Approximating $\overline{z}$ in Hardy and Bergman Norms

Zdeňka Guadarrama and Dmitry Khavinson

Abstract. We consider the problem of finding the best analytic approximation in Smirnov and Bergman norm to general monomials of the type $z^n\overline{z}^m$. We show that in the case of approximation to $\overline{z}$ in the annulus (and the disk) the best approximation is the same for all values of p. Moreover, the best approximations to $\overline{z}$ in Smirnov and Bergman spaces characterize disks and annuli.

1. Introduction

Throughout this paper, G denotes a bounded domain in $\mathbb{C}$ with boundary Γ consisting of n simple closed analytic curves. $R(\overline{G})$ will stand for the uniform closure of the algebra of rational functions in G with poles outside of $\overline{G}$.

Let ds be the arclength measure on the boundary of G. Recall that a function f belongs to the Smirnov class $\mathbb{E}_p(G)$ for $1 \leq p < \infty$, if it is analytic in G and there exists a sequence of finitely connected domains $\{G_n\}_{n=1}^{\infty}$, $G_1 \subset G_2 \subset G_3 \subset ...$ with rectifiable boundaries Γ_n so that $\bigcup\limits_{n=1}^{\infty} G_n = G$, and a constant $M > 0$ such that $\|f\|_{\mathbb{E}_p} := \sup\limits_n \left[\int_{\Gamma_n} |f(z)|^p ds\right]^{\frac{1}{p}} \leq M < \infty$. For a nice and concise introduction to Smirnov spaces see [6], also cf. [14].

We let $d\sigma$ denote area measure in G. The Bergman space $\mathbb{A}_p(G)$ for $1 \leq p < \infty$ is the set of analytic functions $f(z)$ in G, with finite norm $\|f\|_{\mathbb{A}_p} = \|f\|_{\mathbb{L}_p(d\sigma,G)} = \left[\int_G |f(z)|^p d\sigma\right]^{\frac{1}{p}}$ (cf. [7]).

D. Khavinson, in [10], [12], [13], [15], [16] posed the question of "how far" $\overline{z}$ is from being approximable by rational functions that are analytic in G. In particular, the following concept was introduced in [15], also cf. [4].

1991 *Mathematics Subject Classification.* 30E10, 41A99.
Key words and phrases. Analytic approximation, Hardy spaces, Bergman spaces.
This work was supported in part by NSF Grant DMS-0139008.

Definition 1.1. *The analytic content $\lambda(G)$ of a given domain G is defined as*

$$\lambda(G) := \inf_{g\in R(\overline{G})} \|\overline{z} - g(z)\|_{\mathbb{L}_\infty(ds)} = \inf_{g\in R(\overline{G})} \|\overline{z} - g(z)\|_{\mathbb{L}_\infty(d\sigma)}.$$

It turns out that $\lambda(G)$ can be bounded above and below by basic quantities depending on the geometry of the domain G, specifically, its area and perimeter. If we let $A(G)$ denote the area of G and $P(G)$ the perimeter of its boundary, the following inequality holds:

$$\frac{2A(G)}{P(G)} \leq \lambda(G) \leq \sqrt{\frac{A(G)}{\pi}} \tag{1}$$

The upper bound is due to Alexander [3], and the lower bound is due to D. Khavinson [13]. We will refer to this inequality from now on as the A-K inequality.

It follows immediately from (1) that $A(G) \leq \frac{P^2(G)}{4\pi}$, which is the isoperimetric inequality. Moreover, when we notice that both inequalities in (1) are sharp, since they become equalities when the domain is a disk, we obtain the isoperimetric theorem (cf. [10]).

The question of what are the extremal domains for the lower bound of (1) still remains open. A few equivalent formulations for the equation $\lambda(G) = \frac{2A(G)}{P(G)}$ in terms of geometry and potential theory can be found in [12] and [15]. The reader may consult the survey [4] which focuses on extremal domains for the left inequality in (1). The following conjecture [4], [14] remains open.

Conjecture 1.2. *For a fixed $\lambda(G)$, the only extremal domains for which the lower bound in (1) becomes an equality are the disks of radius $\lambda(G)$, and annuli $\{z : r < |z| < R\}$ with $\lambda(G) = R - r$.*

For an extensive discussion about different forms and various ramifications of this conjecture we refer the reader to [4].

If we denote by $\mathbb{E}_1^1(G)$ the unit ball in $\mathbb{E}_1(G)$. We can write

$$\lambda(G) := \inf_{g\in R(\overline{G})} \|\overline{z} - g(z)\|_{\mathbb{L}_\infty(ds,\Gamma)} = \sup_{f\in\mathbb{E}_1^1(G)} \left|\int_\Gamma \overline{z} f(z)dz\right|,$$

and there exist extremal functions $g^*(z)$ and $f^*(z)$ for which the infimum and the supremum above are attained [19]. If the domain is a disk centered at the origin, the best rational approximation to $\overline{z}$ in G is the zero function. In the case of the annulus centered at the origin, the best approximation is $g^*(z) = \frac{Rr}{z}$ [10].

The main focus of this paper is to extend the concept of analytic content to the context of Smirnov and Bergman spaces for $p \geq 1$.

The paper is organized as follows. In the next section we define the Smirnov $p-$analytic content of a domain and show that the A-K inequality extends to the $\mathbb{E}_p$ case yielding bounds for the Smirnov $p-$analytic content in terms of the area

and perimeter of the domain. In section 3 we find the best approximation to any monomial $z^n\overline{z}^m$ in the Smirnov $p-$norm of the annulus and the disk. For disks and annuli, the best approximation to $\overline{z}$ turns out to be the same rational function for all p. We prove a converse for this result in the case of the disk, and for $p=1$ in the case of the annulus. In section 4 we consider the Bergman $p-$analytic content of a domain and explore similar questions, now for the case of the Bergman space $p-$norm. We conclude with some remarks and open questions.

2. Smirnov p-analytic content

Definition 2.1. *The Smirnov $p-$analytic content of a domain G is defined by* $\lambda_{\mathbb{E}_p}(G) := \inf\limits_{g\in\mathbb{E}_p(G)} \|\overline{z}-g(z)\|_{\mathbb{L}_p(ds,\Gamma)}$.

The following general result summarizes the study of extremal problems in Smirnov classes (cf. [19], Theorem 4.3).

Corollary 2.2. *Let $p\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$, and let $\omega(z)\in\mathbb{L}_p(G)$ then the following hold:*
(i) $\inf\limits_{g\in\mathbb{E}_p(G)} \|\omega(z)-g(z)\|_{\mathbb{L}_p(ds,\Gamma)} = \sup\limits_{f\in\mathbb{E}^1_q(G)} \left|\int_\Gamma \omega(z)f(z)dz\right|$.

(ii) There exist extremal functions $g^*(z)\in\mathbb{E}_p(G)$ and $f^*(z)\in\mathbb{E}_q(G)$ for which the infimum and the supremum are attained in (i).
(iii) $g^*(z)\in\mathbb{E}_p(G)$ and $f^*(z)\in\mathbb{E}_q(G)$ are extremal if and only if, almost everywhere on Γ,

$$f^*(z)(\omega(z)-g^*(z))dz = \frac{e^{i\delta}}{\Lambda_{\mathbb{E}_p}^{p-1}}|\omega(z)-g^*(z)|^p ds, \tag{2}$$

where δ is a real constant and $\Lambda_{\mathbb{E}_p}=\|\omega(z)-g^*(z)\|_{\mathbb{L}_p(ds,\Gamma)}$. We will refer to this last equality as the extremality condition in Smirnov spaces.

For $p>1$ the extremal functions $g^*(z)$ and $f^*(z)$ are unique, the latter up to a factor of $e^{i\alpha}$.

