

# Algebraicity in the Dirichlet problem in the plane with rational data||

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Communicated by Y. J. Leung

(Received in final form 30 September 2006)

We consider the Dirichlet problem for the Laplace operator with rational data on the boundary of a planar domain. Our main results include a characterization of the disk as the only domain for which all solutions are rational, and a characterization of the simply connected quadrature domains as the only ones for which all solutions are algebraic of a certain type. This note is an exposition, and full details will appear in a forthcoming paper.

**Keywords:** Dirichlet problem; Quadrature domain; Laplac operator

**1991 Mathematics Subject Classifications:** 30E34; 35J25

This is an exposition of the authors' work on the Dirichlet problem in the plane with rational boundary data. In the present note, we report our main results and give only an indication of the ideas involved in their proofs. Full details will appear in the forthcoming paper [1].

Let  $\Omega$  be a bounded domain in the plane  $\mathbb{R}^2$  and assume that  $\partial\Omega$ , the boundary of  $\Omega$ , consists of finitely many non-intersecting Jordan curves. We shall consider the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = v & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $v \in C(\partial\Omega)$  (and  $C(A)$  denotes the space of continuous functions on a topological space  $A$ ). It is, of course, well known that this Dirichlet problem has a unique solution  $u$

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||Dedicated to Professor Peter L. Duren on the occasion of his 70th birthday.

in  $C(\bar{\Omega})$ . The case where the data function  $v$  is the restriction of a polynomial in  $x$  and  $y$  is an important special case, since, by the Stone–Weierstrass theorem and the maximum principle, any solution of (1) with continuous data can be approximated uniformly on  $\bar{\Omega}$  by solutions with boundary data that are restrictions of such polynomials. Now, if  $\Omega$  is a disk, or more generally the interior of an ellipse, then the latter solutions are polynomials themselves (the corresponding statement holds even in higher dimensions; see e.g., [2]). It was conjectured in [3] that this property characterizes the ellipses. This conjecture was recently proved by Render (also in higher dimensions; see [4]), under a mild additional assumption. In this note (with full details appearing in the article [1]), we consider the more general situation where the data function  $v$  is the restriction to  $\partial\Omega$  of a rational function  $R(x, y)$  whose polar variety does not meet  $\partial\Omega$ .

If the boundary of  $\Omega$  is real-analytic, then the solution of (1), with a restricted rational function (without poles on  $\partial\Omega$ ) as data, extends harmonically to a larger domain. We are interested in characterizing those domains  $\Omega$  for which this harmonic extension encounters only “mild” singularities. For instance, if  $\Omega$  is a disk, then the solution itself is rational and extends to the Riemann sphere with only a finite set of poles (see subsequently). One of our main results in [1] is that this property characterizes the disk (with a vengeance; see Theorem 1). We also show that if all solutions are algebraic (and hence extend as multi-valued functions to the Riemann sphere minus a finite set of points at which only algebraic singularities are encountered), then  $\Omega$  is simply connected and a Riemann map to the unit disk is algebraic (Theorem 4). We further characterize those simply connected domains with algebraic Riemann maps for which all solutions are algebraic with singularities that are controlled by those of the Riemann map (in a sense made precise subsequently; see Theorem 5).

We now proceed to formulate our results more precisely. Our first result, which was alluded to above, is the following.

**THEOREM 1** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$  whose boundary consists of finitely many non-intersecting Jordan curves. The following are equivalent:*

- (i)  $\Omega$  is a disk.
- (ii) *The solution  $u(x, y)$  of (1) is rational for every  $v \in C(\partial\Omega)$  that is the restriction of a rational function  $R(x, y)$  whose polar variety does not meet  $\partial\Omega$ .*

The implication (i)  $\implies$  (ii) is easy (see e.g. [5] for a proof; see also [6]). The opposite implication follows from the more general result Theorem 2 subsequently. To state it in a more convenient way, we shall identify  $\mathbb{R}^2$  with the complex plane  $\mathbb{C}$  in the usual way, i.e., via  $z = x + iy$ . By the relations  $2x = z + \bar{z}$  and  $2iy = z - \bar{z}$ , any real-analytic function  $v(x, y)$  can be expressed as a function  $v(z, \bar{z})$ . We shall abuse the notation slightly and write either  $v(x, y)$  or  $v(z, \bar{z})$  (i.e., dropping the  $\sim$ ) for the same function  $v$ . Clearly,  $v(x, y)$  is rational as a function of  $x$  and  $y$  if and only if  $v(z, \bar{z})$  is rational as a function of  $z$  and  $\bar{z}$ .

