

Chapter 9

The Reflection Principle

From now on by a non-singular analytic Jordan arc γ we will understand an object satisfying either one of the following (equivalent) definitions:

(i) $\gamma = \{z \in \mathbb{C} : z = f(t), 0 \le t \le 1$, *$f$ is analytic and one to one in some neighborhood of* $[0, 1]\}$.

real-value

(ii) $\gamma = \{(x, y) \in \mathbb{R}^2 : \varphi(x, y) = 0$, *where φ is a restriction of a holomorphic function $\varphi(X, Y)$ in a $\mathbb{C}^2$- neighborhood of γ such that* $\nabla\varphi|_\gamma \neq (0, 0)$ $\}$.

Proposition 9.1 (The Schwarz function of a curve.) *Let γ be an analytic Jordan arc. Then there exists a (unique) function $S(z)$ analytic in a neighborhood of γ such that $\overline{z} = S(z)$ on γ.*

The function S(z) is called the *Schwarz function* of γ.

Example 9.1 If $\gamma = \mathbb{R}$, then $S(z) = z$.

Example 9.2 If $\gamma = \{z : |z - z_0| = R\}$, then

$$\overline{z} = \overline{z_0} + \frac{R^2}{z - z_0} =: S(z).$$

Proof of Proposition 9.1: Indeed, using the definition (ii) and the change of variables $z = X + iY$, $\quad w = X - iY \quad (= \overline{z} \text{ in } \mathbb{R}^2)$, we obtain the equation of γ in the form $\Phi(z, \overline{z}) = 0$, where Φ is a holomorphic function of two complex variables. Note that $\frac{\partial\Phi}{\partial\overline{z}}|_\gamma = \frac{1}{2}(\frac{\partial\varphi}{\partial x} + i\frac{\partial\varphi}{\partial y})|_\gamma \neq 0$. By the implicit function theorem we can solve that equation for $\overline{z}$ and obtain $\overline{z} = S(z)$, with the (implicit) function S being analytic in a neighborhood of γ.

Remark 9.1 If γ is closed, then $S(z)$ is analytic in a ring-like domain containing γ and since $S = \overline{z}$ on γ, S has a single-valued branch near γ.

The following example illustrates this remark.

Example 9.3 Let $\gamma = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a^2 - b^2 = 1\}$ be an ellipse. Solving a quadratic equation we easily find

$$S(z) = (a^2 + b^2)z - 2ab(z^2 - 1)^{\frac{1}{2}}.$$

The following theorem is one of the many ways to formulate the celebrated Schwarz Reflection Principle.

Theorem 9.1 *Let γ be an analytic arc and $z^0 \in \gamma$. There exists a neighborhood V of z^0 homeomorphic to a disk and a bijective map $R_\gamma = R : V \to V$ such that*

(i) R is anti-conformal.

(ii) $R|_\gamma$ = identity.

(iii) If V_1 and V_2 are the components of $V \setminus \gamma$, $R(V_1) = V_2$.

(iv) R is an involution, i.e. $R \circ R$ = identity.

Proof: If we parametrize γ in terms of the arclength s as $z = z(s)$, we note that on γ we have $S(z(s)) = \overline{z(s)}$ and

$$\overline{\frac{dz}{ds}} = \frac{dS(z)}{ds} = S'(z)\frac{dz}{ds} \ ,$$

so $S'(z(s)) = \overline{\tau(s)}^2$, where $\tau(s) = \frac{dz}{ds}$ is the unit tangent vector. In particular, $|S'(z)| \equiv 1$ on γ, so $S(z)$ is univalent near z^0. Now set

$$R(z) := \overline{S(z)}.$$

(i) is obvious and (ii) holds by the definition. $(R \circ R)(z) = R(\overline{S(z)}) = \overline{S(\overline{S(z)})}$ is analytic and $R \circ R = \overline{\overline{S(z)}} = z$ on γ. So $R \circ R \equiv z$. Hence, it remains to show (iii). By Taylor's Theorem we can write

$$S(z) = S(z^0) + S'(z^0)(z - z^0) + o(|z - z^0|),$$

for $|z - z^0|$ small, so that

$$R(z) = \overline{S(z^0)} + \overline{S'(z^0)}(\overline{z} - \overline{z^0}) + o(|z - z^0|).$$

Set

$$R^{\#}(z) = z^0 + \tau^2(\overline{z} - \overline{z^0}).$$

(Without loss of generality we can assume that $z^0 = 0$ and $\tau = (1,0)$). Note then that $R^{\#}$ is simply the reflection about the tangent line to γ at z^0 and (iii) now follows.

Corollary 9.1 (Schwarz' Reflection Principle)

(i) Let $D \subset \mathbb{C}$ be a domain and $\sigma \subset \partial D$ be an analytic arc. Suppose that f is analytic in D and continuous in $\overline{D}$, $f(D) = E$ so that $f(\sigma) = \tau \subset \partial E$ is an analytic arc. Then f extends analytically across σ and for points $z \in D$ sufficiently near σ the following "reflection law" holds

$$f(R_\sigma(z)) = R_\tau(f(z)),$$

where R_σ and R_τ are the reflection maps with respect to the arcs σ and τ respectively.

(ii) Let u be harmonic in D, continuous in $\overline{D}$ and vanishing on σ. Then u extends as a harmonic function across σ and for $z \in D$ sufficiently close to σ the following "reflection law" (RL) holds

$$u(z) + u(R_\sigma(z)) \equiv 0. \qquad (9.1)$$

PROOF: *(i)* Shrinking D if necessary without loss of generality we can assume that $R_\sigma(D)$ is on another side of σ. This is guaranteed by (iii) of Theorem 9.1. Define $\hat{f}$ in $R_\sigma(D)$ by

$$\hat{f}(z) = R_\tau(f(R_\sigma^{-1}(z))) = R_\tau \circ f \circ R_\sigma(z).$$

$\hat{f}$ is analytic in $R_\sigma(D)$ and continuous in $\overline{R_\sigma(D)}$. Moreover, $\hat{f}|_\sigma = f|_\sigma$. Hence, by the Morera theorem $\hat{f}$ is an extension of f across σ.

(ii) First of all, applying a conformal mapping if necessary we can assume that $\sigma \subset \mathbb{R}$. Define $\tilde{u}(x,-y) = -u(x,y)$. Then $\tilde{u}$ is harmonic in $R_\sigma(D)$ and smooth and continuous in $\overline{R_\sigma(D)}$. Moreover,

$$\tilde{u}|_\sigma = u|_\sigma = 0$$

and

$$\frac{\partial \tilde{u}}{\partial y}|_{(x,0)} = \lim_{y\to 0+} \frac{\tilde{u}(x,0)-\tilde{u}(x,-y)}{0-(-y)} = \lim_{y\to 0+} \frac{u(x,y)}{y} = \frac{\partial u}{\partial y}|_{(x,0)}.$$

By the Kovalevskaya theorem (Theorem 4.1) $\tilde{u}$ is an extension of u and *(ii)* follows.

Study's interpretation of the Schwarz Reflection Principle

Consider the "real" wave equation

$$\frac{\partial^2 u}{\partial x \partial y} = 0.$$

Let γ be a smooth curve, say a C^2-curve, given by $\{y = s(x)\}$, where s is a C^2, strictly monotone decreasing function. Let $u = f(x) + g(y)$ be a C^2-solution of the wave equation in the region bounded by γ and the lines $x = a$, $y = b$, continuous up to the boundary, and such that $u|_\gamma = 0$. Let $a' = s^{-1}(b)$, $b' = s(a)$, $A = (s^{-1}(b), b)$, $B = (a, s(a))$.

Then u extends as a C^2 solution to $[a,a'] \times [b,b']$ and the following reflection principle holds:

$$u(P) + u(Q) = 0,$$

where $P = (a,b)$ and $Q = (s^{-1}(b), s(a))$. We call such P and Q *symmetric* with respect to the curve γ.

Indeed,

$$\begin{aligned} u(P) &= f(P) + g(P) \\ &= f(B) + g(A) \\ &= -g(B) - f(A) \text{ (since } u|_\gamma = 0) \\ &= -g(Q) - f(Q) \\ &= -u(Q) \end{aligned}$$

Turning back to the Laplace equation in $\mathbb{R}^2$, let γ be an analytic arc in $\mathbb{R}^2$, u satisfy $\Delta u = 0$ in a neighborhood of γ and $u|_\gamma = 0$. Then u extends as a holomorphic function to a $\mathbb{C}^2$-neighborhood of γ. Also, as we know, γ is the intersection of the holomorphic hypersurface $\Gamma = \{(X,Y) \in \mathbb{C}^2 : \varphi(X,Y) = 0\}$ with $\mathbb{R}^2$.

By changing variables $z = X + iY, w = X - iY$, we can write the equation of Γ in the form $w = S(z)$, where S is the Schwarz function of γ, and if we are sufficiently near γ, we can assume that $S(z)$ is univalent (cf. the proof of Theorem 9.1). Then, u satisfies

$$4\frac{\partial^2 u}{\partial z \partial w} = 0$$

near Γ and, since u is holomorphic, $u|_\Gamma = 0$. Now, $u = f(z) + g(w)$, where f and g are analytic functions of one variable. Then Study's interpretation of the reflection principle is as follows. Take $P = (z^0, \overline{z^0}) \in \mathbb{R}^2$ near γ and pass a complex line $M = \{(z,w) : z = z^0\}$ through P. Then $M \cap \Gamma = B = (z^0, S(z^0))$. Similarly, the complex line $N = \{(z,w) : w = \overline{z^0}\}$ meets Γ at a point $A = (S^{-1}(\overline{z^0}), \overline{z^0})$. Pass through A and B the complex lines M' and N', respectively:

$$M' = \{(z,w) : z = S^{-1}(\overline{z^0})\} \ ;$$

$$N' = \{(z,w) : w = S(z^0)\}.$$

Set $Q := M' \cap N' = (S^{-1}(\overline{z^0}), S(z^0))$. However, $Q \in \overline{\mathbb{R}^2!}$. Indeed, $S^{-1}(\overline{z^0}) = \overline{S(z^0)}$ since according to Theorem 9.1, $z^0 = S(\overline{S(z^0)})$ and hence $\overline{z^0} = S(\overline{S(z^0)})$. The rest of the argument extends word for word and (9.1) follows with $Q = R_\gamma(P)$.

Failure of the Reflection Law for other operators

H. Lewy [Le] showed that if u satisfies a second order p.d.e. with the Laplacian in the principal part in a domain D adjacent to the real axis and $u|_{\mathbb{R}} = 0$, then u extends to the mirror image D$'$ of D with respect to $\mathbb{R}$. Even, if u satisfies a "linear relation" on $\mathbb{R}$, e. g. $u + u_y = 0$ on $\mathbb{R}$, it still extends to D$'$. He has also given a counterexample showing that the latter property fails in $\mathbb{R}^3$. However in regard to point-to-point reflection laws like (9.1) the situation for other operators even slightly varying from the Laplacian (or, the wave operator) is drastically different.

Let γ be an analytic curve in $\mathbb{R}^2$ and u satisfy the "Helmholtz" equation

$$\frac{\partial^2 u}{\partial x \partial y} + \lambda^2 u = 0 \qquad (9.2)$$

near γ, $\lambda > 0$, and $u|_\gamma = 0$.

Theorem 9.2 *If for two points P and Q sufficiently close to γ there exists a constant $k := k(P,Q)$ such that the "reflection law"*

$$u(P) + ku(Q) = 0, \qquad (9.3)$$

holds for all u satisfying (9.2) near γ and vanishing on γ, then γ must be a straight line.

We will need a lemma that goes back to B. Riemann. Recall that if

$$L = \frac{\partial^2}{\partial x \partial y} + a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + c(x,y)$$

is a hyperbolic differential operator with a, b, c entire functions of two variables its adjoint is defined by

$$L^\# u = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial}{\partial x}(au) - \frac{\partial}{\partial y}(bu) + cu.$$

The Riemann function $R_L := R(x,y;\xi,\eta)$ at a point (ξ,η) for the operator L is defined as the solution of the following Cauchy-Goursat problem

$$\begin{cases} L^\# R = 0 \quad \text{near} \quad (\xi,\eta) \quad ; \\ R(\xi,y;\xi,\eta) = exp(\int_\eta^y a(\xi,\tau)d\tau) \quad ; \\ R(x,\eta;\xi,\eta) = exp(\int_\xi^x b(\tau,\eta)d\tau). \end{cases} \qquad (9.4)$$

Note that from (9.4) it follows that $r(y) := R(\xi,y;\xi,\eta)$ satisfies $r_y - ar = 0$ on $\{x = \xi\}$, while $r(x) := R(x,\eta;\xi,\eta)$ satisfies $r_x - br = 0$ on $\{y = \eta\}$. Furthermore, observe that $R(\xi,\eta;\xi,\eta) = 1$ and as we noted in Chapter 6, $R(x,y;\xi,\eta)$ is an entire function. Moreover, since it is not hard to verify the symmetry property of the Riemann function namely, $R_L(x,y;\xi,\eta) = R_{L^\#}(\xi,\eta;x,y)$ it follows that $R(x,y;\xi,\eta)$ is in fact an entire function of all four variables.

Lemma 9.1 (Riemann's Lemma). *Let $\gamma = \{(x,y) \ : \ y = s(x)\}$ be a real-analytic curve that is non-characteristic with respect to L. So, in particular, s and s^{-1} are real-analytic near γ. Let u be a solution of $Lu = 0$ near γ. For all points $P(x^0,y^0)$ sufficiently close to γ we have*

$$u(P) = \frac{1}{2}(uR)|_A + \frac{1}{2}(uR)|_B - \int_A^B (U\,dy - V\,dx),$$

where $R = R(x, y; x^0, y^0)$, $A = (s^{-1}(y^0), y^0)$, $B = (x^0, s(x^0))$ *and*

$$U = aRu + \frac{1}{2}Ru_y - \frac{1}{2}R_y u \quad ;$$

$$V = bRu + \frac{1}{2}Ru_x - \frac{1}{2}R_x u.$$

PROOF: The proof is a rather straightforward calculation. Let us separate the following assertion.

Assertion.

$$0 = R(Lu) - u(L^{\#}R) = U_x + V_y.$$

Assuming the assertion let G denote the region bounded by PA, PB and the arc AB. We have (applying Green's formula):

$$\begin{aligned} 0 &= \int\int_G (RLu - uL^{\#}R) \\ &= \oint_{\partial G} U\,dy - V\,dx \\ &= \int_A^B (U\,dy - V\,dx) - \int_P^B U\,dy - \int_P^A V\,dx. \end{aligned} \tag{9.5}$$

Now, using the properties of the Riemann function we compute

$$\begin{aligned} -\int_P^B U\,dy &= -\int_P^B [(aR - R_y)u + \frac{1}{2}(Ru)_y]dy = -\frac{1}{2}(Ru)|_P^B \\ &= \frac{1}{2}R(P;P)u(P) - \frac{1}{2}R(B;P)u(B) \\ &= \frac{1}{2}u(P) - \frac{1}{2}(Ru)|_B, \end{aligned}$$

and similarly,

$$-\int_P^A V\,dx = -\int_P^A [(bR - R_x)u + \frac{1}{2}(Ru)_x]dx = \frac{1}{2}u(P) - \frac{1}{2}(Ru)|_A. \tag{9.6}$$

So, combining (9.5)-(9.6),

$$u(P) = \frac{1}{2}(uR)|_A + \frac{1}{2}(uR)|_B - \int_A^B (U\,dy - V\,dx). \tag{9.7}$$

The proof of the assertion is straightforward and left as an exercise.