For $p=1$ the extremal function $f^*(z)$ is unique up to a factor of $e^{i\alpha}$. If the domain G is simply connected, then $g^*(z)$ is also unique. If the domain G is $n-$connected, $n>1$, then the extremal function $g^*(z)$ is unique provided that $f^*(z)$ has more than $n-2$ zeros in G or that on a certain set $T\subset\Gamma$, $meas(T)>0$, $|f^*(z)|<1$. Otherwise, the extremal function $g^*(z)$ may fail to be unique. (cf. Part 3, Theorem 3.2, in [19], *and Theorem 3.6 (ii) below).*

In our first Theorem we show that following the same strategy used in [10] we can find bounds for $\lambda_{\mathbb{E}_p}(G)$ in terms of the perimeter and the area of the domain G, obtaining the A-K inequality as a limiting case when p approaches infinity.

THEOREM 2.3. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let G be a multiply connected domain in $\mathbb{C}$ bounded by n simple closed analytic curves, as before, $A(G)$ denotes the area of G and $P(G)$ the perimeter. Then*

$$\frac{2A(G)}{\sqrt[q]{P(G)}} \leq \lambda_{\mathbb{E}_p}(G) \leq \sqrt{\frac{A\,(G)}{\pi}} P(G)^{\frac{1}{p}} \tag{3}$$

If $p = 1$, then $2A(G) \leq \lambda_1(G) \leq \sqrt{\frac{A(G)}{\pi}} P(G)$.

PROOF. We first address the lower bound in (3) for $p > 1$.

$$\lambda_{\mathbb{E}_p}(G) := \|\overline{z} - g^*(z)\|_{\mathbb{L}_p(ds,\Gamma)} = \left(\int_\Gamma |\overline{z} - g^*(z)|^p \frac{ds}{P(G)}\right)^{\frac{1}{p}} (P(G))^{\frac{1}{p}}.$$

By Jensen's inequality, since $p > 1$, we have

$$\begin{aligned}
\left(\int_\Gamma |\overline{z} - g^*(z)|^p \frac{ds}{P(G)}\right)^{\frac{1}{p}} (P(G))^{\frac{1}{p}} &\geq \left[\left(\int_\Gamma |\overline{z} - g^*(z)| \frac{ds}{P(G)}\right)^{\frac{1}{p}}\right]^p (P(G))^{\frac{1}{p}} \\
&= P(G)^{-\frac{1}{q}} \int_\Gamma |\overline{z} - g^*(z)| ds \\
&\geq P(G)^{-\frac{1}{q}} \left|\int_\Gamma (\overline{z} - g^*(z)) dz\right|.
\end{aligned}$$

Applying the divergence theorem in the form $\int_\Gamma (\overline{z} - g^*(z)) dz = 2i \int\int_G \frac{\partial}{\partial \overline{z}} (\overline{z} - g^*(z)) d\sigma$, we obtain

$$\lambda_{\mathbb{E}_p}(G) \geq \frac{2A\,(G)}{\sqrt[q]{P(G)}}.$$

For $p = 1$ we have $\lambda_{\mathbb{E}_1}(G) = \int_\Gamma |\overline{z} - g^*(z)| ds \geq \left|\int_\Gamma (\overline{z} - g^*(z)) dz\right| = 2A\,(G)$.

Now for the upper bound, and any $p \geq 1$, we will use Corollary 2.2 (i) and by duality rewrite $\lambda_{\mathbb{E}_p}(G)$ as:

$$\lambda_{\mathbb{E}_p}(G) = \left|\int_\Gamma \overline{z} f^*(z) dz\right| = \left|2i \iint_G \frac{\partial}{\partial \overline{z}} (\overline{z} f^*(z))\, d\sigma\right| = \left|2i \iint_G f^*(z) d\sigma\right|.$$

Since the boundary of the domain is analytic and $\overline{z}$ is real analytic on Γ, then by S. Ya. Khavinson's results on the regularity of extremal functions (see Theorem 5.13 in [19]) we know that $f^*(z)$ is analytic across Γ. Hence we can express $f^*(z)$ as the Cauchy integral of its boundary values, $f^*(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f^*(w)}{w-z} dw$. Substituting this in the last equality, using Fubini's theorem, and bringing absolute values inside the integral we obtain

$$\begin{aligned}
\lambda_{\mathbb{E}_p}(G) &= \left|\iint_G \left(\frac{1}{\pi} \int_\Gamma \frac{f^*(w)}{w-z} dw\right) d\sigma\right| \\
&= \left|\int_\Gamma f^*(w) \left(\frac{1}{\pi} \iint_G \frac{1}{w-z} d\sigma\right) dw\right| \\
&\leq \int_\Gamma |f^*(w)| \left|\frac{1}{\pi} \iint_G \frac{1}{w-z} d\sigma\right| dw.
\end{aligned}$$

Hölder's inequality yields

$$\begin{aligned}\lambda_{\mathbb{E}_p}(G) &\leq \|f^*\|_{\mathbb{L}_q(ds)} \left\| \frac{1}{\pi} \iint_G \frac{1}{w-z} d\sigma \right\|_{\mathbb{L}_p(ds,\Gamma)} \\ &= \left\| \frac{1}{\pi} \iint_G \frac{1}{w-z} d\sigma \right\|_{\mathbb{L}_p(ds,\Gamma)}.\end{aligned}$$

Let $F_G(z) = \frac{1}{\pi} \iint_G \frac{1}{w-z} d\sigma$, $z \in \mathbb{C}$. Now, the Ahlfor-Beurling estimate [2] (see [10] for a simple proof) implies that for a fixed $z \in \mathbb{C}$ and among domains with the same area, the function $|F_G(z)|$ attains its maximum value when the domain is a disk of radius ρ passing through z, which we denote by D_ρ. So, $|F_G(z)| \leq |F_{D\rho}(z)| \leq \sqrt{\frac{A(D_\rho)}{\pi}} = \sqrt{\frac{A(G)}{\pi}}$. Therefore,

$$\begin{aligned}\lambda_{\mathbb{E}_p}(G) &= \left\| \frac{1}{\pi} \iint_G \frac{1}{w-z} d\sigma \right\|_{\mathbb{L}_p(ds,\Gamma)} \\ &= \left(\int_\Gamma \left| \frac{1}{\pi} \iint_G \frac{1}{w-z} d\sigma \right|^p ds \right)^{\frac{1}{p}} \\ &\leq \left(\int_\Gamma \left(\frac{A(G)}{\pi} \right)^{\frac{p}{2}} ds \right)^{\frac{1}{p}} \\ &= \sqrt{\frac{A(G)}{\pi}} P(G)^{\frac{1}{p}}.\end{aligned}$$

The theorem is proved. □

3. Characterization of disks and annuli in terms of approximations to $\overline{z}$ in $\mathbb{E}_p$ norm

Proposition 3.1. *Let $p \geq 1$ and let $G = \{z \in \mathbb{C} : |z| < r\}$. Then:*
(i) The best approximation in $\mathbb{E}_p(G)$ to a general monomial of the type $\omega(z) = z^n \overline{z}^m$ for $m > n$ is the zero function. (For $m \leq n$, it is clear that $z^n \overline{z}^m = r^{2m} z^{n-m}$ is its own best approximation.)
(ii) $\Lambda_p(G) = \|z^n \overline{z}^m\|_{\mathbb{L}_p(ds,\Gamma)} = \sqrt[p]{2\pi r^{p(n+m)+1}}$.
(iii) The best approximation to $\overline{z}$ in $\mathbb{E}_p(G)$ is the zero function and the $p-$analytic content of a disk of radius r is $\lambda_{\mathbb{E}_p}(G) = \|\overline{z}\|_{\mathbb{L}_p(ds,\Gamma)} = \sqrt[p]{2\pi r^{p+1}}$.

The proof is trivial, we only sketch it for the reader's convenience.