**THEOREM 2** *Let  $\Omega$  be a bounded domain in  $\mathbb{C}$  whose boundary consists of finitely many non-intersecting Jordan curves and let  $a \in \Omega$ . Suppose that the solution  $u(z, \bar{z})$  of (1) is rational for every  $v \in C(\partial\Omega)$  that is the restriction of  $R(z, \bar{z})$ , where  $R(z, \bar{z})$  ranges over all polynomials of  $z$  and  $\bar{z}$ , and the single function*

$$R(z, \bar{z}) = \frac{1}{(z - a)}. \quad (2)$$

*Then,  $\Omega$  is a disk.*

*Remark 3* (a) Our proof of Theorem 2 actually shows that if  $\Omega$ , in addition, is assumed to be *simply connected*, then it suffices to let  $R(z, \bar{z})$  range over the four functions  $z\bar{z}$ ,  $z^2\bar{z}$ ,  $z^3\bar{z}$ , and (2). The conclusion is again that  $\Omega$  is a disk. (b) We would also like to mention the paper [7], in which a problem similar to the one studied here is considered. In [7], the Dirichlet problem (1) is considered for data functions  $v$  that are holomorphic rational functions  $R(z)$ , i.e., rational functions of  $z$  alone without poles on the boundary  $\partial\Omega$ . This is, of course, a smaller class of data functions than that considered in the present article. The main result in [7], paralleling that given in Theorems 1 and 2 earlier, is that all solutions of (1) with rational holomorphic data are rational precisely when a Riemann map  $\varphi: \Omega \rightarrow \mathbb{D}$  is rational. Thus, the class of rational holomorphic data functions is not large enough to characterize the disk. In a subsequent paper [8], the authors obtain a constructive algorithm for solving the Dirichlet problem with rational holomorphic data in terms of a Riemann map to the unit disk. (c) Finally, we point out that the solution of (1) with data  $v$  given by the restriction of (2) is closely related to the Bergman kernel of the domain  $\Omega$  (see [9]).

We also consider the case where the solutions  $u(z, \bar{z})$  to (1) with rational data (in the sense described by earlier Theorem 1) are only real-algebraic; i.e.,  $u(z, \bar{z})$  satisfies a polynomial relation  $P(z, \bar{z}, u(z, \bar{z})) = 0$ , where  $P(z, w, t)$  is a polynomial of three variables. (Note that  $u(z, \bar{z})$  is rational as a function of  $z$  and  $\bar{z}$  precisely when it is real-algebraic and  $P(z, w, t)$  has degree one in  $t$ . Also, note that a function  $u(z, \bar{z})$  is real-algebraic if and only if the polarized, or complexified, holomorphic function  $u(z, \zeta)$  is algebraic). We have the following result.

**THEOREM 4** *Let  $\Omega$  be a bounded domain in  $\mathbb{C}$  whose boundary consists of finitely many non-intersecting Jordan curves, and let  $a \in \Omega$ . Suppose that the solution  $u(z, \bar{z})$  of (1) is real-algebraic for every  $v \in C(\partial\Omega)$  that is the restriction of  $R(z, \bar{z})$ , where  $R(z, \bar{z})$  ranges over all polynomials of  $z$  and  $\bar{z}$ , and the single function (2). Then,  $\Omega$  is simply connected and every Riemann map  $\varphi: \Omega \rightarrow \mathbb{D}$  is algebraic.*

We shall now describe the proof of Theorem 2. Suppose that a domain  $\Omega$  satisfies the hypothesis of Theorem 2. It follows that  $\Omega$  must have a real algebraic boundary. Indeed, the hypotheses yield that  $|z|^2$  may be decomposed as

$$|z|^2 = R(z) + \overline{R(z)} \quad \text{for } z \in b\Omega,$$

where  $R(z)$  is rational with no poles on  $\bar{\Omega}$ . This shows that the boundary of  $\Omega$  is contained in the real-algebraic curve defined by the zero set of  $|z|^2 - 2\operatorname{Re} R(z)$ . Consequently, the Jordan curves that define the boundary of  $\Omega$  are piecewise real analytic. This boundary regularity allows us to see that the Bergman kernel  $K(z, w)$  associated to  $\Omega$  is equal to  $(\partial/\partial\bar{z})$  of the harmonic extension of the rational function which is the Cauchy kernel (see [10, p. 97] for the smooth case and [1] for the more general case we need here). To be precise, let  $a \in \Omega$ , and let  $u$  denote the harmonic extension to  $\Omega$  of the boundary values of  $(2\pi i)^{-1}/(z - a)$ . Then