For the proof of Theorem 9.2 we shall need the following:

Corollary 9.2 *In particular, if* $u = 0$ *on* γ,

$$u(P) = \frac{1}{2}\int_A^B R(\frac{\partial u}{\partial x}dx - \frac{\partial u}{\partial y}dy). \tag{9.8}$$

PROOF OF THEOREM 9.2: The operator L now is $L := \frac{\partial^2}{\partial x \partial y} + \lambda^2$. We assume that P and Q are sufficiently close to γ so that all the solutions of Cauchy problems

$$\begin{cases} Lu = 0 \ ; \\ \partial^\alpha u = \partial^\alpha[(y - s(x))g(x, y)], \quad |\alpha| \le 1, \end{cases} \tag{9.9}$$

with g being a polynomial, are real-analytic at P and Q. Recall (cf. Chapter 6) that for our operator L

$$R(x, y; x^0, y^0) = J_0(\lambda\sqrt{(x - x^0)(y - y^0)}),$$

where J_0 is the zero Bessel function. Replacing u on γ by the Cauchy data in (9.9) we obtain

$$\left(\frac{\partial u}{\partial x}dx - \frac{\partial u}{\partial y}dy\right)|_\gamma = -2gs'(x)dx|_\gamma.$$

So, in view of (9.8) we have

$$u(P) = -\int_{A_P}^{B_P} J_0(\lambda\sqrt{(x-x^0)(y-y^0)}g(x,y)s'(x)dx, \tag{9.10}$$

where we put $A = A_P = (s^{-1}(y^0), y^0), B = B_P = (x^0, s(x^0))$ to stress their dependence on P. First, we show that (9.3) implies that P and Q must be "symmetric" with respect to γ. Suppose they are not. Without loss of generality we can assume then that A_Q is inside A_PB_P while B_P is inside A_QB_Q (other possible configurations can be treated similarly). By the Weierstrass' approximation theorem we find polynomials g so that $|g(x,y)|$ is arbitrarily small on (A_Q, B_Q) while the integral $\int_{A_P}^{A_Q} J_0(\lambda\sqrt{(x-x^0)(y-y^0)}g(x,y)s'(x)dx$ is larger than some fixed positive number η. This is possible since $s' \neq 0$ and $J_0 > 0$ near the origin. We see then, using (9.10) for P and Q respectively, that $|u(Q)|$ can be made arbitrarily small, say $\leq \varepsilon$, while $|u(P)| \geq \eta - \varepsilon$. Hence, $u(P)$ cannot equal $-ku(Q)$. Thus, P and Q must be symmetric with respect to γ, i.e. $Q = (s^{-1}(y^0), s(x^0))$.

The Reflection Law 9.3 implies that for all polynomials g we have $(A = A_P = A_Q, B = B_P = B_Q)$:

$$\int_A^B [J_0(\lambda\sqrt{(x-x^0)(y-y^0)}) + kJ_0(\lambda\sqrt{(x-s^{-1}(y^0))(y-s(x^0))})]gs'(x)dx = 0.$$

Therefore,

$$J_0(\lambda\sqrt{(x-x^0)(y-y^0)}) + kJ_0(\lambda\sqrt{(x-s^{-1}(y^0))(y-s(x^0))}) \equiv 0 \tag{9.11}$$

on γ. In particular, taking $(x,y) = (x^0, s(x^0))$ and using the fact that $J_0(0) = 1$, we conclude that $k = -1$. Hence

$$J_0(\lambda\sqrt{(x-x^0)(y-y^0)}) = J_0(\lambda\sqrt{(x-s^{-1}(y^0))(y-s(x^0))})$$

on γ and since J_0 is monotone decreasing on the positive semi-axis near the origin we must have $(x-x^0)(y-y^0) = (x-s^{-1}(y^0))(y-s(x^0))$ on γ. Therefore, γ must be a line.

By using Study's change of variables $z = X + iY, w = X - iY$, that reduces the "real" Helmholtz's operator $\Delta + \lambda^2$ to the complex hyperbolic operator $4\frac{\partial^2}{\partial z \partial w} + \lambda^2$ and applying a similar argument we obtain the following

Corollary 9.3 *Let γ be an analytic curve on $\mathbb{R}^2$. If for all solutions u of $\Delta u + \lambda^2 u = 0$ vanishing on γ the Reflection Law*

$$u(P) + ku(Q) = 0 \tag{RL}$$

holds just for two points P and Q sufficiently close to γ, then k = -1 and γ must be a line passing through the midpoint of PQ and orthogonal to PQ.

Remark 9.2 For the operator $L = \frac{\partial^2}{\partial x \partial y}$ the Riemann function equals 1 identically. Hence the representation formula (9.8) becomes

$$u(P) = \frac{1}{2}\int_A^B (\frac{\partial u}{\partial x}dx - \frac{\partial u}{\partial y}dy).$$

Therefore arguing as in the proof of the theorem we can show that if (RL)

$$u(P) + ku(Q) = 0$$

holds for all u satisfying $Lu = 0, u|_\gamma = 0$, then P and Q must be symmetric with respect to γ and $k = -1$.

Similarly, for $L = \Delta$, the Laplace operator, the Reflection Law (i.e. the Schwarzian Reflection Law) holds near γ if and only if P and Q are symmetric with respect to γ, i.e. $Q = R(P)$, or equivalently, $K_P \cap \Gamma = K_Q \cap \Gamma$ where $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$, and

$$K_P = \{(X,Y) \in \mathbb{C}^2 : (X - x_P)^2 + (Y - y_P)^2 = 0\}$$

and

$$K_Q = \{(X,Y) \in \mathbb{C}^2 : (X - x_Q)^2 + (Y - y_Q)^2 = 0\}$$

are isotropic cones in $\mathbb{C}^2$ emanating from P and Q respectively, and $\Gamma = \{(X,Y) : X - iY = S(X + iY)\}$ is the " complexified" curve γ while S is, as usual, the Schwarz function of γ.

Corollary 9.4 *Let $\Gamma = \{(x_1, x_2, x_3) : (x_1, x_2, 0) \in \gamma, \gamma$ being an analytic curve$\}$ be a cylinder in $\mathbb{R}^3$ with base γ. If the Reflection Law*

$$u(P) + ku(Q) = 0$$

holds for all harmonic functions u vanishing on Γ for P and Q sufficiently close to Γ, then Γ is a plane and P, Q are symmetric with respect to Γ.

PROOF: If $x_3^P \neq x_3^Q$, take $u(x_1, x_2, x_3) = (x_3 - x_3^P)v(x_1, x_2)$ where v is harmonic near γ, vanishes on γ and $v(x_1^0, x_2^0) \neq 0$. Then u is harmonic, vanishes on γ, $u(P) = 0$, and $u(Q) \neq 0$. So, we can assume $x_3^P = x_3^Q = 0$. By the previous remark P and Q must also be symmetric with respect to γ. For all $\lambda > 0$, consider functions

$$u(x_1, x_2, x_3) = u^0(x_1, x_2)e^{\lambda x_3},$$

so that $u^0(x_1, x_2)|_\gamma = 0$ and satisfy $\Delta_{(x_1,x_2)}u^0 + \lambda^2 u^0 = 0$. Then all such u's are harmonic and vanish on Γ. Then, the Reflection Law implies that the hypothesis of Corollary 9.3 is satisfied for all u^0, and therefore γ must be a line.

Notes

The term "Schwarz Function" was introduced by Ph. Davis in his book [Da]. In presenting the Schwarz reflection principle via Theorem 9.1 we follow H. Shapiro's book [Sh1] although Corollary 9.1 is very

close to the reasoning in [Da]. E. Study's beautiful interpretation of the reflection principle via the wave equation is from his paper [St], though he reasons right away for the "complex case". Apparently, his paper fell into oblivion and the idea was rediscovered again in the 1950's by a number of mathematicians, most notably by H. Lewy [Le] and P. Garabedian [G1]. In their papers they also associated the Schwarzian Reflection with that for solutions of real wave equations. Theorem 9.2 or, more precisely, its complex analogue Corollary 9.3, is implicitly contained in [KS3] as an Ansatz in proving Corollary 9.4. Riemann's Lemma can be found, e.g., in Hadamard's classic [Ha]. Detailed discussions pertinent to Remark 9.2 can be found in [KS3], and in a more expanded form, in [Sh1]. A far reaching extension of Corollary 9.4 is contained in [EK] (also see the following chapter). Finally a "positive result" going the opposite direction from Corollary 9.3 has been found in [SSS]. There, the value at a point P of a solution of the Helmholtz equation vanishing on a curve γ is calculated by means of certain integrals along paths joining point Q, symmetric to P, to γ.

Chapter 10

The Reflection Principle (continued)

In the previous section we have studied the Reflection Law (RL) for two points P, Q in the plane and a real-analytic arc Γ. Namely, we were interested in existence of a constant k only depending on P and Q such that

$$u(P) + ku(Q) = 0 \tag{RL}$$

holds for every function u harmonic near the arc Γ and vanishing on it.

In this section we shall discuss the possibility of extending the (RL) to higher dimensions. Recall now two cases for which the Reflection Principle does hold in all dimensions.

1. Taking for a "reflecting" surface the hyperplane $\Gamma := \{x \in \mathbb{R}^n : x_n = 0\}$, we have

   $$u(x', x_n) + u(x', -x_n) = 0,$$

   for all u such that $\Delta u = 0$ near Γ and $u|_\Gamma = 0$.

2. For the sphere $\Gamma = \{x : \sum_{i=1}^n x_i^2 = R^2\}$ we have,

   $$u(x) + |x|^{2-n} u(R^2 \frac{x}{|x|^2}) = 0,$$

   for all u harmonic near Γ such that $u|_\Gamma = 0$.

Using the previous terminology it is easy to check that in both cases, the "Study Relation" (cf. Chapter 9) also hold for symmetric points P and Q, namely:

$$K_P \cap \widehat{\Gamma} = K_Q \cap \widehat{\Gamma}, \tag{SR}$$

where K_P and K_Q are, respectively, the isotropic cones emanating from P and Q, i.e.

$$\begin{aligned} K_P &= \{z \in \mathbb{C}^n : \sum_{j=1}^n (z_j - x_j^P)^2 = 0\}, \\ K_Q &= \{z \in \mathbb{C}^n : \sum_{j=1}^n (z_j - x_j^Q)^2 = 0\}, \end{aligned}$$

$P = (x_j^P)$, $Q = (x_j^Q)$, $j = 1, \ldots, n$ and $\widehat{\Gamma}$ is the "complexification" of Γ (i.e. $\widehat{\Gamma} := \{z \in \mathbb{C}^n : z_n = 0\}$ in the first case and $\widehat{\Gamma} := \{z \in \mathbb{C}^n : \sum_{i=1}^n z_j^2 = R^2\}$ in the second).

Then, the first question one may pose here is the following: if the reflection law (RL) holds for P, Q near a real-analytic surface Γ in $\mathbb{R}^n$, $n \geq 3$, for all harmonic functions u vanishing on Γ (and defined in a fixed neighbourhood of Γ containing P and Q), does it imply that the Study Relation (SR) holds? Or, in other words, is (SR) necessary for the (RL) to hold?

The answer is in the affirmative [EK]. Here, however, we will only present a much simpler result [KS3]. Let us first introduce the following definition.

Definition 10.1 *We say that a harmonic function $u(x)$ has a polar singularity at a point x^0 if it is harmonic in a punctured neighbourhood of x^0 and there exists $N > 0$ such that $|x - x^0|^N u(x)$ is bounded near x^0.*

Theorem 10.1 *Suppose that the Reflection Law (RL) holds for an "enlarged" class of test functions, i. e. for harmonic functions $u(x)$ vanishing on Γ and admiting finitely many polar singularities near Γ. Then the Study Relation (SR) holds for the points P, Q.*

We shall need the following lemma.

Lemma 10.1 *Let u be harmonic in the punctured ball $B(x^0,R)\backslash\{x^0\}$ and suppose that u has a polar singularity at x^0. Then u can be written in the form*

$$u(x)=\frac{Q(x)}{|x-x^0|^m}+v(x),$$

where m is a positive integer, $v(x)$ is harmonic in $B(x^0,R)$ and Q is a polynomial relatively prime with $|x-x^0|^2$.

PROOF OF THE LEMMA 10.1: By the growth condition on $u(x)$ near the singularity $x=x^0$, u can be extended as a distribution $u\in\mathcal{D}'(B(x^0,r))$ to the whole ball $B(x^0,R)$ [1]. Thus,

$$\operatorname{supp}\Delta u\subset\{x^0\}.$$

By the well-known theorem of L. Schwartz (cf., e. g., [R, Theorem 6.25], we can write then

$$\Delta u=R(\partial)\left(\delta_{\{x^0\}}\right),$$

where R is a polynomial and $\delta_{\{x^0\}}$ is the point-mass at x^0. Also we have

$$\delta_{\{x^0\}}=\Delta\left\{\frac{c_n}{|x-x^0|^{n-2}}\right\}.$$

Thus,

$$\Delta\left(u-R(\partial)\left(\frac{c_n}{|x-x^0|^{n-2}}\right)\right)=0$$

[1] For example one can define such an extension simply by

$$(u,\varphi)_{\mathcal{D},\mathcal{D}'}=\int_{B(x^0,R)}u(x)\left(\varphi(x)-\sum_{|\alpha|\le N-1}\frac{1}{\alpha!}\partial^\alpha\varphi(x^0)(x-x^0)^\alpha\right)\,dx$$

for every $\varphi\in C_o^\infty(B(x^0,R))$.

in the sense of distributions and according to Weyl's Lemma it must be harmonic in $B(x^0,R)$. The lemma is proved.

PROOF OF THE THEOREM 10.1: Let us first observe that u must have a polar singularity at Q provided it has a polar singularity at P. Applying Lemma 10.1 we can write u in the form

$$u(x)=\frac{Q_1(x)}{|x-x_P|^{m_1}}+\frac{Q_2(x)}{|x-x_Q|^{m_2}}+v(x),\tag{10.1}$$

where $Q_1(x)$, $Q_2(x)$ are polynomials and $v(x)$ is harmonic. Let $\xi\in K_P\cap\widehat{\Gamma}$ and assume that $\xi\notin K_Q\cap\widehat{\Gamma}$. Let us denote by V a domain in $\mathbb{C}^n$ to which u can be extended holomorphically with the exception of the singularity set $K_P\cup K_Q$. Since $u\equiv 0$ on $\widehat{\Gamma}\backslash(K_P\cup K_Q)$ then if $z\to\xi$ in V with $z\in\widehat{\Gamma}\backslash(K_P\cup K_Q)$, while the left hand side in (10.1) remains zero, the right hand side becomes unbounded. Indeed, the first term tends to infinity, the second remains bounded since $\xi\notin K_Q\cap\widehat{\Gamma}$ and the third term is always bounded. Therefore we arrive at a contradiction and hence ξ must lie in $K_Q\cap\widehat{\Gamma}$.

The following result ([KS3]) shows that unlike in the case $n=2$, there is no generic reflection in higher dimensions.

Theorem 10.2 *Let $\Gamma\subset\mathbb{R}^n$ be a smooth algebraic surface. If*

$$\text{Volume }\{P\text{ near }\ \Gamma:\exists\ Q\text{ such that }\ P,Q\text{ satisfy }\ (SR)\}>0,$$

then Γ must be either a sphere, or a plane.

It is an easy excercise to check that spheres and planes satisfy this hypothesis. Recall that for planes Q is simply the symmetric image of P while for spheres Q is the inversion of P.

The following result [EK] provides a complete answer as to the validity of the (RL) for odd-dimensional spaces.

Theorem 10.3 *If the Reflection Law (RL) holds for two points P, Q sufficiently close to the real-analytic hypersurface $\Gamma\subset\mathbb{R}^{2n+1}$, then Γ must be either a plane or a sphere.*

As for the (RL) in even number of variables the situation is far more delicate. The following example already shows that the (RL) holds in a much greater variety of cases than in odd dimensions.