Sketch of proof. Let $G = \{z \in \mathbb{C} : |z| < r\}$, $p > 1$ and let $f(z) = \frac{|z^n \overline{z}^m|^p}{z^n \overline{z}^m} \frac{ds}{dz}$ for $m > n$. The function $f(z)$ annihilates $\overline{\mathbb{E}_p(G)}$ since, for $k \geq 0$

$$\int_\Gamma \left(\frac{|z^n \overline{z}^m|^p}{z^n \overline{z}^m} \right) z^k ds = \int_0^{2\pi} r^{(p-1)(n+m)+k+1} e^{i(k+m-n)\theta} d\theta = 0.$$

Set $f^*(z) = \frac{f(z)}{\|f\|_{\mathbb{L}_q(ds,\Gamma)}}$, so that $\|f^*\|_{\mathbb{L}_q(ds,\Gamma)} = 1$. Then with $g^*(z) = 0$ the extremality condition is satisfied:

$$\begin{aligned} f^*(z)\ z^n\overline{z}^m dz &= \frac{\frac{|z^n\overline{z}^m|^p}{z^n\overline{z}^m}\frac{ds}{dz}}{\left\|\frac{|z^n\overline{z}^m|^p}{z^n\overline{z}^m}\right\|_q} z^n\overline{z}^m dz \\ &= \frac{\left|z^n\overline{z}^m\right|^p}{\left\|\left(z^n\overline{z}^m\right)^{\frac{p}{q}}\right\|_q} ds. \end{aligned}$$

For $p = 1$, let $f(z) = -iz^{m-n-1}\frac{ds}{dz}$ and $g^*(z) = 0$. Then $\int_\Gamma -iz^{m-n-1+k}dz = 0$, and $-iz^{m-n-1}(z^n\overline{z}^m)dz = r^{2m}d\theta$. Hence $f^*(z) = \frac{f(z)}{\|f(z)\|_{\mathbb{L}_1(ds,\Gamma)}}$ and $g^*(z)$ are both extremal.

Now, $\Lambda_p^p(G) = \|z^n\overline{z}^m\|^p_{\mathbb{L}_p(ds,\Gamma)} = \int_0^{2\pi}\left|r^{n+m}e^{i(n-m)\theta}\right|^p r d\theta = 2\pi r^{p(n+m)+1}$.

Taking $n = 0$ and $m = 1$ we obtain (iii). □

THEOREM 3.2. *Let G be a multiply connected bounded domain with the boundary consisting of n simple closed analytic curves. The zero function is the best approximation to $\overline{z}$ in $\mathbb{E}_p(G)$ if and only if G is a disk.*

PROOF. Necessity is obvious. For the converse, suppose that 0 is the best approximation to $\overline{z}$ in $\mathbb{E}_p(G)$. Then the extremality condition (2), for $p \geq 1$, can be written as

$$f^*(z)\overline{z}dz = const|z|^p ds$$

on each boundary component of the domain G. Without loss of generality we will assume the constant is positive. Dividing by z we can rewrite the equation above as

$$\frac{f^*(z)}{z}dz = const|z|^{p-2}ds. \tag{4}$$

Notice that $0 \in G$, otherwise $\int_\Gamma \frac{f^*(z)}{z}dz = 0$, yet $\int_\Gamma |z|^{p-2}ds \neq 0$ since this is a positive measure. For the same reason $f^*(0) \neq 0$, hence $\frac{f^*(z)}{z}$ has a pole at the origin.

Because the boundary of the domain is analytic, for each boundary component we can find a Schwarz function $S(z) = \overline{z}$, that is, a unique analytic function which at every point along the boundary component takes on the value $\overline{z}$ [14], [20]. Now, $(ds)^2 = dzd\overline{z} = S'(z)dz^2$, so $\frac{ds}{dz} = \sqrt{S'(z)}$ on Γ and we obtain that

$$\frac{f^*(z)}{z^{\frac{p}{2}}} = const\ S(z)^{\frac{p}{2}-1}\sqrt{S'(z)}.$$

Squaring both sides yields

$$\frac{[f^*(z)]^2}{z^p} = const\ \frac{d}{dz}\left[S(z)^{p-1}\right]. \tag{5}$$

This last equation implies that for each contour $S(z)^{p-1}$ is analytic throughout the domain, except at the origin.

We will now consider a few cases.

CASE 1. $p = 1$

When $p = 1$, $|f^*| \leq 1$ in G and $|f^*| = 1$ on Γ. Therefore, $f^*(z)$ is either a unimodular constant or the cover mapping of G onto the unit disk.

Suppose $f^*(z)$ is not constant. From Corollary 2.2 we have that $f^*(z) = e^{i\delta}\frac{|\overline{z}|ds}{\overline{z}dz}$ and $|f^*(z)| = 1$ almost everywhere on the boundary of G. By S. Ya. Khavinson's regularity results (Theorem 5.13 in [19]) $|f^*(z)| = 1$ everywhere on the boundary and $f^*(z)$ extends analytically across each boundary component. Therefore, $f^*(z)$ maps G onto the unit disk $\mathbb{D}$ taking each value in the disk k times, and wrapping each boundary component of Γ around the unit circle at least once, and always following the same positive direction. If that were not the case, and we suppose that at some point $w \in \Gamma$, $f^*(w)$ changes direction, at that point $\frac{df^*}{ds}(w) = 0$, so $\frac{df^*}{dz}(w) = \frac{df^*}{ds}\frac{ds}{dz} = 0$.

Hence, for z near w, $f^*(z) = f^*(w) + \mathcal{O}((z-w)^2)$. So $f^*(z)$ maps the "half" neighborhood of w that is in G onto a full neighborhood of $f^*(w)$, which means that $|f^*|$ can be greater than 1 near w, and that is a contradiction. Now, in order to wrap each boundary component of G around the circle, $f^*(z)$ has to go around the unit circle k times with $n \leq k$.

If we let $\Delta_{\arg} f^*(z)$ denote the change in the argument of $f^*(z)$ as z goes around the boundary of G, then $\Delta_{\arg} f^*(z) \geq n$. Moreover, the tangent vector to Γ traverses the boundary of G once in the clockwise direction, and $n-1$ times in the counterclockwise direction. Hence, remembering that Γ is analytic and $0 \in G$, by the argument principle we obtain that

$$\Delta_{\arg}\left(\frac{f^*(z)}{z}dz\right) = \Delta_{\arg} f^*(z) + \Delta_{\arg}\frac{1}{z} + \Delta_{\arg} dz \geq n - 1 + 2 - n = 1,$$

while $\Delta_{\arg}|z|^{p-2}ds = 0$ and from (4) we obtain a contradiction. Hence, for $p = 1$, $f^*(z)$ is a unimodular constant so from the equation preceding (4) we invoke that on Γ, $\overline{z}\frac{dz}{ds} = e^{ia}|z|$, where a is a real constant. Writing on each boundary component $z(s) = r(s)e^{ib(s)}$, substituting and separating real and imaginary parts yields $r' = \cos a$. Since each component is a closed curve, it cannot be a spiral, $\cos a$ must be zero, thus each component is a circle centered at the origin. Moreover, the case of the annulus is ruled out because $\frac{dz}{ds}$ changes directions between the two boundary circles, hence $|z| = const$ on Γ, and G is a disk centered at the origin.

CASE 2. $p > 1$, $p \notin \mathbb{N}$

If p is not an integer $S(z)^{p-1}$ may be multivalued. Yet, since the left hand side of (5) is $\mathcal{O}\left(\frac{1}{z^p}\right)$ near zero, it follows that $S(z)$ is $\mathcal{O}\left(\frac{1}{z}\right)$ in a neighborhood of the origin.