$$K(z, a) = -2i \left( \frac{\partial}{\partial\bar{z}} \right) \bar{u}.$$

Note that since the boundary values of  $u$  are rational, the hypotheses yield that  $u$  is equal to  $R_1(z) + \overline{R_2(z)}$  where  $R_j$  are rational,  $j=1,2$ . Hence, it follows that the Bergman kernel  $K(z, a)$  is rational in  $z$  for fixed  $a$ . The formula for  $K(z, a)$  can be differentiated with respect to  $\bar{a}$  and the argument repeated to see that  $(\partial/\partial\bar{a})K(z, a)$  is also rational in  $z$  for fixed  $a$ . It was proved in [9] that the Bergman kernel  $K(z, a)$  cannot be rational in  $z$  for each fixed  $a$  in a domain of connectivity bigger than one, and so we may conclude that  $\Omega$  is simply connected. Now, old formulas of Stefan Bergman show that the Riemann map  $f_a: \Omega \rightarrow D_1(0)$  which maps  $a$  to the origin is given as  $c_1 + c_2 Q(z)$  where  $Q(z)$  is the quotient  $\partial/\partial\bar{a}K(z, a)/K(z, a)$ . Hence,  $f_a$  is rational. (The proof of this part given in [1] is longer, but more self-contained and straightforward. It does not use results from [9].)

Thus, we may claim that  $\Omega$  is simply connected and there is a rational mapping  $h$  which maps  $\Omega$  one-to-one onto the upper half plane.

The extended complex plane  $\mathbb{P}$  (or complex projective plane or Riemann sphere) is subdivided into regions  $G_1, G_2, \dots, G_N$  by the piecewise real-analytic curves that comprise  $h^{-1}(\mathbb{R})$ . Let  $G_1 = \Omega$ . Think of the regions  $G_j$  as countries on a spherical planet (with no lakes or oceans).

The mapping  $h$  is a proper holomorphic mapping of each domain  $G_j$  onto either the upper or the lower half plane, and as such, is a finite-to-one branched covering map between the two domains. A branch of  $h^{-1}$  can be defined by choosing either the upper half plane or the lower half plane and thinking of  $h^{-1}$  as the continuation of a local inverse of  $h$ , where  $h$  is viewed as a proper holomorphic mapping of one of the  $G_j$ 's onto the half plane. Note that we may continue any branch of  $h^{-1}$  as a finite valued holomorphic function with only finitely many algebraic singularities in the half plane.

Let  $S(z)$  denote the Schwarz function for a smooth part of the boundary of  $\Omega$  near a point  $z_0 \in b\Omega$ . The anti-holomorphic *Schwarz reflection function* for the boundary of  $\Omega$  near  $z_0$  is given by

$$\overline{S(z)} = h^{-1}(\overline{h(z)}),$$

where  $h^{-1}$  is holomorphic near  $h(z_0) \in \mathbb{R}$  and is the inverse to  $h$  viewed as a one-to-one map on a neighborhood of  $z_0$ . This mapping, defined near a point  $z_0$  in a smooth part of the boundary of  $\Omega$ , analytically continues to all of  $\hat{\mathbb{C}}$  as an antiholomorphic algebraic function.

The boundary data  $\bar{z}z^n$  has a harmonic extension to  $\Omega$  given by  $R_n(z) + \overline{Q_n(z)}$  where  $R_n$  and  $Q_n$  are rational functions of  $z$  with no poles on  $\bar{\Omega}$ . (There are some minor details to attend to in this statement which are addressed fully in [1].) The Schwarz function is holomorphic on a neighborhood of  $b\Omega$  near  $z_0$  and satisfies  $S(z) = \bar{z}$  on  $b\Omega$ . We may insert this fact into the identity

$$\bar{z}z^n = R_n(z) + \overline{Q_n(z)}$$

and its conjugate

$$z\bar{z}^n = \overline{R_n(z)} + Q_n(z),$$

which hold on  $b\Omega$ , to obtain the identities

$$\begin{aligned} S(z)z^n &= R_n(z) + \overline{Q_n(\overline{S(z)})}, \text{ and} \\ zS(z)^n &= \overline{R_n(\overline{S(z)})} + Q_n(z), \end{aligned}$$

which hold for  $z \in b\Omega$ . Since the functions on both sides of these identities are holomorphic on a neighborhood of the point  $z_0 \in b\Omega$ , the identities extend to hold on this neighborhood, and they analytically continue to hold as we continue  $S(z)$  along any curve.