Example 10.1 Consider in $\mathbb{R}^4$ an axially symmetric surface

$$\Gamma = \{(x_1, x_2, x_3, x_4) : f(x_1, \rho) = 0, \rho = \sqrt{x_2^2 + x_3^2 + x_4^2}\},$$

where f is, for instance, an entire function and ρ stands for the distance to the x_1 axis. The meridian curve γ defined by the equation $f(x_1, \rho) = 0$ in the plane (x_1, ρ), i. e. $\gamma := \{(x_1, \rho) : f(x_1, \rho) = 0\}$, is then symmetric about the x_1 - axis.

By making the change of variables:

$$\xi = x_1 + i\rho, \qquad \xi^* = x_1 - i\rho,$$

and solving the equation $f(x_1, \rho) = 0$ for ξ^* we can write the "complexification" $\widehat{\Gamma}$ of Γ in terms of the complex variables ξ, ξ^* as $\widehat{\Gamma} := \{x \in \mathbb{C}^4 : \xi^* = S(\xi)\}$. The analytic function $S = S(\xi)$ is the Schwarz function of the curve γ (cf. Chapter 9).

First, observe the following

Lemma 10.2 *The Schwarz function S preserves the x_1-axis, in other words, if $P = (z, 0, 0, 0)$ then $S(P) = (Z, 0, 0, 0)$.*

Proof: In view of the definition of S and since γ is symmetric about the x_1 - axis, i. e. $\bar{\xi} \in \gamma$ provided that $\xi \in \gamma$, it follows that

$$\bar{\xi} = S(\xi) \quad \text{and} \quad \xi = S(\bar{\xi}) \quad \text{on} \quad \gamma. \tag{10.2}$$

From (10.2) we have $\overline{S(\xi)} = S(\bar{\xi})$ on γ, or conjugating,

$$S(\xi) = \overline{S(\bar{\xi})} \quad \text{on} \quad \gamma. \tag{10.3}$$

Since both functions in (10.3) are analytic and equal on γ they coincide thus giving $S(P) = \overline{S(\bar{P})} = \overline{S(P)}$ for all P on the x_1-axis implying the lemma.

Let us now state a positive result regarding the (RL) for the case of axially symmetric hypersurfaces in $\mathbb{R}^4$.

Theorem 10.4 *For points $P = (a, 0, 0, 0)$ on the axis of symmetry, the following (RL) holds:*

$$u(P) + (-S'(P))u(S(P)) = 0 \tag{10.4}$$

for all u harmonic near Γ and vanishing on it.

Before giving the proof of the theorem let us see how examples (1), (2) fit into the context.

Example 10.2 Planes. In this case γ coincides with the imaginary axis $\gamma = \{i\rho\}$ (here $\Gamma := \{x_1 = 0\}$). The equation for γ is $\xi^* = -\xi$ and so, $S(\xi) = -\xi$. If $P = (a, 0, 0, 0)$, $S(P) = (-a, 0, 0, 0)$, $S'(P) = -1$. Then (10.4) simply yields for this case the well-known reflection law

$$u(a, 0, 0, 0) + u(-a, 0, 0, 0) = 0.$$

Example 10.3 Spheres. In this case $\Gamma = \{x \in \mathbb{R}^4 : x_1^2 + \rho^2 = 1\}$. The equation for γ becomes $\bar{\xi}\xi = 1$ and thus $S(\xi) = \frac{1}{\xi}$. If $P = (a, 0, 0, 0)$, $S(P) = (\frac{1}{a}, 0, 0, 0)$ and, since $S'(\xi) = -\frac{1}{\xi^2}$ the relation (10.4) now becomes the familiar Kelvin reflection in the sphere in $\mathbb{R}^4$

$$u(a, 0, 0, 0) + \frac{1}{a^2} u(\frac{1}{a}, 0, 0, 0) = 0.$$

Let us also check the Study Relation for points

$$P = (a, 0, 0, 0) \qquad Q = S(P) = (S(a), 0, 0, 0)$$

on the x_1-axis in the last example. We have (as before $\xi = x_1 + i\rho, \xi^* = x_1 - i\rho$):

$$K_P = \{x \in \mathbb{C}^4 : (x_1 - a)^2 + \rho^2 = 0\} = \{x \in \mathbb{C}^4 : (\xi - a)(\xi^* - a) = 0\}$$

and

$$K_{S(P)} = \{x \in \mathbb{C}^4 : (\xi - S(a))(\xi^* - S(a)) = 0\}.$$

Thus if $(\xi, \xi^*) \in K_P \cap \widehat{\Gamma}$ then $\xi^* = S(\xi)$ and $\xi = a$ or $S(\xi) = a$. If $\xi = a$ then $S(\xi) = S(a)$ and $(\xi, \xi^*) \in K_{S(P)} \cap \widehat{\Gamma}$. If $S(\xi) = a$ then $\xi = S^{-1}(a)$. Since $a = S(S(a))$ and so $S^{-1}(a) = S(a)$, $\xi = S(a)$ and hence $(\xi, \xi^*) \in K_{S(P)} \cap \widehat{\Gamma}$. It just remains to note that all the steps are reversible.

Proof of the Theorem 10.4: Let u be harmonic in a neighbourhood of Γ, $u|_\Gamma = 0$. Let us separate the following claim.

Assertion. *The function u can be written in the form $u = u_0 + u_1$ where the functions u_0 and u_1 are both harmonic, vanish on Γ and $u_0(x) = u_0(x_1, \rho)$ is axially symmetric while $u_1(x_1, 0, 0, 0) = 0$.*

Assuming the assertion let us continue with the proof of the theorem. Since u_1 vanishes on the x_1-axis, it suffices to prove (10.4) for axially symmetric harmonic functions $u(x) = u(x_1, \rho)$. It is easy to check that an axially symmetric function $u(x_1, \rho)$ is harmonic in $\mathbb{R}^4$ iff

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} = 0. \tag{10.5}$$

(It is crucial here that the dimension of the space is 4! For other dimensions, say n, the axially symmetric harmonic functions satisfy the equation similar to (10.5) but the constant 2 is replaced by $n-2$, so the following trick fails). Thus, if $u(x_1, \rho)$ is harmonic then $v(x_1, \rho) = \rho\, u(x_1, \rho)$ is harmonic as a function of the variables x_1, ρ in the meridian plane.

Indeed, (10.5) can be written as

$$\rho \Delta_{(x_1,\rho)} + 2\nabla_{(x_1,\rho)} u \cdot \nabla_{(x_1,\rho)} \rho = 0,$$

or, in other words $\Delta_{(x_1,\rho)}(\rho\ u) = 0$. But now $v(x_1, \rho)$ satisfies the Schwarz Reflection Principle with respect to γ, thus

$$v(\xi) + v(\overline{S(\xi)}) = 0$$

for all ξ sufficiently close to γ. In other words,

$$\rho u(x_1 + i\rho) + \mathcal{I}m\overline{S(x + i\rho)}u\left(\overline{S(x_1 + i\rho)}\right) = 0. \tag{10.6}$$

Differentiating (10.6) with respect to ρ we obtain

$$u(\xi) + i\rho\frac{\partial u}{\partial \cancel{\xi}} + \frac{\partial}{\partial \rho}\left(Im\, \overline{S(\xi)}\right) u\left(\overline{S(\xi)}\right) + Im\, \overline{S(\xi)}\frac{\partial u}{\partial \rho}\left(\overline{S(\xi)}\right) = 0. \tag{10.7}$$

ρ

Since by Lemma 10.2 $Im\,(S(\xi)) = 0$ when $\rho = 0$ the relation (10.4) follows from (10.7) and the fact that according to the Cauchy-Riemann equations and Lemma 10.2

$$\frac{\partial}{\partial \rho}\left(Im\, \overline{S(\xi)}\right)|_{\rho=0} = -S'(x_1).$$

Proof of the Assertion: Let us now proceed with the proof of the assertion. If $u(x_1, x_2, x_3, x_4)$ is the given harmonic function, define

$$u_0(x_1, \rho) = \frac{1}{4\pi\rho^2}\int_{S(x_1,\rho)} u(x_1, y)\, dS_{(x_1,\rho)}(y), \quad \rho \neq 0,$$

where $y = (y_2, y_3, y_4) \in \mathbb{R}^3$, $S(x_1, \rho) = \{(x_1, y) : y \in \mathbb{R}^3 : y_2^2 + y_3^2 + y_4^2 = \rho^2\}$ is the 2-dimensional sphere centered at $(x_1, 0, 0, 0)$ of radius ρ, dS is Lebesgue measure on $S(x_1, \rho)$. It is clear then that $\lim_{\rho \to 0+} u_0(x_1, \rho) = u(x_1, 0, 0, 0)$ and it is also clear that $u_0|_\Gamma = 0$. Let us now show that u_0 is harmonic. As observed above we must check that u_0 satisfies

$$\frac{\partial^2 u_0}{\partial x_1^2} + \frac{\partial^2 u_0}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u_0}{\partial \rho} = 0.$$

If $S^2(0,1)$ denotes the unit sphere in $\mathbb{R}^3$ and dS_y stands for Lebesgue measure on it, u_0 can be expressed as,

$$u_0(x_1,\rho) = \frac{1}{4\pi}\int_{S^2(0,1)} u((x_1,0,0,0)+\rho\,(0,y))\,dS_y.$$

Hence,

$$\frac{\partial^2 u_0}{\partial x_1^2} = \frac{1}{4\pi}\int_{S^2(0,1)} \frac{\partial^2 u}{\partial x_1^2}((x_1,0,0,0)+\rho\,(0,y))\,dS_y. \qquad (10.8)$$

On the other hand,

$$\frac{\partial u_0}{\partial \rho} = \frac{1}{4\pi}\int_{S^2(0,1)} \nabla_{x'} u((x_1,0,0,0)+\rho\,(0,y))\cdot y\,dS_y,$$

where $x'=(x_2,x_3,x_4)$. Taking into account that for each $y\in S^2(0,1)$ the vector $\vec{y}$ is the exterior unit normal ν with respect to the ball $B(0,\rho)=\{(x_1,y): y\in\mathbb{R}^3 : \|y\|\le\rho\}$, the divergence theorem implies:

$$\begin{aligned}\frac{\partial u_0}{\partial \rho} &= \frac{1}{4\pi\rho^2}\int_{S(x_1,\rho)} \frac{\partial u}{\partial \nu}\,dS_{(x_1,\rho)}(y)\\ &= \frac{1}{4\pi\rho^2}\int_{B(0,\rho)} \Delta_{x'} u((x_1,x'))\,dx'. \end{aligned} \qquad (10.9)$$

(Here $\Delta_{x'} := \sum_{j=2}^4 \frac{\partial^2}{\partial x_j^2}$). Differentiating again with respect to ρ we obtain using (10.9)

$$\begin{aligned}\frac{\partial^2 u_0}{\partial \rho^2} &= -\frac{2}{4\pi\rho^3}\int_{B(0,\rho)} \Delta_{x'} u(x_1,x')\,dx'\\ &+ \frac{1}{4\pi\rho^2}\frac{\partial}{\partial\rho}\left\{\int_{B(0,\rho)} \Delta_{x'} u(x_1,x')\,dx'\right\} \qquad (10.10)\\ &= -\frac{2}{\rho}\frac{\partial u_0}{\partial \rho} + A.\end{aligned}$$

Now since u is harmonic $-\frac{\partial^2 u}{\partial x_1^2} = \Delta_{x'} u$ and we calculate using spherical coordinates and (10.8):

$$\begin{aligned} A &= \frac{1}{4\pi\rho^2}\frac{\partial}{\partial\rho}\left\{\int_0^\rho \left(\int_{|y|=1} \Delta_{x'} u(x_1,ry)\,dS_y\right) r^2\,dr\right\}\\ &= \frac{1}{4\pi}\int_{S^2(0,1)} \Delta_{x'} u((x_1,\rho y)\,dS_y \qquad (10.11)\\ &= -\frac{1}{4\pi}\int_{S^2(0,1)} \frac{\partial^2 u}{\partial x_1^2}((x_1,0,0,0)+\rho\,(0,y))\,dS_y\\ &= -\frac{\partial^2 u_0}{\partial x_1^2}.\end{aligned}$$

From (10.10-11) we obtain

$$\frac{\partial^2 u_0}{\partial \rho^2} = -\frac{2}{\rho}\frac{\partial u_0}{\partial \rho} - \frac{\partial^2 u_0}{\partial x_1^2}$$

and the proof of harmonicity of u_0 is concluded. Therefore the assertion follows by writing $u(x)=u_0(x)+u_1(x)$ with $u_1(x):=u(x)-u_0(x)$.

The general even-dimensional case

For a point $x^0\in\mathbb{R}^n$ set

$$g(\cdot - x^0) := (z_1-x_1^\circ)^2+\cdots+(z_n-x_n^\circ)^2,$$

the defining function of the isotropic cone $K_{\{x^0\}}$ with vertex at x^0.

Let Γ be a real-analytic hypersurface in $\mathbb{R}^n$ and $\widehat{\Gamma}$ its extension to $\mathbb{C}^n$. Consider two points $x^0,x^1\in\mathbb{R}^n\backslash\Gamma$ that satisfy the Study Relation (SR), i.e.

$$K_{\{x^0\}}\cap\widehat{\Gamma} = K_{\{x^1\}}\cap\widehat{\Gamma}. \qquad \text{(SR)}$$

Let us now introduce the following definition.

Definition 10.2 *Let the dimension n be an even number. We say that the points $x^0, x^1 \in \mathbb{R}^n$ satisfy the Strong Study Relation ((SSR) for short) if x^0, x^1 satisfy (SR) and*

$$g(\cdot - x^0)|_{\widehat{\Gamma}} = (\lambda^{\frac{1}{p}} + h(\cdot)\,(g(\cdot - x^1))^p)g(\cdot - x^1)|_{\widehat{\Gamma}}, \qquad \text{(SSR)}$$

where $p = \frac{n-2}{2}$, λ is a constant, and h is some holomorphic function.

The following theorem [EK] then completely settles the problem of reflection in even-dimensional spaces.

Theorem 10.5 *Let the dimension n be even. The (RL)*

$$u(x^0) + ku(x^1) = 0$$

holds, for all functions u harmonic near Γ and vanishing on Γ, if and only if x^0 and x^1 satisfy the (SSR) with $\lambda = \frac{1}{k}$.