Also notice that if p is not an integer $S(z)$ cannot vanish anywhere in G. If it did it would be possible to obtain an unbounded singularity on the right hand side of (4) by differentiation, while the left hand side would remain bounded. Therefore the Schwarz function for every boundary component of G is analytic in the whole domain and has a simple pole at the origin. Moreover, since $\frac{[f^*(z)]^2}{z^p}$ remains the same when it is continued analytically throughout G, $S(z)$ has to be the same analytic function for each boundary component. So $S(z)^{p-1} = \int_\Gamma \frac{[f^*(z)]^2}{z^p}dz$ and from this we obtain that $S(z) = \frac{const}{z} + g(z)$, where $g(z)$ is analytic in G and is independent of which boundary component we consider. $S(z) = \overline{z}$ on the boundary. $S(z)z = |z|^2$ is real, positive on the boundary and analytic inside the domain G, hence it is constant. The boundary of the domain is therefore a circle centered at the origin.

CASE 3. $p > 1$, $p \in \mathbb{N}$

When p is even, i.e. $p = 2k$, (4) becomes

$$\frac{f^*(z)}{z^k} dz = const|z|^{k-1} ds, \tag{6}$$

which in turn yields

$$\frac{[f^*(z)]^2}{z^{2k}} = const \frac{d}{dz}\left[S(z)^{2k-1}\right]. \tag{7}$$

(6) implies that $S(z)^{2k-1}$ is analytic throughout G and has a pole of order $2k-1$ at the origin in G, so $S(z)$ has to have a simple pole at the origin. Following the same reasoning as in case 2 we can conclude that $S(z)^{2k-1}$ is the same for every boundary component.

Then $\overline{z}^{2k-1} = S(z)^{2k-1} = \frac{const}{z^{2k}-1} + g(z)$ for $g(z)$ analytic in G. Multiplying through by z^{2k-1} we have once again that $|z| = const$.

For p odd, i.e. $p = 2k+1$, (5) can be written as

$$\frac{[f^*(z)]^2}{z^{2k+1}} = const \frac{d}{dz}\left[S(z)^{2k}\right] \tag{8}$$

So $S(z)^{2k} = \frac{const}{z^{2k}} + g(z)$, with $g(z)$ analytic in G. Hence, once more, the boundary of the domain is a circle. □

Definition 3.3. ([6], Ch. 10) *Let G be a Jordan domain with rectifiable boundary Γ, let $z = \phi(w)$ map G onto $|w| < 1$. Since $\phi' \in H^1$ and has no zeros, it has a canonical factorization $\phi'(w) = S(w)\Phi(w)$ where S is a singular inner function and Φ is an outer function. G is said to satisfy the Smirnov condition if $S(w) = 1$, i.e. if ϕ' is purely outer.*

It is the case that G is a Smirnov domain if and only if $\mathbb{E}^p(G)$ coincides with the $\mathbb{L}^p(\Gamma)$ closure of the polynomials. We will use repeatedly the property that if a function $f \in \mathbb{E}^p(G)$ belongs to $\mathbb{L}^p(\Gamma)$ with $q > p$, then $f \in \mathbb{E}^q(G)$.

Remark 3.4. For a simply connected domain we can significantly relax the assumption of analyticity of the boundary in Theorem 3.2 and obtain that the domain is a disk invoking the following result from [8].

Theorem 3.5. *(Thm. 3.29 in* [8]*). Let G be a Jordan domain in $\mathbb{R}^2 \cong \mathbb{C}$ containing 0 and with the rectifiable boundary Γ satisfying the Smirnov condition. Suppose the harmonic measure on Γ with respect to 0 equals $c|z|^\alpha ds$ for $z \in \Gamma$, where ds denotes arclength measure on Γ, $\alpha \in R$ and c is a positive constant. Then*
(i) For $\alpha = -2$, the solutions are precisely all disks G containing 0.
(ii) For $\alpha = -3, -4, -5, ...$ there are solutions G which are not disks.
(iii) For all other values of α, the only solutions are disks centered at 0.

To apply this result in our context we need first to notice that the positive measure $\frac{f(z)}{z}dz = const|z|^{p-2}\,ds$ annihilates all analytic functions vanishing at the origin and hence is, after normalizing by a scalar multiple, a representing measure for analytic functions at the origin. Moreover, since the domain is simply connected, we can separate real and imaginary parts and then conclude that this latter measure is precisely the harmonic measure at 0. Because $p-2 \geq -1$, part (iii) applies and the domain is a disk centered at the origin.

THEOREM 3.6. *Let $p, q \geq 1$, $\frac{1}{p}+\frac{1}{q}=1$.*
Let G be an annulus $\{z : 0 < r < |z| < R\}$ and $\Gamma = \gamma_1 \cup \gamma_2$ be its boundary.
(i) For $p > 1$ the best analytic approximation to $\omega = z^n\overline{z}^m$ in $\mathbb{E}_p(G)$ is unique and equal to $g^*(z) = cz^{n-m}$ where $c(n,m,p) = \frac{r^{2m+q(n-m)+\frac{q}{p}}+R^{2m+q(n-m)+\frac{q}{p}}}{r^{q(n-m)+\frac{q}{p}}+R^{q(n-m)+\frac{q}{p}}}$.
(ii) For $p = 1$, and $n-m = -1$, the set of functions that are closest to $\omega = z^n\overline{z}^m$ in $\mathbb{E}_1(G)$ consist of all functions of the form $g^*(z) = cz^{n-m}$ where c is any constant such that $r^{2m} \leq c \leq R^{2m}$.
(iii) For $p > 1$, the distance from $z^n\overline{z}^m$ to $\mathbb{E}_p(G)$ is

$$\begin{aligned}\Lambda_p(G) \quad &: \quad = \|z^n\overline{z}^m - g^*(z)\|_{\mathbb{L}_p(ds,\Gamma)} \\ &= \frac{(rR)^{n-m}\left(R^{2m}-r^{2m}\right)}{r^{q(n-m)+\frac{q}{p}}+R^{q(n-m)+\frac{q}{p}}}\sqrt[p]{2\pi\left[\left(r^{\frac{1}{p}}R^{\frac{q}{p}(1-n-m)}\right)^p+\left(R^{\frac{1}{p}}r^{\frac{q}{p}(1-n-m)}\right)^p\right]}.\end{aligned}$$

For $p = 1$, $\Lambda_1(G) = \left|\int_\Gamma z^n\overline{z}^m f^*(z)dz\right| = 2\pi(R^{2m}+r^{2m})$.

Note: For $n-m \neq -1$,we have been unable to find the best approximation in closed form, see the remark at the end of the proof.

PROOF. Consider $f(z) = \frac{\left|z^n\overline{z}^m - cz^{n-m}\right|^p}{z^n\overline{z}^m - cz^{n-m}}\frac{ds}{dz}$. Then

$$\begin{aligned}&\int_\Gamma\left(\frac{\left|z^n\overline{z}^m - cz^{n-m}\right|^p}{z^n\overline{z}^m - cz^{n-m}}\right)z^k ds \\ = &\int_0^{2\pi}\frac{\left|R^ne^{in\theta}R^me^{-im\theta} - cR^{n-m}e^{i(n-m)\theta}\right|^p}{R^ne^{in\theta}R^me^{-im\theta} - cR^{n-m}e^{i(n-m)\theta}}R^{k+1}e^{ik\theta}d\theta \\ &+\int_0^{2\pi}\frac{\left|r^ne^{in\theta}r^me^{-im\theta} - cr^{n-m}e^{i(n-m)\theta}\right|^p}{r^ne^{in\theta}r^me^{-im\theta} - cr^{n-m}e^{i(n-m)\theta}}r^{k+1}e^{ik\theta}d\theta \\ = &\left(R^{\frac{p(n-m)+q(k+1)}{q}}\frac{\left|R^{2m}-c\right|^p}{R^{2m}-c}+r^{\frac{p(n-m)+q(k+1)}{q}}\frac{\left|r^{2m}-c\right|^p}{r^{2m}-c}\right)\int_0^{2\pi}e^{i(k+m-n)\theta}d\theta \\ = &\ 0, \text{ unless } k = n-m.\end{aligned}$$

and

$$2\pi\left(R^{p(n-m)+1}\frac{\left|R^{2m}-c\right|^p}{R^{2m}-c}+r^{p(n-m)+1}\frac{\left|r^{2m}-c\right|^p}{r^{2m}-c}\right) = 0$$

if

$$R^{p(n-m)+1}\frac{\left|R^{2m}-c\right|^p}{R^{2m}-c} = -r^{p(n-m)+1}\frac{\left|r^{2m}-c\right|^p}{r^{2m}-c},$$

which is only possible if $r^{2m} < c < R^{2m}$. In that case,

$$\left(\frac{c - r^{2m}}{R^{2m} - c}\right)^{p-1} = \left(\frac{R}{r}\right)^{p(n-m)+1}$$

and after some algebra we obtain that

$$c = \frac{r^{2m+q(n-m)+\frac{q}{p}} + R^{2m+q(n-m)+\frac{q}{p}}}{r^{q(n-m)+\frac{q}{p}} + R^{q(n-m)+\frac{q}{p}}}.$$