Rewrite the last two formulas as

$$R_n(z) = S(z)z^n - \overline{Q_n(\overline{S(z)})} \quad (3)$$

$$Q_n(z) = zS(z)^n - \overline{R_n(\overline{S(z)})}. \quad (4)$$

These identities reveal how to analytically continue the functions  $R_n$  and  $Q_n$  outside of  $\Omega = G_1$ . Indeed, continue to think of the  $G_j$ 's as countries on a globe, and think of a curve  $\gamma$  as being the path of an "explorer". As the curve  $\gamma$  parametrized by  $z(t)$  moves from  $\Omega$  into a region  $G_j$  adjacent to  $\Omega$ , the "shadow curve"  $\Gamma$  given by  $Z(t) = \overline{S(z(t))}$  moves into  $G_1$ . [To be more precise,  $Z(t)$  is produced by analytically continuing  $\overline{S(z)} = h^{-1}(\overline{h(z)})$  along the trace of  $\gamma$ , i.e., by analytically continuing  $h^{-1}$  along the curve traced out by  $\overline{h(z(t))}$ .] Since  $R_n$  and  $Q_n$  have no poles in the closure of  $G_1$ , equations (3) and (4) reveal that the  $R_n$  and  $Q_n$  may be continued along  $\gamma$  into  $G_j$  and the continuations of  $R_n$  and  $Q_n$  have no poles in the closure of  $G_j$  either. (Note that  $S(z)$  remains bounded because it is the conjugate of  $\overline{S(z)}$ , which stays in the bounded domain  $G_1$ .) This process may be repeated. As we extend  $\gamma$  into "new countries", the shadow curve is always in a country that has previously been explored where  $R_n$  and  $Q_n$  have already been seen to extend. One by one, we may extend  $\gamma$  into new countries, extending  $R_n$  and  $Q_n$  as we go. As long as the point at infinity is not in the new country, equations (3) and (4) reveal that the extensions do not have poles in the new country.

We may categorize the countries on the planet into which we extend  $\gamma$  as follows.  $G_1$  is the "level one" country. Next,  $G_1$  together with all adjacent countries is level 2. The next level is obtained from the previous level by adding to previously explored countries all those adjacent countries which have not yet been explored. The key to the argument is that the shadow curve is always a level behind the exploring curve.

At some stage, when we reach the next level, we will have covered all the countries on the entire Riemann sphere. We shall call this the *last level*. We need it to happen that the point at infinity falls in the interior a country added at the last level. To ensure that this is the case, we may modify our original domain  $\Omega$  using a linear fractional transformation. Indeed, if the point at infinity is not in the interior of the last level domain, then pick any point  $p_0$  that is. Let  $L(z) = 1/(z - p_0)$ , and replace our original domain by  $L(\Omega)$ . This new domain still satisfies the hypothesis of the theorem because linear fractional transformations and their inverses preserve rational functions. Furthermore, the sequence of countries and levels that we constructed earlier is simply picked up and moved by  $L$  on the Riemann sphere.

As we let the exploring curve  $\gamma$  run out to infinity in the country added at the last level via a parametrization  $z(t)$ , the shadow  $\Gamma$  tends to a point in the finite complex plane in the closure of a previously explored country, which is a bounded domain. Equation (3) now shows that  $R_n$  has at worst a pole of order  $n$  at infinity and equation (4) shows that  $Q_n$  has at worst a pole of order 1 at infinity. Since these are the only poles, we conclude that  $R_n$  is a polynomial of degree at most  $n$  and  $Q_n(z) = Az + B$  for some constants  $A$  and  $B$ .

Now, if any function  $Q_n$  were to be the zero function, then identity (3) would yield that  $S(z)$  is a rational function, and it would follow from a theorem of Davis [11] that  $\Omega$  would have to be a disc, and the conclusion of the theorem holds true. So we need only consider the case where  $Q_1$ ,  $Q_2$ , and  $Q_3$  are nonzero. Any three nonzero first degree polynomials are linearly dependent, and so there exist constants,  $c_1$ ,  $c_2$ ,  $c_3$ , not all zero, such that

$$c_1 \overline{Q_1(z)} + c_2 \overline{Q_2(z)} + c_3 \overline{Q_3(z)} \equiv 0.$$

Now, taking this same linear combination of the identities (3), we obtain

$$c_1 R_1(z) + c_2 R_2(z) + c_3 R_3(z) = S(z)(c_1 z + c_2 z^2 + c_3 z^3),$$

and we again see that  $S(z)$  is rational, and Davis' theorem yields that  $\Omega$  must be a disc, and this completes the proof of the theorem.