A proof of this result is beyond the scope of these lectures. However, as an example, let us analyse the (SSR) for the case of axially symmetric hypersurfaces Γ in $\mathbb{R}^4$ already studied in this chapter. As above x_1, ρ denote the meridian complex coordinates, $\xi = x_1 + i\rho$, $\xi^* = x_1 - i\rho$ and the complex extension $\widehat{\Gamma}$ of Γ to $\mathbb{C}^4$ is given by $\xi^* = S(\xi)$. As already seen in Theorem 10.4 the (RL) holds for all pairs of points $x^0 = (a, 0, 0, 0)$ and $x^1 = (S(a), 0, 0, 0)$. In this case:

$$g(\cdot - x^0) = (\xi - a)(\xi^* - a), \quad g(\cdot - x^1) = (\xi - S(a))(\xi^* - S(a)),$$

and

$$g(\cdot - x^0)|_{\widehat{\Gamma}} = (\xi - a)(S(\xi) - a), \quad g(\cdot - x^1)|_{\widehat{\Gamma}} = (\xi - S(a))(S(\xi) - S(a)).$$

As before (cf. Example 10.2)) one easily verifies the (SR):

$$\begin{aligned} K_{\{x^0\}} \cap \widehat{\Gamma} &= K_{\{x^1\}} \cap \widehat{\Gamma} \\ &= \{x_1 = \tfrac{a+S(a)}{2} = \tfrac{a+S^{-1}(a)}{2}, \rho = \pm\tfrac{(a-S(a))}{2i}\} \\ &= \{(a, S(a))\} \cup \{(S^{-1}(a), a)\}. \end{aligned}$$

We now have $n = 4$, so $p = 1$. Since $p = 1$, (SSR) simply says that $dg(\cdot - x^0)|_{\widehat{\Gamma}\cap K_{\{x^0\}}} = \lambda dg(\cdot - x^1)|_{\widehat{\Gamma}\cap K_{\{x^1\}}}$. We have

$$\begin{aligned} dg(\cdot - x^0)|_{\widehat{\Gamma}} &= ((S(\xi) - a) + S'(\xi)(\xi - a))\ d\xi \\ dg(\cdot - x^1)|_{\widehat{\Gamma}} &= ((S(\xi) - S(a)) + S'(\xi)(\xi - S(a)))\ d\xi. \end{aligned}$$

Thus

$$\begin{aligned} dg(\cdot - x^0)_{(a,S(a))} &= (S(a) - a)\ d\xi \\ dg(\cdot - x^1)_{(a,S(a))} &= -S'(a)(S(a) - a)\ d\xi, \end{aligned}$$

and (SSR)

$$dg(\cdot - x^0)_{(a,S(a))} = \lambda dg(\cdot - x^1)_{(a,S(a))}, \tag{10.12}$$

holds with $\lambda = -\frac{1}{S'(a)}$ (cf. Theorem 10.4). The same value for λ is obtained in (10.12) if we calculate both differentials at points on $\widehat{\Gamma} \cap K_{\{x^0\}}$ where $(\xi, \xi^*) = (S^{-1}(a), a))$ and recall (Lemma 10.2) that $S^{-1}(a) = S(a)$ and thus $S'(S(a)) = \frac{1}{S'(a)}$.

Notes

The notion of Study Relation (SR) appeared in [KS3]. Theorem 10.2 has been proved in [KS3] (for $n = 3$). Theorems 10.3 and 10.5 that completely settle the problem of point-to-point reflection in higher dimensions have been established rather recently in [EK]. The notion of Strong Study Relation (SSR) has also been introduced in [EK].

Theorem 10.4 appeared earlier in [Kh2].

Chapter 11

Behaviour of Solutions of Cauchy Problems in the Large

Let $n = 2$ and $\mathcal{L} = \frac{\partial^2}{\partial z \partial w} + a(z,w)\frac{\partial}{\partial z} + b(z,w)\frac{\partial}{\partial w} + c(z,w)$ be a differential operator with all coefficients assumed to be entire functions.

Let Γ be a hypersurface in $\mathbb{C}^2$, non-characteristic with respect to $\mathcal{L}$, given by $\{w = S(z)\}$ or $\{z = S^{-1}(w)\}$ with S, S^{-1} holomorphic near Γ.

The following theorem was noticed by several authors, e.g., Garabedian, Vekua, Henrici, H. Lewy to name a few.

Theorem 11.1 *The solution u of a Cauchy Problem*

$$\begin{cases} \mathcal{L}u = 0\,; \\ \partial^\alpha(u - f) = 0, \quad |\alpha| \le 1 \text{ on } \Gamma \end{cases} \tag{11.1}$$

with entire data f extends (perhaps as a multivalued function) along any path in $\mathbb{C}^2$ starting at a non-characteristic point on Γ provided that $S(z), S^{-1}(w)$ extend as holomorphic functions along that path.

PROOF: Take (z^o, w^o) sufficiently close to Γ, so the solution u is holomorphic at (z^o, w^o) by the C-K theorem. Represent $u(z^o, w^o)$ by Riemann's formula ((9.7), Lemma 9.1) applied in the context of $\mathbb{C}^2$. However, since on Γ we have $w = S(z)$, $z = S^{-1}(w)$, $u = f(z, S(z)) = f(S^{-1}(w), w)$, $\frac{\partial u}{\partial w} = \frac{\partial f}{\partial w}(z, S(z)) = \frac{\partial f}{\partial w}(S^{-1}(w), w)$, $\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}(z, S(z)) = \frac{\partial f}{\partial z}(S^{-1}(w), w)$ and f, R are entire (the latter in view of Corollary 6.4 and the remark preceding Lemma 9.1), the right hand side of (9.7) extends holomorphically to any point (z^o, w^o) in $\mathbb{C}^2$ as long as $S(z)$ and $S^{-1}(w)$ extend to that point.

Consider now a special (CP)

$$\begin{cases} \frac{\partial^2 u}{\partial z \partial w} = 0\,; \\ \partial^\alpha(u - zw) = 0, \quad |\alpha| \le 1 \text{ on } \Gamma. \end{cases} \tag{11.2}$$

Denote the (unique) solution by u_Γ. Then, $u_\Gamma = F(z) + G(w)$, where F, G are analytic functions of one variable, and so

$$\frac{\partial u_\Gamma}{\partial z}|_\Gamma = F'(z)|_\Gamma = w|_\Gamma = S(z)|_\Gamma.$$

Hence, $\frac{\partial u_\Gamma}{\partial z} = S(z)$. Similarly, $\frac{\partial u_\Gamma}{\partial w} = S^{-1}(w)$.

Therefore Theorem 11.1 can be restated to say that solutions to all Cauchy problems (11.1) with entire data extend holomorphically along any path starting at a non-characteristic point on Γ and avoiding the singularities of the solution u_Γ of (11.2).

Restricting ourselves to $\mathbb{R}^2 = \{(z,w) : \bar{w} = z\}$ while changing variables to $z = x + iy, \bar{z} = x - iy$, and accordingly, the operator to $\mathcal{L} := \Delta + A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C$ (where A, B, C are entire functions), with $\gamma := \Gamma \cap \mathbb{R}^2 \subset \{(z, \bar{z}) : \bar{z} = S(z), S(z) \text{ is the Schwarz function of } \gamma\}$ and recalling (cf. Chapter 9) that $S^{-1}(\bar{z}) = \overline{S(z)}$ we are led to the following corollary.

Corollary 11.1 *All solutions of (CP)*

$$\begin{cases} \mathcal{L}u = 0\,; \\ \partial^\alpha(u - f) = 0, \quad |\alpha| \le 1 \text{ on } \gamma \end{cases} \tag{11.3}$$

with entire data f (as a function of two complex variables) extend along all paths in $\mathbb{R}^2$ provided that $S(z)$ is analytically continuable along those paths, or equivalently, u_Γ ($=u_\gamma$) defined by

$$\begin{cases} \Delta u_\Gamma = 0\,; \\ \partial^\alpha(u_\Gamma - |z|^2) = 0, \quad |\alpha| \le 1 \text{ on } \gamma \end{cases} \tag{11.4}$$

is continuable along those paths.

Example 11.1 Let $\mathcal{L} := \Delta$.

(i) Take $\gamma = \mathbb{R}$, then $S(z) = z$. Thus, all solutions of (11.3) are entire harmonic functions (already noted in connection with the Bony-Schapira theorem - cf. Corollary 7.3).

(ii) Take $\gamma = \{z : |z| = R|\}$, then $S(z) = \frac{R^2}{z}$. Thus, all solutions of (11.3) extend harmonically to $\mathbb{R}^2 \setminus \{0\}$.

However, solutions can become multi-valued when continued in the complex space, e.g., for $R = 1$, $u_\Gamma = 2log|z| + 1$ and is multi-valued in $\mathbb{C}^2$.

(iii) Consider any quadratic curve, e.g. an ellipse, $\gamma = \{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b,\ a^2 - b^2 = 1\}$. Then $S(z) = (a^2 + b^2)z - 2ab\sqrt{z^2 - 1}$ and all solutions of (11.3) extend harmonically to $\mathbb{R}^2 \setminus \{\pm 1\}$.

Recall the problem A stated in Chapter 1 but now posed in $\mathbb{R}^2$. Consider a domain Ω bounded by a simple analytic curve γ and define the potential of Ω by

$$u_\Omega(z) = \frac{1}{2\pi} \int_\Omega log|\xi - z|\, dA(\xi).$$

(dA is the Lebesgue measure). Then u_Ω solves (in the distributional sense) the problem $\triangle u_\Omega = \chi_\Omega$ and as is easily seen is C^1-smooth in $\mathbb{R}^2$. The problem is then to find a harmonic extension of u_Ω across γ into Ω. Let u_γ be the solution of the (CP)

$$\begin{cases} \triangle u_\gamma = 0\,; \\ \partial^\alpha(u_\gamma - \frac{1}{4}|z|^2) = 0, \ \ |\alpha| \le 1 \text{ on } \gamma. \end{cases} \tag{11.5}$$

Let us denote by $v_\gamma := \frac{1}{4}|z|^2 - u_\gamma$ the solution of

$$\begin{cases} \triangle v_\gamma = 1\,; \\ \partial^\alpha v_\gamma = 0, \ \ |\alpha| \le 1 \text{ on } \gamma. \end{cases} \tag{11.6}$$

Proposition 11.1 *$u_\Omega(z) - v_\gamma(z)$ is harmonic in Ω and provides the desired extension of u_Ω inside Ω.*

PROOF: In Ω and near γ we have $\triangle(u_\Omega - v_\gamma) = 1 - 1 = 0$. On γ, $\partial^\alpha(u_\Omega - v_\gamma) = \partial^\alpha u_\Omega$ for $|\alpha| \le 1$. Then we have the functions u_Ω and $u_\Omega - v_\Gamma$ harmonic in $\mathbb{R}^2 \setminus \Omega$ and in Ω respectively and having the same Cauchy data on γ. Hence, by Kovalevskaya's theorem (Theorem 4.1), $u_\Omega - v_\gamma$ provides the desired extension.

We consider now the above problem in greater generality. Define

$$u_{\Omega,p}(z) = \frac{1}{2\pi} \int_\Omega p(\xi)\, log|\xi - z|\, dA(\xi),$$

where $p(\xi)$ is, say, a polynomial (or, an entire function) of two real variables.

Let P be another polynomial such that $\triangle P = p$ and $u_{\gamma,P}$ be solution of

$$\begin{cases} \triangle u_{\gamma,P} = 0\,; \\ \partial^\alpha(u_{\gamma,P} - P) = 0, \ \ |\alpha| \le 1 \text{ on } \gamma. \end{cases} \tag{11.7}$$

Define $v_{\gamma,P} = P - u_{\gamma,P}$. It solves the problem

$$\begin{cases} \triangle v_{\gamma,P} = p\,; \\ \partial^\alpha v_{\gamma,P} = 0, \ \ |\alpha| \le 1 \text{ on } \gamma. \end{cases} \tag{11.8}$$

Then, arguing as in the proof of Proposition 11.1 we obtain

Proposition 11.2 *$u_{\Omega,p}(z) - v_{\gamma,P}(z)$ is harmonic in Ω and provides the desired extension of $u_{\Omega,p}$ inside Ω.*

As a corollary of this result and Corollary 11.1 we obtain Herglotz' theorem mentioned in Chapter 1.

Corollary 11.2 *All potentials $u_{\Omega,p}$ with polynomial, or even entire densities extend harmonically into Ω along any path free of singularities of the Schwarz function of the boundary γ of Ω or, equivalently, free of singularities of u_γ the solution of the Cauchy problem*

$$\begin{cases} \triangle u_\gamma = 0\,; \\ \partial^\alpha(u_\gamma - \frac{1}{4}|z|^2) = 0, \ \ |\alpha| \le 1 \text{ on } \gamma. \end{cases} \tag{11.9}$$

The function u_γ turns out to be responsible for a great many potential-theoretic properties of the domain Ω.

In particular, let us note in passing the following.

Remark 11.1 Suppose the Schwarz function S of γ extends as a distribution inside Ω. Since it is analytic near γ, in the sense of distributions we have $\frac{\partial S}{\partial \bar z} = T$, where T is a distribution compactly supported in Ω. Then the following quadrature identity holds:

$$\int_\Omega f\, dA = \int_\Omega f\, dT = < f, T >$$

for all analytic functions f in $\overline{\Omega}$.

Indeed, by Green's formula

$$\int_\Omega f\, dA = \frac{1}{2i}\int_\gamma f\, \bar z\, dz = \frac{1}{2i}\int_\gamma f\, S(z)\, dz = \int_\Omega f\frac{\partial S}{\partial \bar z}\, dA = < f, T > .$$

Similarly, if $\Delta u_\gamma = T$, we have the quadrature identity

$$\int_\Omega v\, dA = < v, T >$$

hold for all v harmonic in $\overline{\Omega}$.

Ω is then called a *quadrature domain* with respect to the distribution T.

For example, if Ω is the interior of an ellipse $\gamma := \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b,\ a^2 - b^2 = 1\}$, then (cf. Corollary 13.5)

$$\int_\Omega v\, dA = const \cdot \int_{-1}^{1} v(t)\sqrt{1-t^2}\, dt$$

for all harmonic polynomials v.

In order to investigate a possibility of extending Corollaries 11.1, 11.2 to higher dimensions let us give the following definition.

Definition 11.1. *We shall call the Schwarz Potential (SP) u_Γ of a non-singular, real analytic hypersurface $\Gamma \subset \mathbb{R}^n$, $n \geq 2$ the solution of the (CP)*

$$\begin{cases} \Delta u_\Gamma = 0\ ; \\ \partial^\alpha(u_\Gamma - \frac{1}{2}\|x\|^2) = 0, \quad |\alpha| \leq 1 \text{ on } \Gamma. \end{cases} \tag{11.10}$$

The following conjecture if true would be an analogue of Corollary 11.1.

The (SP)-Conjecture. *All solutions of the Cauchy problems*

$$\begin{cases} \Delta u = 0\ ; \\ \partial^\alpha(u - f) = 0, \quad |\alpha| \leq 1 \text{ on } \Gamma \end{cases}$$

with entire data f, extend harmonically along all paths free of singularities of u_Γ.

For the sake of convenience let us adapt the following notation.

Notation. We shall write $f \equiv g$ on Γ for $\partial^\alpha(g - f) = 0$, $|\alpha| \leq 1$ on Γ.

To see why the (SP)-Conjecture may be at least plausible let us note the following proposition.

Proposition 11.3 *Let $\widehat{\Gamma}$ be, as usual, the complexification of Γ, i.e., $\widehat{\Gamma} = \{z \in \mathbb{C}^n : \phi(z) = 0\}$ where ϕ is holomorphic and non-singular in a neighborhood of Γ. Let $z^o \in \widehat{\Gamma}$ be a characteristic point (with respect to Δ). Then u_Γ must be singular at z^o.*

PROOF: Suppose u_Γ is holomorphic near z^o. Then, so is $v_\Gamma := \frac{1}{2}\sum_{j=1}^n z_j^2 - u_\Gamma$.

Now, $v_\Gamma \equiv 0$ on $\widehat{\Gamma}$ and $\Delta v_\Gamma = n \neq 0$. Since $v_\Gamma \equiv 0$ on $\widehat{\Gamma}$, the Weierstrass' Preparation Theorem implies that $v_\Gamma = \phi^2 g$ for some g holomorphic near z^o.

But on the other hand since $\phi(z^o) = 0$ we have

$$n = \Delta(\phi^2 g)|_{z^o} = 2\sum_{j=1}^{n}(\frac{\partial \phi}{\partial z_j})^2 g|_{z^o}.$$

This gives a desired contradiction since z^o is a characteristic point and hence $\sum_{j=1}^{n}(\frac{\partial\phi}{\partial z_j})^2|_{z^o} = 0$.