Therefore, $f(z)$ annihilates $\overline{\mathbb{E}_p(G)}$.
Now let $f^*(z) = \frac{f}{\|f\|_{\mathbb{L}_q(ds)}}$ so that $\|f^*\|_{\mathbb{L}_q(ds)} = 1$ and let $g^*(z) = cz^{n-m}$. Then,

$$\begin{aligned} f^*(z)(z^n\overline{z}^m - g^*(z))dz &= \frac{\frac{\left|z^n\overline{z}^m - cz^{n-m}\right|^p}{z^n\overline{z}^m - cz^{n-m}}}{\left\|\frac{|z^n\overline{z}^m - cz^{n-m}|^p}{z^n\overline{z}^m - cz^{n-m}}\right\|_q}(z^n\overline{z}^m - cz^{n-m})ds \\ &= \frac{\left|z^n\overline{z}^m - cz^{n-m}\right|^p}{\left\|\left(z^n\overline{z}^m - cz^{n-m}\right)^{\frac{p}{q}}\right\|_q}ds \\ &= \frac{\left|z^n\overline{z}^m - cz^{n-m}\right|^p}{\|z^n\overline{z}^m - cz^{n-m}\|_p^{p-1}}ds, \end{aligned}$$

which is condition (iii) in Corollary 2.2. Therefore $f^*(z)$ and $g^*(z)$ are extremal.
For $p = 1$, and $n - m = -1$, by Corollary 2.2, $f^*(z)$ and $g^*(z)$ are extremal if and only if they satisfy that $f^*(z)(z^n\overline{z}^m - g^*(z))dz = |z^n\overline{z}^m - g^*(z)|ds$ on each boundary component of the annulus.
Consider $f^*(z) = -i$ and $g^*(z) = \frac{c}{z}$.
On $\gamma_1 = \{z \in \mathbb{C} : |z| = r\}$, with clockwise orientation on the boundary we have

$$-i(z^n\overline{z}^m - \frac{c}{z})dz = -(r^{2m} - c)d\theta$$

and on the other hand

$$|z^n\overline{z}^m - \frac{c}{z}|ds = \left|r^{2m} - c\right| d\theta$$

The same analysis on $\gamma_2 = \{z \in \mathbb{C} : |z| = R\}$, where the orientation on the boundary is counterclockwise, yields

$$-i(z^n\overline{z}^m - \frac{c}{z})dz = (R^{2m} - c)d\theta$$

and

$$|z^n\overline{z}^m - \frac{c}{z}|ds = \left|R^{2m} - c\right| d\theta.$$

Which means

$$-(r^{2m} - c)d\theta = \left|r^{2m} - c\right| d\theta$$

and

$$(R^{2m} - c)d\theta = \left|R^{2m} - c\right| d\theta.$$

These two equations hold simultaneously for any constant c in the interval $[r^{2m}, R^{2m}]$.
Finally we compute $\Lambda_p(G)$.

For $p>1$, recalling that $c(n,m,p)=\frac{r^{2m+q(n-m)+\frac{q}{p}}+R^{2m+q(n-m)+\frac{q}{p}}}{r^{q(n-m)+\frac{q}{p}}+R^{q(n-m)+\frac{q}{p}}}$, we have

$$\begin{aligned}\Lambda_p^p(G) &= \left\|z^n\overline{z}^m - cz^{n-m}\right\|^p_{\mathbb{L}_p(ds,\Gamma)}\\ &= 2\pi\left[r^{p(n-m)+1}\left(c-r^{2m}\right)^p + R^{p(n-m)+1}\left(R^{2m}-c\right)^p\right]\\ &= 2\pi\left[\left(r^{\frac{1}{p}}R^{\frac{q}{p}(1-n-m)}\right)^p + \left(R^{\frac{1}{p}}r^{\frac{q}{p}(1-n-m)}\right)^p\right]\left(\frac{(rR)^{n-m}\left(R^{2m}-r^{2m}\right)}{r^{q(n-m)+\frac{q}{p}}+R^{q(n-m)+\frac{q}{p}}}\right)^p.\end{aligned}$$

Therefore $\Lambda_p(G)=\frac{(rR)^{n-m}(R^{2m}-r^{2m})}{r^{q(n-m)+\frac{q}{p}}+R^{q(n-m)+\frac{q}{p}}}\sqrt[p]{2\pi\left[\left(r^{\frac{1}{p}}R^{\frac{q}{p}(1-n-m)}\right)^p+\left(R^{\frac{1}{p}}r^{\frac{q}{p}(1-n-m)}\right)^p\right]}$.

Now, for $p=1$ and $n-m=-1$

$$\begin{aligned}\Lambda_1(G) &= \left|\int_\Gamma z^n\overline{z}^m f^*(z)dz\right|\\ &= \left|\int_0^{2\pi} r^{2m}+R^{2m}d\theta\right|\\ &= 2\pi(r^{2m}+R^{2m}).\end{aligned}$$

The proof of Theorem 3.6 is now complete. □

REMARK 3.7. When $p=1$ and $n-m\neq -1$, because the boundary is analytic and $f^*(z)$ is continuous on $\overline{G}$, $|f^*(z)|=1$ everywhere on the boundary. Therefore, $f^*(z)$ is either constant or a $k-$sheeted covering of the unit disk. It is not a constant since $\int_\Gamma z^n\overline{z}^m dz=0$ unless $n-m=-1$. So $f^*(z)$ maps G onto a $k-$sheeted cover of the unit disk with $k\geq n$. Hence the best approximation to $z^n\overline{z}^m$ cannot be a monomial cz^{n-m}. Moreover, it follows from the duality relations that $f^*(z)$ has to be a transcendental function.

By letting $n=0$ and $m=1$ we have the following corollary.

COROLLARY 3.8. *Let* $\frac{1}{p}+\frac{1}{q}=1$.
Let G be an annulus $\{z:0<r<|z|<R\}$.
(i) For $p>1$ the best analytic approximation to $\omega=\overline{z}$ in $\mathbb{E}_p(G)$ is $g^*(z)=\frac{rR}{z}$.
(ii) For $p=1$, all functions $g^*(z)=\frac{c}{z}$ for any constant $c\in[r^2,R^2]$, serve as the best approximation to $\overline{z}$ in $\mathbb{E}_1(G)$.
(iii) For $p\geq 1$ the $p-$analytic content of G is $\lambda_{\mathbb{E}_p}(G)=(R-r)(2\pi(R+r))^{\frac{1}{p}}$

Notice that the best approximation to $\overline{z}$ in $\mathbb{E}_p(G)$ is $g^*(z)=\frac{rR}{z}$ independent of p!

Next we will prove a partial converse for Theorem 3.6 in the case when $p=1$. For that we will need the following lemma.

LEMMA 3.9. *Let G be a multiply connected domain in $\mathbb{C}$ with analytic boundary consisting of n components. If $g^*(z) = \frac{c}{z}$ is the best approximation to $\overline{z}$ in $\mathbb{E}_1(G)$ and $\overline{z}$ does not coincide with $\frac{c}{z}$ on any of the boundary components then:*
i) $f^*(z)$, the extremal function in $\mathbb{E}^1_\infty(G)$ for which $\sup\limits_{f\in\mathbb{E}^1_\infty(G)} \left|\int_\Gamma \overline{z} f(z)dz\right|$ is attained, is a unimodular constant.
ii) The number n of boundary components of G is 2.

PROOF. Replicating the argument used in Theorem 3.2, case 1, we can show that unless $f^*(z)$ is a constant, it is a $k-$sheeted covering of the unit disk, with $k \geq n$, thus $\Delta_{\arg} f^*(z) \geq n$. Moreover, the tangent vector to Γ goes along the boundary of G once in the clockwise direction, and $n-1$ times in the counterclockwise direction. So $\Delta_{\arg} f^*(z)dz \geq n+2-n = 2$. Now, since the boundary of the domain is analytic and we are assuming that $\frac{1}{z}$ is analytic in G, $\frac{1}{z}$ has no poles in G. By the argument principle we can say that

$$\Delta_{\arg}\left(\overline{z} - \frac{c}{z}\right) = \Delta_{\arg}\frac{|z|^2 - c}{z} = \Delta_{\arg}(|z|^2 - c) + \Delta_{\arg}\frac{1}{z} = 0.$$

So $\Delta_{\arg}(\overline{z} - \frac{c}{z})f^*dz = \Delta_{\arg} f^*(z)dz + \Delta_{\arg}\left(\overline{z} - \frac{c}{z}\right) \geq 2$. Yet Corollary 2.2 (*iii*) yields that $\Delta_{\arg}(\overline{z} - \frac{c}{z})f^*dz = 0$ since it has constant argument on Γ, so we have a contradiction. Hence, $f^*(z)$ has to be constant.

With $f^*(z)$ constant we have that $\Delta_{\arg}(\overline{z} - \frac{c}{z})f^*dz = \Delta_{\arg}dz = 2 - n = 0$ therefore the number of boundary components of G is $n = 2$. □

THEOREM 3.10. *Let G be a multiply connected domain in $\mathbb{C}$ with analytic boundary Γ. If $g^*(z) = \frac{c}{z}$ is the best approximation to $\overline{z}$ in $\mathbb{E}_1(G)$ and the hypotheses of Lemma 3.9 are satisfied, then G is an annulus.*

PROOF. (That the best analytic approximation to $\overline{z}$ in $\mathbb{E}_1$ of the annulus is $g^*(z) = \frac{c}{z}$ follows from Corollary 3.8.)
We infer from Lemma 3.9 that $f^*(z) = e^{i\alpha}$. By the duality relations we obtain

$$\lambda_{\mathbb{E}_1} = \left|\int_\Gamma \overline{z}dz\right| = \left|\int_\Gamma (\overline{z} - \frac{c}{z})dz\right| \leq \int_\Gamma \left|\overline{z} - \frac{c}{z}\right| ds = \lambda_{\mathbb{E}_1}. \quad (9)$$

Therefore equality holds throughout.
Now, since $|z|^2 - c$ is real and the boundary is analytic, (9) implies that $\arg\left(\frac{dz}{z}\right)$ is constant on every boundary component of G. On the other hand we have from Lemma 3.9 (*ii*) that G has two boundary components γ_1 and γ_2, with opposite orientation. So letting $z(s) = r(s)e^{ib(s)}$, with s being the arclength parameter, since $\left|\frac{dz}{ds}\right| = 1$, by differentiating we obtain

$$\frac{dz}{ds} = (ir(s)b'(s) + r'(s))e^{ib(s)} = e^{ia_j + ib(s)},\ j = 1, 2$$

were a_j, $j = 1, 2$ are constants on γ_1 and γ_2 respectively. This yields that

$$ir(s)b'(s) + r'(s) = e^{ia_j},\ j = 1, 2.$$

Differentiating again, we obtain

$$r''(s) + i(r(s)b'(s))' = 0,$$

hence ($r(s)$ and $b(s)$ are real-valued functions) $r''(s) = 0$ and $r(s)$ is a linear function. Recalling that the boundary of the domain consists of two closed curves we conclude that $r(s)$ is linear and periodic, hence it is constant on each boundary component. So the boundary of the domain consists of two concentric circles and the domain is an annulus. □

REMARK 3.11. In Lemma 3.9, if $\overline{z}$ does coincide with $\frac{c}{z}$ on one of the boundary components, say γ_o, i.e. if that component is a circle, then on that boundary component $|f^*| \leq 1$ while on the remaining components $|f^*| = 1$. In this case we can only infer that $\underset{\Gamma\setminus\gamma_o}{\Delta \arg} f^* \geq n-1$ and the argument above fails. We conjecture that Theorem 3.10 holds for all $p \geq 1$ and without the additional hypothesis in Lemma 3.9. Yet, we have not been able to prove it.

4. The Bergman Space case: Characterization of disks and annuli in terms of the best analytic approximation to $\overline{z}$ in $\mathbb{A}_p$ norm.

Let $d\sigma$ be area measure on G.

We use the standard notation $\mathbb{W}^{1,q}(G)$ and $\mathbb{W}^{1,q}_o(G)$ for Sobolev spaces and Sobolev spaces with vanishing boundary values. The reader may consult [9, Ch.5], [1] for details.

Khavin's lemma (see [20]) describes the annihilator of $A_p(G)$ as follows:
For $p > 1$,

$$\begin{aligned} Ann(\mathbb{A}_p(G)) \quad &: \quad = \left\{ f \in \mathbb{L}_q(d\sigma, G) : \int_G fg d\sigma = 0 \text{ for all } g \in \mathbb{A}_p(G) \right\} \\ &= \quad \left\{ \frac{\partial u}{\partial \overline{z}},\ u \in \mathbb{W}^{1,q}_o(G) \right\}. \end{aligned}$$

For $p = 1$,

$$Ann(\mathbb{A}_1(G)) := \left\{ weak(*) \text{ closure of } \frac{\partial u}{\partial \overline{z}},\ u \in \mathbb{W}^{1,\infty}(G),\ \text{in } \mathbb{L}_\infty(d\sigma, G) \right\}.$$

DEFINITION 4.1. *The Bergman $p-$analytic content of a domain G is*

$$\lambda_{\mathbb{A}_p}(G) := \inf_{g \in \mathbb{A}_p(G)} \|\overline{z} - g(z)\|_{\mathbb{L}_p(d\sigma, G)}.$$

By the Hahn-Banach theorem,

$$\lambda_{\mathbb{A}_p}(G) = \max_{f \in Ann(\mathbb{A}_p(G)), \|f\| \leq 1} \left| \int_G \overline{z} f d\sigma \right|.$$

A similar result to Corollary 2.2 holds in the context of Bergman spaces; we state it as Corollary 4.2 for completeness. See [17] Theorem 3.1, Remarks (i) and (iv). Also see [18] p. 940.