According to an intuitive feeling that singularities of solutions to a Cauchy problem appear only as a result of propagation through $\mathbb{C}^n$ of singularities originated on the initial surface, where by the above proposition the (SP) picks up all of them, we "conclude" that the (SP) should in principle exhibit all possible singularities for solutions of all Cauchy problems (for the Laplace equation).

Remark 11.2 If this conjecture was indeed true it would, in particular, have answered Herglotz' question in all dimensions. Namely, the same word for word argument as that used in two dimensions would show that all potentials $u_{\Omega,p}$ with a polynomial (or, even an entire) density p extend inside Ω to any region free of singularities of $u_{\partial\Omega}$ (the (SP) of the boundary $\partial\Omega$ of Ω).

The following simple example is the first in a row of similar examples continued in the next sections for which the (SP)-Conjecture does hold.

Example 11.2 Suppose $\Gamma = \{x \in \mathbb{R}^n \;:\; \sum_{j=1}^{n} a_j x_j - c = 0\}$ be a hyperplane. First, let us find u_Γ. It is easier to start out with $v_\Gamma = \frac{1}{2}\|x\|^2 - u_\Gamma$ that solves

$$\begin{cases} \Delta v_\Gamma = n\,; \\ v_\Gamma \equiv 0 \text{ on } \Gamma. \end{cases}$$

One easily guesses that

$$v_\Gamma = \frac{n}{\|a\|^2}\left(\sum_{j=1}^{n} a_j x_j - c\right)^2,$$

where $a = (a_1, ..., a_n)$. Hence,

$$u_\Gamma = \frac{1}{2}\|x\|^2 - \frac{n}{2\|a\|^2}\left(\sum_{j=1}^{n} a_j x_j - c\right)^2.$$

The (SP)-Conjecture then would imply that solutions to all Cauchy problems with entire data on Γ are in fact entire harmonic functions; this is indeed the case in view of the Bony-Schapira theorem (cf. Corollaries 7.1 and 8.1) since every characteristic with respect to the Laplacian complex hyperplane intersects $\widehat{\Gamma}$, the complexification of Γ. Another more direct proof of this can be obtained by applying the estimates in Chapter 2.

Remark 11.3 Of course, the previous example can be extended to the case of a complex hyperplane $\widehat{\Gamma} = \{z \in \mathbb{C}^n \;:\; \sum_{j=1}^{n} a_j z_j - c = 0\}$ as long as $\widehat{\Gamma}$ is not characteristic, i.e., $\sum_{j=1}^{n} a_j^2 \neq 0$. In fact, Proposition 11.3 shows that for characteristic $\widehat{\Gamma}$, $u_{\widehat{\Gamma}}$ does not even exist.

Notes

An important yet simple Theorem 11.1 is contained in various forms in [G2], [Hen], [Le], [V]. Here we followed the presentation in [Kh1] where the reader can find more references. Most frequently, it is stated in the form of Corollary 11.1 (first statement)- cf., e.g., [Da]. Propositions 11.1 - 11.2 and Corollary 11.2 are from Herglotz' often overlooked memoir [Her]. Yet some versions of it, in all dimensions, were noted earlier by E. Schmidt and later by Wavre. We refer the reader to a detailed historic account and references in Shapiro's book [Sh1].

The remark following Corollary 11.2 is a subject on its own. Someone interested in pursuing this direction further is referred to works by Aharonov, Gustafsson, Sakai, Shapiro and references therein. Here we simply refer the reader to [Da], [KS1], [Sa], [Sh1] where a great many references are given. Definition 11.1 and the (SP)-Conjecture appeared in [KS1] and so did Proposition 11.3 although, to some degree, the (SP)-Conjecture was anticipated in [Sh2]. Also, cf. [Kh1] and [Sh1]. First few nontrivial cases (spheres, cylinders, cones and ellipsoids) for which the (SP)-Conjecture holds although restricted to polynomial data were investigated in [KS1] by the ad-hoc, "hands on" methods. Some of those results are discused in the following sections. There is by now a considerable progress achieved in verifying the Conjecture mostly due to the efforts by Sternin and Shatalov and their school (cf., e.g., their book [SS]) and, independently, by G. Johnsson [Jo]. The latter, in particular, verified the conjecture for all quadratic surfaces.

A direct argument handling the last example can be found, for instance, in Hörmander's book [Hör1, Lemma 9.1.4].

Chapter 12

The Schwarz Potential Conjecture for Spheres

Let $\Gamma = \{x \in \mathbb{R}^N : |x| = R\}$. First let us calculate the SP u_Γ by solving

$$\begin{cases} \Delta u_\Gamma = 0; \\ u_\Gamma \equiv \frac{1}{2}|x|^2 \text{ on } \Gamma. \end{cases}$$

The rotational symmetry of Δ, Γ and the data together with uniqueness of solutions imply that one should seek a solution in the form

$$u_\Gamma = \begin{cases} c_1 \log|x| + c_2, & \text{when } N = 2; \\ c_1|x|^{2-N} + c_2, & \text{when } N \geq 3. \end{cases}$$

Calculating the constants we find

$$u_\Gamma = \begin{cases} R^2(\log|x| + \frac{1}{2} - \log R) & \text{when } N = 2; \\ -\frac{R^N}{N-2}\frac{1}{|x|^{N-2}} + \frac{N}{2(N-2)}R^2 & \text{when } N \geq 3. \end{cases}$$

So the SP conjecture would imply the following

Theorem 12.1 *All solutions of the (CP)*

$$\begin{cases} \Delta u = 0; \\ u \equiv f \text{ on } \Gamma \end{cases}$$

with entire data f extend harmonically to all of $\mathbb{R}^N \setminus \{0\}$.

From now on we shall assume $N \geq 3$ and $R = 1$.

Our strategy is rather straightforward.

Let $P_m :=$ {polynomials in N variables of degree $\leq m$} and let $H_k \subset P_k$ be a subspace of homogeneous polynomials of degree k.

Let $f(x) = \sum_{m=0}^{\infty} f_m(x)$, where $f_m \in H_m$.

We are going to solve the (CP)

$$\begin{cases} \Delta U_m = 0; \\ U_m \equiv f_m \quad on\ \Gamma \end{cases}$$

for all $m \geq 0$ and show that the series $\sum_{m=0}^{\infty} U_m(x)$ converges uniformly on compact subsets of $\mathbb{R}^N \setminus \{0\}$.

Lemma 12.1 *Let $f_m \in P_m$. Then, $f_m \equiv u_m + (|x|^2-1)v_m$ on Γ, where $u_m \in P_m$, $v_m \in P_{m-2}$ (or $=0$ if $m = 0,1$) are harmonic polynomials.*

PROOF: Consider operator $T : P_m \to P_m$ defined by

$$T(p) = \Delta[(|x|^2 - 1)p].$$

Claim. T is surjective.

Since $dim P_m < +\infty$ it is sufficient to check that T is injective. Let $p \in Ker T$, then $(|x|^2-1)p$ is harmonic. But it vanishes on Γ, hence the maximum principle implies that it vanishes identically. Therefore $p \equiv 0$.

For $m = 0, 1$, the lemma is obvious, so let $m \geq 2$. Find $p \in P_{m-2}$ such that $\Delta[(|x|^2-1)p] = \Delta f_m$. Then $u_m := f_m - (|x|^2-1)p \in P_m$ is harmonic. Now, by the same taken $p = v_m + (|x|^2-1)q$, where $v_m \in P_{m-2}$ is harmonic and $q \in P_{m-4}$. So,

$$f_m = u_m + (|x|^2-1)v_m + (|x|^2-1)^2 q$$

and the lemma follows.

Let

$$\begin{aligned} u_m &= u_{m,0} + u_{m,1} + \cdots + u_{m,m}, \\ v_m &= v_{m,0} + v_{m,1} + \cdots + v_{m,m-2} \end{aligned} \tag{12.1}$$

denote the decomposition of u_m and v_m into homogeneous polynomials; thus $u_{m,j}$, $v_{m,j}$ are harmonic and in H_j.

Lemma 12.2 *The solution U_m of the (CP)*

$$\begin{cases} \Delta U_m = 0; \\ U_m \equiv f_m \ on\ \Gamma, \quad f_m \in H_m \end{cases}$$

is given by

$$U_m = \sum_{k=0}^{m} u_{m,k} + \sum_{k=0}^{m-2} \frac{2}{2-N-2k}\left(\frac{v_{m,k}}{|x|^{N-2+2k}} - v_{m,k}\right), \tag{12.2}$$

where $u_{m,k}$ and $v_{m,k}$ are the same as in (12.1) (and Lemma 12.1). (In the trivial cases of $m = 0, 1$ $v_{m,k} = 0$).

PROOF: Recall Euler's formula: if $g \in H_k$, then

$$\sum_{1}^{N} x_j \frac{\partial g}{\partial x_j} = kg. \tag{12.3}$$

Let $h \in H_k$ be harmonic. We have

$$\begin{aligned} \frac{\partial^2}{\partial x_j^2}\left(\frac{h}{(|x|^{N-2-2k})}\right) &= h_{x_j x_j} |x|^{N-2-2k} + 2(2-N-2k)x_j|x|^{-N-2k}h_{x_j} \\ &-(2-N-2k)(N+2k)\ h\ x_j^2|x|^{-N-2-2k} \\ &+(2-N-2k)|x|^{-N-2k}\ h. \end{aligned}$$

Summing over j and using (12.3) we conclude that $\frac{h}{|x|^{N-2+2k}}$ is harmonic. Now $\sum_{k=0}^{m} u_{m,k} = u_m$ while

$$\frac{2}{2-N-2k}\left(\frac{v_{m,k}}{|x|^{N-2-2k}} - v_{m,k}\right) \equiv (|x|^2-1)v_{m,k} \quad \text{on } \Gamma. \tag{12.4}$$

(12.4) and Lemma 12.1 complete the proof of Lemma 12.2.

It's worthwhile to pause here to observe the following corollary.

Corollary 12.1 *All solutions of (CP) with polynomial data on the sphere Γ extend harmonically to $\mathbb{R}^N \setminus \{0\}$ thus confirming the (SP) conjecture for polynomial data.*

Remark 12.1 With a little bit more work, one can show that any solution U_m of the (CP) with data $f_m \in P_m$ can be written in the form

$$U_m = u_m + R(D)(u_\Gamma),$$

where $u_m \in P_m$ is harmonic and R is a polynomial.

Lemma 12.3 *Let $F_m := \max\{|f_m(x)| : x \in \Gamma\}$ and $G_m := \max\{\|\nabla f_m(x)\| : x \in \Gamma\}$. Then,*

$$\lim_{m\to\infty} F_m^{\frac{1}{m}} = \lim_{m\to\infty} G_m^{\frac{1}{m}} = 0.$$

PROOF: We shall conduct the argument for G_m, the proof for F_m is similar.

For any $x \in \Gamma$ fixed and any $t \in \mathbb{C}$, the function $t \to f(tx)$ is an entire analytic function.

We have

$$f(tx) = \sum_0^\infty t^m f_m(x).$$

For each j, $1 \le j \le N$, we have

$$\frac{\partial f(tx)}{\partial x_j} = \sum_0^\infty t^m \frac{\partial f_m(x)}{\partial x_j},$$

i.e. $\dfrac{\partial f_m(x)}{\partial x_j}$ are the Taylor coefficients of the entire function $t \to \dfrac{\partial f(tx)}{\partial x_j}$.

The Cauchy-Hadamard estimate then implies

$$\left|\frac{\partial f_m(x)}{\partial x_j}\right| \le \frac{\max\{|\frac{\partial f(tx)}{\partial x_j}| : |t| \le T\}}{T^m}$$

for all $T > 0$. Therefore,

$$\max\{\left|\frac{\partial f_m(x)}{\partial x_j}\right| : x \in \Gamma\} \le \frac{\max\{|\frac{\partial f(z)}{\partial z_j}| : z \in \mathbb{C}^N, \|z\| \le T\}}{T^m}$$

and

$$\max\{\|\nabla f_m(x)\| : x \in \Gamma\} \le \frac{\left(\sum_{j=1}^N \{\max\{|\frac{\partial f(z)}{\partial z_j}| : \|z\| \le T\}\}^2\right)^{\frac{1}{2}}}{T^m}.$$

Taking m^{th}-root and letting $m \to \infty$ gives

$$\overline{\lim_{m\to\infty}}\, G_m^{\frac{1}{m}} \le \frac{1}{T}.$$

Since T was arbitrary, the lemma follows.

From now on A_N, B_N, C_N, etc. denote constants that only depend on the dimension of the space.

Lemma 12.4 *Let $h \in P_m$ be harmonic and $h = h_0 + h_1 + \cdots + h_k + \cdots + h_m$ be its decomposition into homogeneous harmonic polynomials. Then,*

$$\max_{x\in\Gamma} |h_k(x)| \le C_N k^{\frac{N}{2}} \max_{x\in\Gamma} |h(x)|, \quad 1 \le k \le m.$$

Also,

$$|h_0(x)| = |h_0(0)| \le \max_{x\in\Gamma} |h(x)|.$$

PROOF: Without loss of generality suppose $\max_{x\in\Gamma} |h(x)| = 1$. The case $k = 0$ is obvious, so consider $k \ge 1$.

Recall that $h_i \perp h_j$ if $i \ne j$ in $L^2(\Gamma, d\sigma)$, where $d\sigma$ is the surface measure on Γ. Hence

$$\int_\Gamma |h_k|^2 d\sigma \le \int_\Gamma |h|^2 d\sigma \le |\Gamma|,$$

where $|\Gamma|$ is the $(N-1)$-dimensional measure of Γ.

Therefore, it follows easily that

$$\int_B |h_k|^2 d\sigma \le A_N, \tag{12.5}$$

where $B := \{x : |x| < 1\}$ is the unit ball.

Fix $y \in \Gamma$. Then, for $0 < r < 1$, the mean value theorem applied to the ball B' centered at ry with radius $(1-r)$ and the subharmonic function $|h_k|^2$ gives the estimate

$$|h_k(ry)|^2 \le \frac{1}{|B'|}\int_{B'} |h_k(x)|^2 dx \le \frac{A_N}{A'_N(1-r)^N} = \frac{A''_N}{(1-r)^N}.$$

From homogeneity of h_k we derive then

$$|h_k(y)| \le \left[\frac{A''_N}{r^{2k}(1-r)^N}\right]^{\frac{1}{2}}$$

for all $0 < r < 1$. Taking $r = 1 - \frac{1}{2k}$ we obtain the lemma.

Lemma 12.5 *Let $f_m \equiv u_m + (|x|^2 - 1)v_m$ on Γ be as in Lemma 12.1. Then*

$$V_m := \max_{x\in\Gamma}\{|v_m(x)|\} \le C_N(G_m + m^{2N}F_m),$$

where $F_m = \max_{x\in\Gamma}\{|f_m(x)|\}$, $G_m = \max_{x\in\Gamma}\{\|\nabla f_m(x)\|\}$ are the same as in Lemma 12.3. In particular, $lim_{m\to\infty} V_m^{\frac{1}{m}} = 0$.

Proof: On Γ by the hypothesis we have for $1 \le j \le N$

$$\frac{\partial f_m}{\partial x_j} = \frac{\partial u_m}{\partial x_j} + 2x_j v_m(x).$$

Therefore,

$$4\sum_1^N x_j^2 |v_m|^2 = 4|v_m|^2 \le 2(\|\nabla f_m\|^2 + \|\nabla u_m\|^2)$$

on Γ, i.e, for $x \in \Gamma$,

$$|v_m(x)| \le C(G_m + \|\nabla u_m(x)\|). \tag{12.6}$$

Lemma 12.6 *Let $h \in H_k$ be a homogeneous polynomial of degree k. Then*

$$\max_{x\in\Gamma}\|\nabla h(x)\| \le k\sqrt{2}\max_{x\in\Gamma}|h(x)|.$$

Assume Lemma 12.6 for the moment.