Corollary 4.2. *Let $\frac{1}{p}+\frac{1}{q}=1$, and let $\omega(z)\in\mathbb{L}_p(d\sigma,G)$. Then the following hold:*
(i) $\inf\limits_{g\in\mathbb{A}_p(G)}\|\omega(z)-g(z)\|_{\mathbb{L}_p(d\sigma,G)}=\sup\limits_{f\in Ann(\mathbb{A}_p(G)),\|f\|\le 1}\left|\int_G\omega(z)fd\sigma\right|.$
(ii) There exist extremal functions $g^*(z)\in\mathbb{A}_p(G)$ and $f^*(z)\in Ann(\mathbb{A}_p(G))$ for which the infimum and the supremum are attained in (i).
(iii) When $p>1$, $g^*(z)\in\mathbb{A}_p(G)$ and $f^*(z)\in Ann(\mathbb{A}_p(G))$ are extremal if and only if, for some real number δ,

$$\begin{aligned} e^{i\delta}f^*(z)(\omega(z)-g^*(z)) &\ge 0 \text{ in } G,\\ \Lambda^p_{\mathbb{A}_p}|f^*(z)|^q &= |\omega(z)-g^*(z)|^p \text{ in } G,\end{aligned}$$

where $\Lambda_{\mathbb{A}_p}=\|\omega(z)-g^*(z)\|_{\mathbb{L}_p(d\sigma,G)}$.

When $p=1$ the conditions above become $e^{i\delta}f^*(z)(\omega(z)-g^*(z))=|\omega(z)-g^*(z)|$ a.e. in G.
(iv) For $p>1$ the best approximations $g^*(z)\in\mathbb{A}_p(G)$ and $f^*(z)\in Ann(\mathbb{A}_p(G))$ are always unique. For $p=1$ and $\omega(z)$ continuous in G, the best approximation $g^*(z)\in\mathbb{A}_p(G)$ is unique. For discontinuous $\omega(z)$ the best approximation need not be unique. Also, in the case where $p=1$ the duality condition in (iii) implies that $f^*(z)\in Ann(\mathbb{A}_1(G))$ is unique, up to a unimodular constant, provided that $\omega(z)$ does not coincide with an analytic function on a set of positive area measure.

Remark 4.3. For the case of the disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<r\}$ it was shown in [17] Proposition 2.3, that the best rational approximation in $\mathbb{A}_p(\mathbb{D})$ to $\omega=z^n\overline{z}^m$ for $p\ge 1$ and $m>n$ is $g^*(z)=0$. When $m\le n$, $g^*(z)=cz^{n-m}$, where $c=c(n,m,p)$ is an appropriate constant.

In that case we can compute the Bergman $p-$analytic content of $\mathbb{D}$ as follows:

$$\begin{aligned}\lambda_{\mathbb{A}_p}(\mathbb{D}) &= \sqrt[p]{\int_0^{2\pi}\int_0^r|te^{-it}|^p\,tdtd\theta}\\ &= \sqrt[p]{2\pi\int_0^r|t|^p\,tdt}\\ &= \sqrt[p]{\frac{2\pi r^{p+2}}{p+2}}.\end{aligned}$$

Following the argument in [17] we find the extremal functions for the case of the annulus.

Proposition 4.4. *Let $p,q\ge 1$, $\frac{1}{p}+\frac{1}{q}=1$. Let G be an annulus $\{z:r<|z|<R,\ r<R\}$. The best analytic approximation to $\omega=z^n\overline{z}^m$ in $\mathbb{A}_p(G)$ is $g^*(z)=cz^{n-m}$, with $c=c(n,m,p)$ satisfying*

$$\int_r^R t^{p(n-m)+1}\left|t^{2m}-c\right|^{p-1}sgn(t^{2m}-c)dt=0$$

and $\Lambda_{\mathbb{A}_p}(G)=\sqrt[p]{2\pi\int_r^R|t^{2m}-c|^p\,t^{p(n-m)+1}dt}.$

In particular, if $n = 0$ and $m = 1$, for $p \geq 1$, the Bergman $p-$analytic content of G in $\mathbb{A}_p$ is $\lambda_{\mathbb{A}_p}(G) = \sqrt[p]{2\pi \int_r^R |c - t^2|^p \, t^{-\frac{p}{q}} dt}$ where $c(0,1,p)$ is such that $\int_r^R t^{1-p} \left|t^2 - c\right|^{p-1} sgn(t^2 - c) dt = 0$.

Proof. Consider $f(z) = \frac{\left|z^n \overline{z}^m - c z^{n-m}\right|^p}{z^n \overline{z}^m - c z^{n-m}}$,

$$\begin{aligned}
\int_G \left(\frac{\left|z^n \overline{z}^m - c z^{n-m}\right|^p}{z^n \overline{z}^m - c z^{n-m}} \right) z^k d\sigma &= \int_0^{2\pi} \int_r^R \frac{\left|z^n \overline{z}^m - c z^{n-m}\right|^p}{z^n \overline{z}^m - c z^{n-m}} z^k t dt d\theta \\
&= \int_r^R \frac{t^{\frac{p}{q}(n-m)+k+1} \left|t^{2m} - c\right|^p}{t^{2m} - c} \left(\int_0^{2\pi} e^{-i(k+(n-m))\theta} d\theta \right) dt \\
&= 0, \text{ unless } k = n - m.
\end{aligned}$$

If we choose $c = c(p)$, so that $\int_r^R t^{p(n-m)+1} \left|t^{2m} - c\right|^{p-1} sgn(t^{2m} - c) dt = 0$, then $f(z) \in Ann(\mathbb{A}_p(G))$.

Defining $f^*(z) = \frac{f}{\|f\|_{\mathbb{L}_q(d\sigma, G)}}$ we can check that necessary and sufficient (Corollary 4.2 (iii)) conditions for extremality hold and the result follows.

To compute $\Lambda^p_{\mathbb{A}_p}(G)$:

$$\begin{aligned}
\Lambda^p_{\mathbb{A}_p}(G) := \int_G \left|z^n \overline{z}^m - c z^{n-m}\right|^p d\sigma &= \int_0^{2\pi} \int_r^R \left| t^{n+m} e^{i(n-m)\theta} - c t^{n-m} e^{i(n-m)\theta} \right|^p t dt d\theta \\
&= \int_0^{2\pi} \int_r^R t^{p(n-m)+1} \left|t^{2m} - c\right|^p dt d\theta = 2\pi \int_r^R t^{p(n-m)+1} \left|t^{2m} - c\right|^p dt.
\end{aligned}$$

□

Theorem 4.5. *Let G be a bounded domain with analytic boundary. The best analytic approximation to $\overline{z}$ in $\mathbb{A}^p(G)$ is $g^*(z) = 0$ if and only if G is a disk.*

Proof. If G is a disk, the best rational approximation to $\overline{z}$ *in* $\mathbb{A}_p(G)$ is $g^*(z) = 0$ by Remark 4.3.

Now suppose 0 is the best approximation to $\overline{z}$ in $\mathbb{A}^p(G)$. First assume $p > 1$. In this case Corollary 4.2 (iii) can be written as $|z|^p = \lambda^p |f^*|^q$, so $f(z) = \frac{|\overline{z}|^p}{\overline{z}}$ annihilates $\overline{\mathbb{A}^p(G)}$, $f^*(z) = \frac{f(z)}{\|f(z)\|_q}$, and by Khavin's Lemma $f^*(z) = \frac{\partial u}{\partial \overline{z}}$ for some $u \in \mathbb{W}^{1,q}_o(G)$. Hence,

$$\frac{\partial u}{\partial \overline{z}} = const \frac{|\overline{z}|^p}{\overline{z}}.$$

Integrating with respect to $\overline{z}$ we obtain

$$u(z) = \int \frac{\partial u}{\partial \overline{z}} d\overline{z} = const \int z^{\frac{p}{2}} \overline{z}^{\frac{p}{2}-1} d\overline{z} = const |z|^p + h(z),$$

where $h(z)$ is analytic.

Since $u(z) \in \mathbb{W}^{1,q}_o(G)$ and $|z|^p$ is real analytic near Γ, it is easy to see that for any sequence of domains G_j, $\cup G_j = G$ with rectifiable boundaries Γ_j, $\|u\|_{\mathbb{L}_q(\Gamma_j, ds)}$ are bounded, so $h(z) \in \mathbb{E}_q(G)$. Now, $u(z) = 0$ a.e. on Γ, hence $h(z) = -|z|^p$ a.e. on Γ. Since Γ is analytic, and $h(z) \in \mathbb{E}_q(G)$ for $q \geq 1$ and has bounded boundary

values, $h(z)$ is bounded in G. But $h(z)$ has real boundary values on Γ a.e., hence $h(z)$ is constant. Thus $|z|^p$ is constant a.e. on Γ, so G is a disk centered at the origin. The case $p = 1$ and $q = \infty$ requires only small modifications that are left to the reader. $\square$

Along the same lines we also have:

THEOREM 4.6. *Let G be a finitely connected domain with analytic boundary, $p \geq 1$. G is an annulus centered at the origin if and only if the best analytic approximation to $\overline{z}$ in $\mathbb{A}^p(G)$ is $g^*(z) = \frac{c}{z}$.*

PROOF. If G is an annulus we have already seen in Proposition 4.4 that the best rational approximation to $\overline{z}$ in $\mathbb{A}_p(G)$, $p \geq 1$, is $g^*(z) = \frac{c}{z}$.

To prove the converse, suppose the best approximation to $\overline{z}$ in $\mathbb{A}^p(G)$ is $g^*(z) = \frac{c}{z}$. Once again for the sake of clarity we focus on the case $p > 1$, the remaining case only requires small modifications that are left to the reader. Corollary 4.2 (iii) yields that $f^*(z) = \frac{|\overline{z}-\frac{c}{z}|^p}{\overline{z}-\frac{c}{z}} \in Ann(\overline{\mathbb{A}^p(G)})$, and Khavin's Lemma yields that $f^*(z) = \frac{\partial u}{\partial \overline{z}}$ for some $u \in \mathbb{W}^{1,q}_o(G)$.
Denoting $|z|$ by r we have

$$\frac{\partial u}{\partial \overline{z}} = \frac{|\overline{z} - \frac{c}{z}|^p}{\overline{z} - \frac{c}{z}} = \frac{|\overline{z}z - c|^p}{\overline{z}z - c} \frac{z}{z^{\frac{p}{2}}\overline{z}^{\frac{p}{2}}} = \frac{|r^2 - c|^{p-1}}{r^{p-2}\overline{z}} sign(r^2 - c).$$

Integrating we have that

$$\begin{aligned} \int \frac{\partial u}{\partial \overline{z}} d\overline{z} &= \int \frac{|r^2 - c|^{p-1}}{r^{p-2}\overline{z}} sign(r^2 - c) d\overline{z} \\ &= \int \frac{(r^2 - c)^{p-1}}{r^{p-2}} sign(r^2 - c) \frac{d}{d\overline{z}} \log |z\overline{z}| d\overline{z} \\ &= 2 \int \frac{(r^2 - c)^{p-1}}{r^{p-2}} sign(r^2 - c) d \log r. \end{aligned} \tag{10}$$

Since $0 \notin G$, this integral is bounded away from zero and yields a real-valued function $F(r)$ for all $r > 0$. So $u(z) = F(r) + h(z)$ with $h(z)$ analytic.

As in the proof of Theorem 4.5, $h(z)$ extends across the boundary and hence belongs to $\mathbb{H}^\infty(G)$. Now, because $u(z) \in \mathbb{W}^{1,q}_o(G)$, $u(z) = F(r) + h(z) = 0$ a.e. on the boundary of G. So $h(z)$ is real valued almost everywhere on the boundary. Hence it has to be real inside the domain as well and therefore constant.

Now note that $u = F(r) + const$ and $u = 0$ on Γ. Moreover from (10) it readily follows that $F'(r) = 0$ only at one point $r_o = \sqrt{c}$, where F' changes sign. Hence F may take the same value only twice, so Γ consists of two components and on each one the value of $F(r)$ is the same, i.e. Γ consists of two concentric arcs centered at the origin. Since G is bounded and $0 \notin G$ ($\frac{1}{z}$ is analytic in G!), G must be an annulus. $\square$

5. Final Remarks

For the Bergman norm, assuming that G is a multiply connected domain with analytic boundary, we were able to prove that for all $p \geq 1$ the domain is an annulus whenever the best approximation to $\overline{z}$ is $\frac{c}{z}$ (and that the domain is a disk, whenever the best approximation to $\overline{z}$ is a constant function). Our proof relies on the assumption of analyticity of the boundary. However, it is easy to see from the proof that this assumption can be relaxed and we only need assume that the domain G is Smirnov. We do not know whether the result holds for domains with arbitrary rectifiable boundaries.

The Smirnov norms case turns out to be more difficult. One of the reasons is that knowing the best approximation in Bergman norm determines the extremal function in the dual problem throughout the domain (although in a vast set $Ann(\mathbb{A}_p)$). In the $\mathbb{E}_p$ setting, it only determines the extremal function in the dual problem, although analytic in the domain, on parts of the boundary where $\overline{z}$ does not coincide with its best approximation, which unfortunately could happen a priori. In the Bergman setting this can never happen because two real analytic functions can never coincide on a set of positive area measure without being identical.

If a constant is the best approximation to $\overline{z}$ in $\mathbb{E}_p$, we showed in Theorem 3.2, for multiply connected domains and under the assumption of analyticity of the boundary, that the domain is a disk. We were able to reach the same conclusion for Jordan domains with rectifiable boundaries satisfying the Smirnov condition, but only when the domain is simply connected. We think it should be possible to generalize Theorem 3.2 to multiply connected domains with weaker regularity conditions imposed on the boundary.

The following question seems natural in connection with Remark 3.4 (and Thm. 3.2). Let G be a finitely connected domain containing the origin and assume that

$(*)$ *the measure* $const|z|^{\alpha}ds$, $\alpha \in R$, *on the boundary* Γ *is a representing measure at the origin for analytic functions in* G, *say, continuous in* $\overline{G}$.

Does condition $(*)$ alone imply that G must be simply connected? If so, (cf. Remark 3.4) then for $\alpha > -2$, G must be a disk centered at the origin. Perhaps, condition $(*)$ implies that G is simply connected only for specific values of α, what are these values and what happens in the remaining cases? Under a less restrictive regularity assumption, say assuming the boundary of G merely rectifiable, even for $\alpha = 0$, there exist highly nonregular, non-Smirnov domains, so called pseudocircles, for which $(*)$ still holds (cf. [6], Ch. 10).

When the domain is an annulus we found the best $\mathbb{E}_p$-approximation to any monomial $z^n\overline{z}^m$ explicitly for all $p > 1$ and for $p = 1$ when $n - m = -1$. Yet, when $p = 1$ and $n - m \neq -1$, the extremal function f^* in the dual problem is a trascendental function, hence we can only conclude that the best approximation to $z^n\overline{z}^m$ is not a monomial. It would be worthwhile to study the best approximation of such monomials in $\mathbb{E}_1$ of the annulus in greater detail.

In Theorem 3.10 we show for $p = 1$ that the domain is an annulus whenever the best approximation to $\overline{z}$ is $\frac{c}{z}$. In the proof we use Lemma 3.9 which, via the argument principle, shows that the boundary of the domain consists of two boundary components. However, the hypothesis of the lemma assumes that the boundary of the domain is analytic, and that $\overline{z} \neq \frac{c}{z}$ on every boundary component of the domain. If $\overline{z} = \frac{c}{z}$ on some component, then that boundary contour is a circle but already our argument that the boundary has two components fails since the argument principle can only estimate the change in the argument of f^* on the remaining components. It should be possible to coach the proof of Theorem 3.10 to include the case when $\overline{z} = \frac{c}{z}$ on a boundary component and to extend it to all $p \geq 1$, but we have not been able to do it.

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Department of Mathematics, University of Arkansas Fort Smith, Fort Smith, Arkansas 72913

E-mail address: zguadarr@uafortsmith.edu

Department of Mathematics, University of South Florida, Tampa, Florida 33620

E-mail address: dkhavins@cas.usf.edu