From Lemmas 12.4, 12.6 and the fact that $u_m = f_m$ on Γ we obtain for $x \in \Gamma$

$$\left|\frac{\partial u_m}{\partial x_j}\right| \le \sum_{k=0}^m C_N k^{\frac{N}{2}+1} F_m$$

and, finally,

$$\|\nabla u_m(x)\| \le C_N m^{\frac{N+4}{2}} F_m \le C_N m^{2N} F_m. \tag{12.7}$$

Now, (12.6) and (12.7) imply Lemma 12.5.

Proof of Lemma 12.6: Fix $x \in \Gamma$. Then , in view of (12.3) the normal derivative of h at x equals

$$\frac{\partial h}{\partial n}(x) = \sum_1^N x_j \frac{\partial h}{\partial x_j} = kh(x). \tag{12.8}$$

Fix vector $\vec{y}$, $\|\vec{y}\| = 1$ tangent to Γ at x. Consider the 2-dimensional plane Π spanned by vectors $\vec{x}$, $\vec{y}$. Π intersects Γ over a unit circle T. h restricted to the plane Π is a homogeneous polynomial H_k of degree k. In particular, by Lemma 12.1, the restriction of h on T, $H_k|_T$, is a harmonic polynomial of two variables of degree $\le k$. Thus, if we use polar coordinates (r, θ) in the plane Π, we have on T

$$H_k|_T = \sum_0^k (a_j \cos j\theta + b_j \sin j\theta) = H_k(\theta),$$

i.e. H_k becomes a trigonometric polynomial of order k.

Then, invoking classical Chebyshev's inequality, we obtain

$$
\begin{aligned}
|D_{\vec{y}} h(x)| &= \left| \frac{dH_k(\theta)}{d\theta} |_x \right| \leq k \max_{z \in T} |H_k(z)| \\
&= k \max_{x \in T} |h(x)| \leq k \max_{x \in \Gamma} |h(x)| \qquad (12.9)
\end{aligned}
$$

for an arbitrary vector $\vec{y}$ tangent to Γ at x. From (12.8) and (12.9) the lemma follows.

Remark 12.2 It would be interesting to find the sharp constant in Lemma 12.6. (12.8) suggests that $\sqrt{2}$ probably is not optimal: the maximum of normal and tangential derivatives cannot be attained at the same point.

Proof of the Theorem 12.1: We want to show that $\sum_{m=0}^{\infty} |U_m(x)| < \infty$, for all $x, 0 < |x| < \infty$, where U_m is defined by (12.2).

For that, it suffices to show that the series:

(I) $\sum_{m=0}^{\infty} |u_m(x)|$,

(II) $\sum_{m=0}^{\infty} \sum_{k=0}^{m-2} \frac{1}{|2-N-2k|} \frac{|v_{m,k}(x)|}{|x|^{N-2+2k}}$,

and

(III) $\sum_{m=0}^{\infty} \sum_{k=0}^{m-2} \frac{1}{|2-N-2k|} |v_{m,k}(x)|$

all converge.

(I) Lemma 12.3 implies that $F_m = \varepsilon_m^m$, where $lim\ \varepsilon_m = 0$. Fix $x : \frac{1}{R} < |x| < R, R > 1$. From Lemma 12.4 it follows

$$|u_{m,k}| \leq C_N k^{\frac{N}{2}} \varepsilon_m^m |x|^k \leq C'_N A_N^k \varepsilon_m^m R^k, \qquad (12.10)$$

where $A_N > 1$ is a constant. Thus,

$$\sum_{k=0}^{m} |u_{m,k}(x)| \leq C'_N \varepsilon_m^m \sum_{k=0}^{m} (A_N R)^k \leq C''_N \varepsilon_m^m (A_N R)^{m+1} \qquad (12.11)$$

and hence,

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} |u_{m,k}(x)| \leq C''_N A_N R \sum_{m=0}^{\infty} (A_N \varepsilon_m R)^m < A(R) < +\infty$$

because $\varepsilon_m \to 0$ when $m \to \infty$.

It is worth pausing here to observe the following

Corollary 12.2 *The solution* $u_0 := \sum_{m=0}^{\infty} \sum_{k=0}^{m} u_{m,k}(x)$ *of the Dirichlet problem in the unit ball* B

$$\begin{cases} \Delta u = 0; \\ u = f \ on\ \Gamma \end{cases}$$

with entire data f is an entire harmonic function.

Note that the above argument immediately implies the convergence of (III) as well. Indeed, Lemma 12.5 implies that $V_m := \max_{x \in \Gamma} |v_m(x)| = \delta_m^m$, $\delta_m \to 0$ while Lemma 12.3 provides the estimate

$$|v_{m,k}(x)| \leq C_N k^{\frac{N}{2}} \delta_m^m |x|^k \qquad (12.12)$$

which is identical with (12.10).

Finally, to establish the convergence of (II), we fix $x : \frac{1}{R} < |x| < R$. Without loss of generality, we can assume $|x| < 1$, since for $|x| > 1$ convergence of (II) is implied by that of (III).

Then (12.12) yields (cf. (12.11))

$$
\begin{aligned}
\sum_{k=0}^{m-2} \frac{|v_{m,k}(x)|}{|x|^{N-2+2k}} &\leq C_N \delta_m^m \sum_{k=0}^{m-2} k^{\frac{N}{2}} R^{N-2+k} \\
&\leq C'_N \delta_m^m R^{N-2} (A_N R)^{m-1}.
\end{aligned}
$$

Therefore, as before

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m-2} \frac{|v_{m,k}(x)|}{|x|^{N-2+2k}} < A(R) < +\infty$$

and hence series (II) converges as well.

From the estimates we have given it follows at once that the series $u(x) := \sum_{m=0}^{\infty} U_m(x)$ giving the solution of our Cauchy problem on the sphere converges absolutely everywhere in $\mathbb{C}^N \setminus \{z : \sum_1^N z_j^2 = 0\}$. Thus we obtain the following corollary establishing the (SP) conjecture for spheres in the $\mathbb{C}^N$-context.

Corollary 12.3 *The solution of the (CP)*

$$\begin{cases} \Delta u = 0; \\ u \equiv f \text{ on } \Gamma \end{cases}$$

with entire data f extends (perhaps, as a multi-valued function for odd N) to the whole complement in $\mathbb{C}^N$ of the isotropic cone $\Gamma_0 := \{z : \sum_1^N z_j^2 = 0\}$.

Notes

Theorem 12.1 and Corollary 12.3 put in a context of the (SP) Conjecture are contained in G. Johnsson's Thesis [Jo]. In fact, Johnsson has even solved the problem for all second order operators that have the Laplacian as their principal part. Johnsson's work is rather deep and is based on the globalizing family arguments (cf. Chapters 6 and 7) blended with local uniformization of solutions of Cauchy's problems pioneered by Leray [L].

Similar and even somewhat more general results based on a set of interesting topological ideas (R. Thom's theorem) have been independently developed by V.Shatalov, B. Sternin and their school (cf. [SS] and references therein).

The elementary proof of Theorem 12.1 we present here was found in [Kh3]. Lemmas 12.1, 12.2 and following elementary Corollary 12.1 were obtained earlier in [KS1]. Lemmas 12.3, 12.4 and Corollary 12.2 are

contained in [KS4] where Corollary 12.2 is shown to hold for arbitrary ellipsoids. Lemmas 12.5, 12.6 that tie the loose ends together are from [Kh3].

Chapter 13

Potential Theory on Ellipsoids

Consider an ellipsoid Γ

$$\Gamma = \{x \in \mathbb{R}^N : \sum_{j=1}^{N} \frac{x_j^2}{a_j^2} - 1 = 0, a_1 > a_2 > \ldots > a_N > 0\}$$

and let Ω be its interior.

Definition 13.1 *A family of ellipsoids $\{\Gamma_\lambda\}$,*

$$\Gamma_\lambda = \{x \in \mathbb{R}^N : \sum_{j=1}^{N} \frac{x_j^2}{a_j^2 + \lambda} - 1 = 0\},$$

where $-a_N^2 < \lambda < +\infty$ is called a confocal family (for N=2 these are ellipses with the same foci).

Observe that when $\lambda \to -a_N^2$,

$$\Gamma_\lambda \to \{x \in \mathbb{R}^N : x_N = 0, \sum_{j=1}^{N-1} \frac{x_j^2}{a_j^2 - a_N^2} - 1 = 0\} =: E.$$

E is called the *focal ellipsoid.*

The following classical theorem goes back to MacLaurin.

Theorem 13.1 *Let u be, say, an entire harmonic function. Then*

$$\frac{1}{|\Omega_\lambda|} \int_{\Omega_\lambda} u(x)dx = constant \tag{13.1}$$

for all $\lambda: \quad \lambda > -a_N^2$.

From now on, for the sake of brevity, we shall only consider the case $N \geq 3$.

Remark 13.1 MacLaurin's theorem is a corollary of the following result of Asgeirsson (1949).

"Suppose $u = u(x,y)$, where $x \in \mathbb{R}^{m_1}, y \in \mathbb{R}^{m_2}$ satisfy the *ultrahyperbolic equation*

$$\Delta_x u = \Delta_y u.$$

Then if $\mu_i(x,y,r)$, $i = 1,2$ denote respectively the mean values of u over m_i-dimensional balls of radius r centered at (x,y), we have $\mu_1(x,y,r) = \mu_2(x,y,r)$."

To see how the Asgeirsson theorem implies MacLaurin's, change variables to

$$x_i = \xi_i \cosh\ \alpha_i + \eta_i \sinh\ \alpha_i, \qquad i = 1, \ldots, N,$$

where α_i are some constants. One easily finds

$$\frac{\partial^2 u}{\partial \xi_i^2} = \frac{\partial^2 u}{\partial x_i^2} \cosh^2 \alpha_i;$$

$$\frac{\partial^2 u}{\partial \eta_i^2} = \frac{\partial^2 u}{\partial x_i^2} \sinh^2 \alpha_i.$$

Subtracting we obtain $\Delta_\xi u = \Delta_\eta u$, and so we can apply Asgeirsson's theorem. MacLaurin's theorem follows at once by noting that spheres in the ξ and η variables respectively centered at the origin, i.e.

$$\sum_{i=1}^{N} \frac{x_i^2}{\cosh^2 \alpha_i} \leq r^2$$

and

$$\sum_{i=1}^{N} \frac{x_i^2}{\sinh^2 \alpha_i} \leq r^2,$$

are nothing else but confocal ellipsoids in the x-variables.

The following notions are due to E. Fischer. Let H_k as before be the space of homogeneous polynomials of degree k. If $f \in H_k$, then

$$f(z) = \sum_{|\alpha|=k} f_\alpha z^\alpha.$$

Introduce an inner product on H_k (called the Fischer inner product), by letting

$$< z^\alpha, z^\beta >= \begin{cases} 0, & \alpha \neq \beta \\ \alpha!, & \alpha = \beta \end{cases} .$$

If $f = \sum_{|\alpha|=k} f_\alpha z^\alpha$, $g = \sum_{|\alpha|=k} g_\alpha z^\alpha$ then

$$< f, g >= \sum_{|\alpha|=k} \alpha! f_\alpha \overline{g_\alpha}.$$

Now consider the operator $\left(\frac{\partial}{\partial z}\right)^\alpha$ that maps H_k into $H_{k-|\alpha|}$, and $z^\alpha(f) = z^\alpha f$ which maps $H_{k-|\alpha|}$ back to H_k.

The main point of introducing such an inner product is the following:

Fact. *The operators $\left(\frac{\partial}{\partial z}\right)^\alpha$ and z^α are adjoint with respect to $<,>$.*

We leave checking this property as an easy exercise for the reader. The most immediate consequence is the following.

Corollary 13.1 *Let P be a polynomial with real coefficients. The operators $P(D)$ and multiplication by P are mutually adjoint with respect to $<,>$.*

PROOF OF MACLAURIN'S THEOREM: It suffices to check (13.1) for harmonic homogeneous polynomials.

We shall need the following lemma. Let $\mathcal{H}_m$ denote the subspace of homogeneous harmonic polynomials of degree m.

Lemma 13.1 *The linear span of harmonic polynomials $(z \cdot \overline{\xi})^m$ for all $\xi \in \Gamma_0 = \{\xi \in \mathbb{C}^N : \sum_{j=1}^N \xi_j^2 = 0\}$ (the isotropic cone) is dense in $\mathcal{H}_m$.*

PROOF: Note that for any polynomial $f \in H_m$,

$$\frac{1}{m!} < f, (z \cdot \overline{\xi})^m >= f(\xi) . \tag{13.2}$$

Indeed, for all multi-indices α, $|\alpha| = m$ we have

$$\frac{1}{m!} < z^\alpha, (z \cdot \overline{\xi})^m >= \frac{1}{\alpha!} < z^\alpha, (z\overline{\xi})^\alpha >= \xi^\alpha$$

and (13.2) follows.

Let us assume that $u \in \mathcal{H}_m$ satisfy

$$< u, (z \cdot \overline{\xi})^m >= 0, \quad \forall \xi \in \Gamma_0.$$

Then $u(\xi) = 0$, for all $\xi \in \Gamma_0$. By Hilbert's Nullstellensatz,

$$u(\xi) = \Big(\sum_{j=1}^N \xi_j^2\Big) q(\xi), \quad q \in H_{m-2}.$$

But then, since u is harmonic, we have

$$0 =< \Delta u, q >=< u, \Big(\sum_{j=1}^N \xi_j^2\Big) q >=< u, u >$$

and u must vanish identically. That proves the lemma.

In view of the lemma, we just have to check (13.1) for polynomials

$$(z \cdot \overline{\xi})^m, \quad \xi \in \Gamma_0.$$

Fix λ. Let $b_i = (a_i^2 + \lambda)^{1/2}$ be the semi-axes of Ω_λ. We have to show that

$$\frac{1}{|\Omega_\lambda|} \int_{\Omega_\lambda} (x \cdot \overline{\xi})^m dx = \frac{1}{|\Omega|} \int_\Omega (y \cdot \overline{\xi})^m dy, \quad \forall \xi \in \Gamma_0.$$

Changing variables in both integrals

$$x_j = b_j x'_j; \quad y_j = a_j x''_j, \ j = 1, \dots, N,$$

we see that it suffices to show the following:

$$\int_B \Big(\sum_{j=1}^N a_j x_j \overline{\xi_j}\Big)^m dx = \int_B \Big(\sum_{j=1}^N b_j x_j \overline{\xi_j}\Big)^m dx,$$

where B is the unit ball in $\mathbb{R}^N$. Or, since for $\xi \in \Gamma_0$

$$\sum_{j=1}^{N}\Big((a_j\xi_j)^2-(b_j\xi_j)^2\Big)=-\lambda^2\sum_{j=1}^{N}\xi_j^2=0,$$

it suffices to check the following assertion.

Assertion. *The polynomial*

$$P(t):=\int_B\Big(\sum_{j=1}^{N}t_jx_j\Big)^m dx$$

depends only on $\sum_{j=1}^N t_j^2$, *for* $t\in\mathbb{C}^N$.

To prove the assertion first note that $P\in H_m$ and without loss of generality we can assume that $m>0$.

Take $t\in\mathbb{R}^N$ and let U be an orthogonal transformation on $\mathbb{R}^N$. Then

$$P(Ut)=\int_B(Ut,x)^m dx=\int_B(t,U^*x)^m dx=\int_B(t,U^*x)^m d(U^*x)=P(t)$$

where $(,)$ is the usual euclidean inner product, and U^* is the adjoint of U with respect to $(,)$.

Then $P(t)=C=const$ on the n-dimensional sphere S^{n-1}. Applying Nullstellensatz again we obtain

$$P(t)-C=\Big(\sum_{j=1}^{N}t_j^2-1\Big)Q(t),$$

where Q is some polynomial. But $P(0)=0$, so $C=Q(0)$ and hence

$$P(t)=\Big(\sum_{j=1}^{N}t_j^2\Big)^r R(t),$$

where R is relatively prime to $\left(\sum_{j=1}^N t_j^2\right)$.

But R is constant on S^{N-1}, so repeating the above argument we conclude that $R(t)$ must be a constant. This proves our Assertion and hence the theorem.

Let as above

$$\Omega=\{x\in\mathbb{R}^N:\sum_{j=1}^{N}\frac{x_j^2}{a_j^2}\le 1, a_1>a_2>\ldots>a_N>0\}$$

and

$$u_\Omega(x):=C_N\int_\Omega\frac{dy}{\mid x-y\mid^{N-2}},\quad x\in\mathbb{R}^N\setminus\Omega$$

be the exterior potential of Ω.

Corollary 13.2 *For* $x\in\mathbb{R}^N\setminus\overline{\Omega}$

$$u_\Omega(x)=C_N\int_E\frac{d\mu(y)}{\mid x-y\mid^{N-2}},$$

where

$$d\mu(y)=2\Big(\prod_{j=1}^{N}a_j\Big)\Big(\prod_{j=1}^{N-1}(a_j^2-a_N^2)\Big)^{-1/2}\Big(1-\sum_{j=1}^{N-1}\frac{y_j^2}{a_j^2-a_N^2}\Big)^{1/2}dy'|_E$$

(dy' is Lebesgue measure on $\{y_N=0\}$).

Proof: We have by MacLaurin's theorem

$$u_\Omega(x)=\lim_{\lambda\to-a_N^2}\frac{\prod_{j=1}^N a_j}{\prod_{j=1}^N(a_j^2+\lambda)^{1/2}}\int_{\Omega_\lambda}v(y)dy,$$

where we set $v(y):=\frac{C_N}{|x-y|^{N-2}}$. Now

$$\frac{\prod_{j=1}^N a_j}{\prod_{j=1}^N(a_j^2+\lambda)^{1/2}}\int_{\Omega_\lambda}v(y)dy=$$

$$\frac{\prod_{j=1}^N a_j}{\prod_{j=1}^N (a_j^2+\lambda)^{1/2}}\Big(\int_{\Omega_\lambda}(v(y)-v(y',0))dy+\int_{\Omega_\lambda}v(y',0)dy\Big).$$

We can apply Fubini's theorem to each of these integrals, and letting

$$A_\lambda(y') = \Big(1-\sum_{j=1}^{N-1}\frac{y_j^2}{a_j^2+\lambda}\Big)^{1/2}(a_N^2+\lambda)^{1/2},$$

we obtain

$$u_\Omega(x) = \lim_{\lambda\to -a_N^2}\frac{\prod_{j=1}^N a_j}{\prod_{j=1}^N (a_j^2+\lambda)^{1/2}}\Big(\int_{\Omega_\lambda\cap\{y_N=0\}}2v(y',0)A_\lambda(y')dy' +$$

$$+\int_{\Omega_\lambda\cap\{y_N=0\}}\int_{-A_\lambda(y')}^{A_\lambda(y')}(v(y)-v(y',0))\ dy_N\ dy'\Big). \qquad (13.3)$$

Now, observing that

$$|\, v(y)-v(y',0)\,| \le \max_\Omega \|\, \nabla v \,\|\, (a_N^2+\lambda)^{1/2},$$

we obtain from (13.3) that

$$u_\Omega(x) = \lim_{\lambda\to -a_N^2}\frac{\prod_{j=1}^N a_j}{\prod_{j=1}^N (a_j^2+\lambda)^{1/2}}\int_{\Omega_\lambda\cap\{y_N=0\}}2v(y',0)A_\lambda(y')dy'$$

or, applying Lebesgue's dominated convergence theorem,

$$u_\Omega(x) = 2\frac{\prod_{j=1}^N a_j}{\prod_{j=1}^{N-1}(a_j^2-a_N^2)^{1/2}}\int_E\Big(1-\sum_{j=1}^{N-1}\frac{y_j^2}{a_j^2+\lambda}\Big)^{1/2}v(y',0)dy'$$

and the corollary follows.

Remark 13.2 For $N=3$, with a little bit more care, the argument implies the statements made in the introduction concerning oblate and prolate spheroids in examples 1.1 and 1.2.

Since the density of the distribution $d\mu$ is real analytic in the interior of E we note the following corollary:

Corollary 13.3 *The potential $u_\Omega(x)$ extends as a (multivalued) harmonic function into $\mathbb{R}^N\setminus\partial E$.*

Since inside Ω the Schwarz potential of the boundary Γ of Ω satisfies

$$u_\Gamma(x) = N(u_\Omega(x)-\widehat{\mu}(x))+\frac{1}{2}\,|\,x\,|^2$$

where $\widehat{\mu}(x) := C_N\int_E \frac{d\mu(y)}{|\,x-y\,|^{N-2}}$, we have

Corollary 13.4 *The Schwarz potential u_Γ of $\partial\Omega$ extends harmonically inside $\Omega\setminus\partial E$.*

The following quadrature identity is worth mentioning.

Example 13.1 Let $\Omega=\{x\in\mathbb{R}^3:\ x_2^2+x_3^2\le 1\}$. Then for all harmonic functions u integrable in Ω the following quadrature formula holds

$$\int_\Omega u(x)dx = \pi\int_{-\infty}^{+\infty}u(x_1,0,0)dx_1. \qquad (13.4)$$

Indeed, it is easy to see that the restriction of all such u on the x_1-axis are integrable so the right-hand side of (13.4) is well defined.

Now applying Corollary 13.2 adopted to the family of prolate ellipsoids

$$\Omega_a = \{\frac{x_1^2}{a^2}+x_2^2+x_3^2\le 1\},$$

we have

$$\begin{aligned}\int_\Omega u(x)dx &= \lim_{a\to\infty}\int_{\Omega_a}u(x)dx\\ &= \lim_{a\to\infty}\pi a(a^2-1)^{-3/2}\int_{-(a^2-1)^{1/2}}^{(a^2-1)^{1/2}}u(x_1,0,0)(a^2-1-x_1^2)dx_1.\end{aligned}$$

Since $a(a^2-1)^{-3/2}(a^2-1-x_1^2)$ tends to 1 as $a \to \infty$, Lebesgue's dominated convergence theorem yields (13.4).

Notation. Set $q_{a(\lambda)}(x) := \sum_{j=1}^N \frac{x_j^2}{a_j^2+\lambda} - 1$.

The proof of MacLaurin's theorem yields the following

Corollary 13.5 *For any $m \in \mathbb{N}$ and any, say, entire harmonic function u the following "mean value property" holds*

$$\frac{1}{|\Omega_\lambda|}\int_{\Omega_\lambda} q^m_{a(\lambda)}(x)u(x)dx = const. \tag{13.5}$$

for all $\lambda > -a_N^2$.

In particular, arguing as before, we obtain another corollary.

Corollary 13.6 *The exterior potential u_{Ω, q_a^m} with density $p = q_a^m$ extends harmonically to $\mathbb{R}^N \setminus E$, and moreover, as a multivalued function to $\mathbb{R}^N \setminus \partial E$.*

Proof of corollary 13.5: Set $b_j = (a_j^2+\lambda)^{1/2}$. As in the proof of Theorem 13.1, by changing variables $x_j = a_j y_j, x_j = b_j y_j$, $j = 1, \dots, N$, we reduce the statement to showing that

$$\int_B (|\, y\,|^2 - 1)^m \Big(\sum_{j=1}^N a_j \xi_j y_j\Big)^k dy = \int_B (|\, y\,|^2-1)^m \Big(\sum_{j=1}^N b_j\xi_j y_j\Big)^k dy$$

for all $\xi \in \Gamma_0$ (the isotropic cone) and all k (B is the unit ball in $\mathbb{R}^N$).

Or, equivalently, recalling that

$$\sum_{j=1}^N \Big((a_j\xi_j)^2 - (b_j\xi_j)^2\Big) = 0,$$

we have to show that

$$P(t) := \int_B (|\, y\,|^2-1)^m \Big(\sum_{j=1}^N t_j y_j\Big)^k dy$$

depends on $\sum_{j=1}^N t_j^2$ only. But since $(|\, y\,|^2 - 1)$ is invariant under rotations, we can extend word for word the argument used in proving the last assertion in the proof of Theorem 13.1.

Remark 13.3 In fact we can replace the density $q^m_{a(\lambda)}(x)$ by $f \circ q_{a(\lambda)}(x)$ where f is a more or less arbitrary analytic function of one variable. In particular, m in (13.5) need not be an integer.

Notes

Theorem 13.1 goes back to MacLaurin who considered prolate spheroids. The general case was treated later by Laplace (cf. [Ma] and the discussion therein). The proof presented here is from [KS1]. A proof of the Asgeirsson theorem and the derivation of Theorem 13.1 from it can be found in [CH, vol. II]. Fischer's product and its applications to partial differential operators with constant coefficients appeared in his seminal paper [Fi]. The reader is also referred to [NS1, NS2] for more applications and other developments. Corollaries 13.2, 13.3, 13.4 and Example 13.1 are taken from [KS1].

Chapter 14

Potential Theory on Ellipsoids (continued)

We keep the same notation as in Chapter 13. Our first theorem and the following corollary extend Corollary 13.6 to the case of an arbitrary polynomial density.

Theorem 14.1 *Let p be a polynomial. Then the potential*

$$u_{p,\Omega}(x) = c_N \int_\Omega \frac{p(y)}{|x-y|^{N-2}} dy \tag{14.1}$$

extends harmonically across $\Gamma := \partial\Omega$ into $\Omega \setminus E$.

PROOF: Observe that in view of Corollary 13.6 the theorem follows inmediately from the following two lemmas.

Lemma 14.1 *Let Q denote the linear space generated by*

$$\left\{ \sum_{j=0}^{m'} \sum_{|\alpha|\le m-2j} A_\alpha \partial^\alpha q_a^{m-j} \right\}_{m=0}^{\infty},$$

where

$$m' = \begin{cases} \frac{m}{2}, & m \text{ even} \\ \frac{m+1}{2}, & m \text{ odd} \end{cases}$$

$(q_a = \sum_1^N \frac{x_j^2}{a_j^2} - 1)$. Then

$$Q = \mathcal{P} := \{\text{all polynomials in } N \text{variables }\}.$$

Lemma 14.2 *For any $m \in \mathbb{N}$, $\alpha : |\alpha| \le m$, $y \in \mathbb{R}^N \setminus \overline{\Omega}$*

$$\int_\Omega \partial_y^\alpha q_a^m(y)|x-y|^{2-N} dy = \partial_x^\alpha \int_\Omega q_a^m(y)|x-y|^{2-N} dy.$$

PROOF OF LEMMA 14.1: The proof is by induction. Obviously $P_0 \subset Q$. Assume that $P_k \subset Q$. We want to show: $x_j x^\alpha$, $|\alpha| = k$, $j = 1, \ldots, N$ are all in Q. Without loss of generality we may assume $j = 1$. By the induction hypothesis we have

$$\begin{aligned} x_1 x^\alpha &= x_1 \sum_{j=0}^{k'} \left(\sum_{|\alpha|\le k-2j} A_j \partial^\alpha q^{k-j}(x) \right) \\ &= \sum_{j=0}^{k'} \left(\sum_{|\alpha|\le k-2j} A_j x_1 \partial^\alpha q^{k-j}(x) \right). \end{aligned}$$

So we only have to check that $x_1 \partial^\alpha q^{k-j} \in Q$. First note that for $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 = 0$ there is nothing to prove. Hence, assume $\alpha_1 \ge 1$. Set $\beta = (\alpha_1 + 1, \alpha_2, \ldots, \alpha_N)$ and $\sigma = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_N)$.

Now

$$x_1 \partial^\alpha q^{k-j} = \frac{a_1^2}{2(k+1-j)} \partial^\beta q^{k+1-j} - \alpha_1 \partial^\sigma q^{k-j}.$$

Indeed, if we write $\alpha = (\alpha_1, 0, \ldots, 0) + \alpha'$, we have

$$\begin{aligned} \frac{a_1^2}{2(k+1-j)} \partial^\alpha \partial_1 q^{k+1-j} &= \partial^{\alpha'} \{ \partial^{\alpha_1 - 1}(q^{k-j} + x_1 \partial_1 q^{k-j}) \} \\ &= \partial^{\alpha'} \{ \partial^{\alpha_1 - 2}(2 q^{k-j} + x_1 \partial_1^2 q^{k-j}) \} \\ &= \cdots \\ &= \alpha_1 \partial^\sigma q^{k-j} + x_1 \partial^\alpha q^{k-j}. \end{aligned}$$

PROOF OF LEMMA 14.2: For $m = 0$ or $\alpha = 0$, there is nothing to prove. So, let $m > 0$. Fix $\alpha : |\alpha| \le m$ with, say, $\alpha_1 > 0$. Let $\sigma =$

$(\alpha_1 - 1, \alpha_2, \ldots, \alpha_N)$. Note that $\partial^\sigma q^m = 0$ on $\Gamma := \partial\Omega$. Therefore, integration by parts yields

$$\int_\Omega \partial_y^\alpha q^m(y)\, |x-y|^{2-N} dy = \int_\Omega \partial_1 (\partial^\sigma q^m(y))\, |x-y|^{2-N} dy$$

$$= \int_\Omega \{\partial_1(\partial^\sigma q^m(y)\, |x-y|^{2-N})\} dy - \int_\Omega (\partial^\sigma q^m(y)) \partial_1 \{|x-y|^{2-N}\} dy$$

$$= \int_\Omega (\partial^\sigma q^m) \partial_{x_1} \{|x-y|^{2-N}\} dy = \partial_{x_1} \int_\Omega (\partial^\sigma q^m) |x-y|^{2-N} dy.$$

Iterating this argument we move the differential operator ∂^α outside the integral. This completes the proof of Lemma 14.2 and, hence, of Theorem 14.1.

From Theorem 14.1 and Corollaries 13.5 and 13.6 it follows

Corollary 14.1 *The exterior potential (14.1) with a polynomial density can be represented as a finite linear combination of potentials of masses supported on the focal ellipsoid*

$$E := \left\{ x \in \mathbb{R}^N : x_N = 0, \sum_1^{N-1} \frac{x_j^2}{a_j^2 - a_N^2} - 1 \le 0 \right\}$$

with real-analytic densities and their derivatives. In particular, all such potentials $u_{p,\Omega}$ extend as (multi-valued) harmonic functions to $\mathbb{R}^N \setminus \partial E$.

The Standard Single Layer Potential

In general, if Γ is a compact, smooth, real hypersurface a single layer potential with density p is defined by

$$u(x) := c_N \int_\Gamma \frac{p(y)}{|x-y|^{N-2}} dS_y,$$

where dS_y is Lebesgue measure on Γ.

Clearly, u is harmonic in $\mathbb{R}^N \setminus \Gamma$ and continuous in $\mathbb{R}^N$. Consider the homothetic ellipsoids Γ_ε defined by

$$\Gamma_\varepsilon := \left\{ x \in \mathbb{R}^N : \sum_1^N \frac{x_j^2}{a_j^2} = 1 + \varepsilon \right\}.$$

Let Ω_ε be the "cavity" between Γ and Γ_ε. Take the potential $u_\varepsilon(x)$ of the uniform distribution on Ω_ε with density $1/\varepsilon$, then letting $\varepsilon \to 0$ we obtain

$$u_\varepsilon(x) \to u(x) = c_N \int_\Gamma \frac{\omega}{|x-y|^{N-2}} dS_y,$$

where $u(x)$ is a single layer potential on Γ with density $\omega = \frac{1}{|\nabla q_a|}$. We shall call $u(x)$ the *standard single layer potential.*

Theorem 14.2 *Let p be a polynomial of degree m. Then, the potential*

$$u(x) = u_p(x) := c_N \int_\Gamma \frac{p\,\omega}{|x-y|^{N-2}} dA(y)$$

equals a harmonic polynomial of degree $\le m$ inside Ω.

Corollary 14.2 *Let $t > 1$, $\widetilde{\Omega} := t\,\Omega \setminus \Omega$ be an ellipsoidal cavity and let p be as above. Then, the volume potential with density p*

$$u_{p,\widetilde{\Omega}}(x) := c_N \int_{\widetilde{\Omega}} \frac{p}{|x-y|^{N-2}} dy$$

equals a harmonic polynomial of degree $\le m$ inside Ω. In particular, for $p = const$, $u_{p,\widetilde{\Omega}}(x) = const$ in Ω, i.e., $\nabla u_{p,\Omega} = 0$ and we obtain the theorem of Newton (cf. [Ke]): there is no attraction inside the ellipsoidal cavity.

Proof of Corollary 14.2: In view of Theorem 14.2 and Fubini's theorem we have for $x \in \Omega$

$$u_{p,\widetilde{\Omega}}(x) = c_N \int_1^t d\tau \left\{ \int_{\Gamma_\tau} \frac{p\,\omega}{|x-y|^{N-2}} dS_\tau(y) \right\} = \int_1^t \left(\sum_{|\alpha| \le m} a_\alpha(\tau)\, x^\alpha \right) d\tau$$

which is a polynomial of degree $\leq m$.

PROOF OF THEOREM 14.2: Fix $x^0 \in \Omega$ and a line l through x^0. Let $l \cap \Gamma = \{x_1, x_2\}$. Take an infinitesimal circular cone of solid angle $d\theta$ with vertex x^0 and axis l. It cuts Γ at two infinitesimal pieces S_1, S_2 and we are going to calculate the attraction at x^0 by S_1, S_2. Each piece S_i attracts x^0 with the force

$$f_i = \frac{m_i}{r_i^2},\ i = 1,2, \tag{14.2}$$

where r_i are distances from S_i to x_0 (along l),

$$m_i = \frac{p(x_i)dA_i}{|\nabla q_a(x_i)|} \tag{14.3}$$

are the masses of S_i and finally, the areas dA_i of S_i equal

$$dA_i = \frac{r_i^2\, d\theta}{\cos\beta_i},\quad i = 1,2\,, \tag{14.4}$$

where β_i is the angle between the positive direction of l (choose it to be $\overline{x^0\, x_1}$) and the outer normal to Γ at x_i. Let t denote the parameter along line l. Without ambiguity we denote the restriction of q_a on l by $q_a(t)$. Then (t_i denotes the value of the parameter corresponding to the point x_i, i=1,2), we have

$$\frac{dq_a}{dt}|_{t_i} = |\nabla q_a(x_i)| \cos\beta_i. \tag{14.5}$$

Combining (14.2-5) and dividing by $d\theta$ we see that the attraction at x^0 by the infinitesimal elements S_i, i=1,2 along line l equals

$$\sum_{i=1}^{2} \frac{p(t_i)}{q_a'(t_i)}. \tag{14.6}$$

Now consider the imbedding of l into a complex line L, i.e. simply allow the parameter t to take on complex values. Then (14.6) is nothing

else but

$$\sum \mathrm{res}\, \frac{p(t)}{q_a(t)}, \tag{14.7}$$

the sum of the residues of the rational function p/q_a on L. Expression (14.6) depends on point x^0 and a complex line L as parameters. Since calculating (14.7) reduces to a contour integral, we can interchange the differentiation of (14.6) with respect to x^0 and the operation of summing up the residues. Hence for any multi-index α we have

$$\partial_{x^0}^{\alpha}\left(\sum \mathrm{res}\, \frac{p}{q_a}|_L\right) = \sum \left(\mathrm{res}\, \partial_{x^0}^{\alpha}\, \frac{p}{q_a}|_L\right).$$

For $\alpha : |\alpha| \geq m :=$ degree of p, we observe that rational functions $\partial_{x^0}^{\alpha}(p/q)|_L$ are holomorphic at infinity and have a zero of order at least two there, i.e.

$$\partial_{x^0}^{\alpha}\left(\frac{p}{q_a}|_L\right) \sim O\left(\frac{1}{t^2}\right),\quad t \to \infty \text{ in } L.$$

Now moving the contour of integration when calculating (14.7) to ∞ we obtain that for all $\alpha : |\alpha| \geq m$

$$\partial_{x^0}^{\alpha}\left[\sum \mathrm{res}\, \left(\frac{p}{q_a}|_L\right)\right] = 0. \tag{14.8}$$

Therefore, all derivatives of order $\geq m$ of the attraction exhibited on x^0 by infinitesimal elements of $l \cap \Gamma$ vanish. Summing (14.8) over all lines l (or, strictly speaking, integrating (14.8) over $\mathbb{RP}^{N-1}$) we conclude that all derivatives of order $\alpha : |\alpha| \geq m$ of the attractive force $\nabla u_p(x^0)$ vanish. Hence, $u_p(x)$ is a polynomial of degree $\leq m$. The proof is now complete.

Remark 14.1 In order to extend the above argument to a more general setting let us introduce the following definition.

Definition 14.1 *Let $\Gamma := \{x \in \mathbb{R}^N : \varphi(x) = 0, \varphi \in \mathcal{P}_k, \text{irreducible}\}$ be a smooth real algebraic surface of degree k. A domain Ω is called a domain of hyperbolicity for Γ if for any $x^0 \in \Omega$ any line l passing through x^0 intersects Γ at precisely k points.*

For example, the interior of an ellipsoid is a domain of hyperbolicity. More generally, if a hypersurface of degree $2k$ consists of an increasing family of k ovaloids then the smallest one is the domain of hyperbolicity.

The standard single layer density on Γ is defined exactly as before with one distinction, the sign $+$ or $-$ is assigned on each connected component of Γ depending whether the number of obstructions for "viewing" this component from the domain of hyperbolicity of Γ is even or odd.

Then the above proof extends word for word to yield the following theorem:

Theorem 14.3 *Let $\Gamma := \{x \in \mathbb{R}^N : \varphi(x) = 0, \varphi \in P_k\}$ be an algebraic surface of degree k and Ω its domain of hyperbolicity. Then for any polynomial $p \in P_m$ the potential of the polynomial layer $p\omega$*

$$u_p(x) := c_N \int_\Gamma \frac{p\omega}{|x-y|^{N-2}} dS(y),$$

where w is the standard single layer density equals a harmonic polynomial of degree $\leq$m-k+2 inside Ω. So, in particular, for m$\leq$k-2 there is no attraction inside the domain of hyperbolicity.

Corollary 14.3 *If Ω is an ellipsoid and $p \in P_m$ then inside Ω the volume potential*

$$u_{p,\Omega} := c_N \int_\Omega \frac{p(y)}{|x-y|^{N-2}} dy$$

is a polynomial of degree $\leq m+2$.

Proof of Corollary 14.3: For $t > 1$, let $\Omega_t := \{x \in \mathbb{R}^N : \sum_1^N \frac{x_j^2}{a_j^2} \leq t^2\}$ denote the ellipsoid homothetic with Ω. Without loss of generality we can assume p to be homogeneus, i.e. $p \in H_m$. Fix $x \in \Omega$. Using homogeneity and a simple change of variables we calculate

$$u_{p,\Omega_t}(x) := c_N \int_{\Omega_t} \frac{p(y)}{|x-y|^{N-2}} dy = t^{m+2} u_{p,\Omega}\left(\frac{x}{t}\right). \tag{14.9}$$

By Corollary 14.2 the potential of the cavity $\Omega_t \setminus \Omega$ inside Ω,

$$u_{p,\Omega_t}(x) - u_{p,\Omega}(x) =: h_m(x) \in P_m, \tag{14.10}$$

is a harmonic polynomial. For any $\alpha : |\alpha| = m+2$ using (14.9-10) we obtain

$$\partial^\alpha u_{p,\Omega_t}(x) = \partial^\alpha \left[t^{m+2} u_{p,\Omega}\left(\frac{x}{t}\right)\right] = \partial^\alpha u_{p,\Omega}\left(\frac{x}{t}\right) = \partial^\alpha u_{p,\Omega}(x),$$

i.e. all such $\partial^\alpha u_{p,\Omega}$ are homogeneus and of degree 0 in Ω. Also, for any $\alpha : |\alpha| > m$ we have, since $\deg p = m$,

$$\Delta\left(\partial^\alpha u_{p,\Omega}(x)\right) = \partial^\alpha\left(\Delta u_{p,\Omega}(x)\right) = \partial^\alpha(-p) = 0.$$

Thus for all $\alpha : |\alpha| = m+2$, $\partial^\alpha u_{p,\Omega}(x)$ are harmonic and homogeneus of degree 0. Hence, all $\partial^\alpha u_{p,\Omega}(x)$ are constants for $\alpha : |\alpha| = m+2$ and the Corollary follows.

Remark 14.2 (i) In particular, the potential of the uniform density is a quadratic polynomial inside the ellipsoid. This is an old theorem of Newton (cf. [Ke],[Ma]).

(ii) It turns out that even a partial converse to Corollary 14.3 is also true. Namely, if a potential of the uniform density in a solid K equals a quadratic polynomial inside K, K must be an ellipsoid. In dimensions 2 and 3 this was proved respectively by Hölder [Höl] and, independently, by Dive and Nikliborc [Di],[Ni]. In full generality the result has been obtained by DiBenedetto and Friedman [DF].

By translation and rotation one easily sees that the potential $u_\Omega(x)$ can be written as $u_\Omega(x) := B - \sum_1^N A_j x_j^2$ inside ellipsoid Ω, where

$B > 0$, $A_j > 0$, $\sum_1^N A_j = \frac{1}{2}$. The main difficulty in proving the DiBenedetto-Friedman theorem is to show that in fact every such polynomial represents an interior potential of some ellipsoid. Recently a simple proof of the latter assertion was obtained by L.Karp [Ka] by making use of L.Brouwer's domain invariance theorem. Assuming this, the following beautiful argument of Dive finishes the proof. Namely, find an ellipsoid Ω whose interior potential is the same quadratic polynomial. By Corollary 14.2 all ellipsoids homothetic with Ω, have interior potentials equal $B' - \sum_1^N A_j x_j^2$ with a different constant term, but the same coefficients A_j. Let $t^0 := \inf\{t : \Omega_t \supset K\}$. Then there exists $x^0 \in \partial\Omega_{t^0}$ such that $x^0 \ni \partial K$ and Ω_{t^0} and K have a common supporting plane H at x^0. The potential of the shell between Ω_{t^0} and K (i.e., $\Omega_{t^0} \setminus K$) is constant inside K. Thus the attractive force vanishes identically inside K. In particular, the attraction in the direction of the normal to H at x^0 vanishes. This leads to an obvious contradiction unless Volume$(\Omega_{t^0} \setminus K) = 0$, i.e. $\Omega_{t^0} = K$.

(iii) It is easy to compute an equilibrium potential of an ellipsoid Ω. Recall (cf. [Ke]) that $V(x)$ is called an equilibrium potential of a total charge σ on $\Gamma := \partial\Omega$ if

$$V(x) := c_N \int_\Gamma \frac{\rho(y)}{|x-y|^{N-2}} dA(y)$$

where $\rho(y)$ is the density function, so that $V(x) \equiv 1$ inside Ω and

$$\lim_{|x|\to\infty} |x|^{N-2} V(x) = \sigma.$$

Then we have (as above, the volume potential $u_\Omega(x) = B - \sum_1^N A_j x_j^2$ inside Ω).

$B - \sum A_j y_j$ x_j^2

Corollary 14.4

$$V(x) = \frac{1}{2B}\left(\widehat{\mu}(x) - \sum_1^N x_j \frac{\partial\widehat{\mu}}{\partial x_j}(x)\right), \tag{14.11}$$

where (cf. Corollary 13.2) E is the focal ellipsoid,

$$\widehat{\mu}(x) := c_N \int_E \frac{d\mu(y')}{|x-y|^{N-2}}$$

and

$$d\mu(y') = 2\left(\frac{\prod_{j=1}^N a_j}{\prod_1^{N-1}\left(a_j^2 - a_N^2\right)^{1/2}}\right)\left(1 - \sum_1^{N-1} \frac{y_j^2}{a_j^2 - a_N^2}\right)^{1/2} dy'|_E.$$

In particular, the equilibrium potential extends harmonically to $\mathbb{R}^N \setminus \partial E$.

Proof: The right side of (14.11) is harmonic in $\mathbb{R}^N \setminus \overline{\Omega}$, vanishes at ∞ and since the exterior potential $u_\Omega(x)$ equals (Corollaries 14.3 and 13.2) $B - \sum_1^N A_j x_j^2$ inside Ω, and $\widehat{\mu}(x)$ outside Ω (and is C^1-smooth in $\mathbb{R}^N$), we obtain that it equals 1 on $\Gamma := \partial\Omega$. Hence, it must coincide with the equilibrium potential $V(x)$ outside Ω.

Remark in passing that the classical theorem of Ivory (cf.,e.g., [Ma]) states that moreover, the equipotential surfaces of V are precisely ellipsoids confocal with $\Gamma := \partial\Omega$.

Finally, we are ready to prove the Schwarz Potential Conjecture with polynomial data on ellipsoidal surfaces.

Corollary 14.5 *All solutions u of Cauchy problems*

$$\begin{cases} \Delta u = 0, \\ u \equiv p \ \ on\, \Gamma, \end{cases}$$

where data p is a polynomial extend harmonically to $\mathbb{R}^N \setminus \partial E$.

Proof: Consider the potential

$$v(x) := c_N \int_\Omega \frac{(-\Delta p)(y)}{|x-y|^{N-2}}.$$

By Corollary 14.3 $v(x)|_\Omega = p + h$, where h is a harmonic polynomial and deg$(h) \leq$ deg(p). Also, Theorem 14.1 implies that $v(x)$ extends harmonically to $\mathbb{R}^N \setminus \partial E$. Call this extension $v_1(x)$. But on $\Gamma := \partial\Omega$ we have:

$$u \equiv p \equiv v - h \equiv v_1 - h.$$

Hence $u := v_1 - h$ provides the desired extension.

Notes

Lemmas 14.1 and 14.2 are from Shahgholian's paper [Sha], as well as Theorem 14.1. However, from there he follows a different route to Corollaries 14.3 and 14.5 elaborating on a beautiful idea of Dirichlet. Theorem 14.2 is due to Ferrers [Fe] for dimension $N = 3$. The elegant proof here and, what is more important, the far reaching extension (Theorem 14.3) are due to A.Givental [Gi]. Yet the crucial step of introducing the notion of a domain of hyperbolicity (Definition 14.1) is due to Arnold [Ar] as well as the partial case (although, by a different argument) of Theorem 14.3 when $m \leq k - 2$ which is a direct extension of Newton's theorem (cf. Corollary 14.2). For further interesting developments we refer the reader to a paper by B. Shapiro and A.Vainstein [SV]. Corollary 14.3 is also due to Ferrers [Fe] in case $N = 3$, although his proof based on heavy computations with ellipsoidal harmonics is hardly extendible to higher dimensions. Corollary 14.4 has been observed in [KS1]. Far reaching generalizations of Corollary 14.5 with arbitrary entire data and for more general differential operators with lower order terms have been obtained by G.Johnsson [Jo] and, independently and by different methods, by V.Shatalov and B.Sternin [SS].

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