Now, suppose that  $\Omega$  is simply connected and that a Riemann map  $\varphi : \Omega \rightarrow \mathbb{D}$  is algebraic. The following result characterizes those domains for which the solutions to (1) with rational data are real-algebraic with singularities controlled by those of the Riemann map. To explain this in greater detail, we need to introduce some more notation. We let  $X$  be the (compact) Riemann surface of  $\varphi$  realized as a branched cover  $\pi : X \rightarrow \mathbb{P}$ , where  $\mathbb{P}$  denotes the Riemann sphere (a.k.a. the extended complex plane), and  $\Phi$  the meromorphic function on  $X$  obtained by lifting  $\varphi$  to  $X$  via the projection  $\pi$ . More precisely, there is a simply connected domain  $\Omega \subset X$  and a meromorphic function  $\Phi$  on  $X$  such that  $\pi|_{\Omega}$  is a biholomorphism of  $\Omega \rightarrow \Omega$  and  $\Phi|_{\Omega} = \varphi \circ \pi|_{\Omega}$ . The function  $\Phi$  is called the lift of  $\varphi$ . If  $f(z)$  is any function holomorphic in  $\Omega$ , then there is a unique holomorphic function  $F$ , called the lift of  $f$ , on  $\Omega$  such that  $F = f \circ \pi$ . If  $F$  extends as a meromorphic function on  $X$ , then we say that  $f$  lifts to a meromorphic function on  $X$ . This means, loosely speaking, that  $f$  is an algebraic function and its analytic continuation along curves on  $\mathbb{P}$  can only encounter branch points at points where the Riemann map  $\varphi$  does. Moreover, the branching of  $f$  at such a point is "no more complicated" than that of  $\varphi$ , i.e., the branching order of  $f$  at such a point divides that of  $\varphi$ .

In order to lift real-analytic functions in  $\Omega$ , we introduce the conjugate Riemann surface  $X^*$  as follows:  $X^*$  equals  $X$  as a smooth manifold, but the coordinate charts on  $X^*$  are of the form  $\{U_{\alpha}, \Psi_{\alpha}(\zeta)\}$ , where  $\{U_{\alpha}, \Psi_{\alpha}(\zeta)\}$  are the coordinate charts on  $X$ . This is equivalent to saying that the holomorphic functions on  $X^*$  are of the form  $\overline{H(\zeta)}$  where  $H(\zeta)$  is holomorphic on  $X$ . We embed  $X$  as the diagonal  $D := \{(\zeta, \tau) \in X \times X^* : \tau = \zeta\}$  in  $X \times X^*$ . Observe that  $D$  is a totally real two-dimensional submanifold of the two-dimensional complex manifold  $X \times X^*$ , since  $\tau \mapsto \tau$  is an anti-holomorphic (conjugate of a holomorphic) mapping  $X^* \rightarrow X$ . Thus, we may think of



$\tilde{\Omega}$  as a relatively open subset of  $D \subset X \times X^*$ . If  $v(z, \bar{z})$  is a real-analytic function in  $\Omega$ , then there is a holomorphic function  $V$  in an open neighborhood of  $\tilde{\Omega} \subset D$  in  $X \times X^*$  such that  $V(\zeta, \tau) = v(\pi(\zeta), \bar{\pi}(\tau))$ . We will say that  $v$  lifts as a meromorphic function on  $X \times X^*$  if  $V$  extends as a meromorphic function on  $X \times X^*$ . Observe that if  $v(z, \bar{z})$  is a harmonic function in the simply connected domain  $\Omega$ , then  $v(z, \bar{z}) = f(z) + \overline{g(z)}$ , where  $f$  and  $g$  are holomorphic in  $\Omega$ . In this case,  $V(\zeta, \tau) = F(\zeta) + \overline{G(\tau)}$ , where  $F$  and  $G$  are the lifts of  $f$  and  $g$ , respectively. It follows that  $u$  lifts to  $X \times X^*$  as a meromorphic function if and only if  $f$  and  $g$  lift to  $X$  as meromorphic functions. In this way, we see that if  $u$  lifts as a meromorphic function on  $X \times X^*$ , then  $u(z, \bar{z})$  is real-algebraic and the singularities of  $u$  are controlled (via  $f$  and  $g$ ) by the singularities of the Riemann map  $\varphi: \Omega \rightarrow \mathbb{D}$ . We have the following result.

**THEOREM 5** *Let  $\Omega$  be a simply connected domain in the plane with smooth boundary. Assume that a Riemann map  $\varphi: \Omega \rightarrow \mathbb{D}$  is algebraic and let  $\pi: X \rightarrow \mathbb{P}$  be the Riemann surface of  $\varphi$  realized as a branched cover. Let  $X^*$  denote the conjugate Riemann surface. If the solution  $u(z, \bar{z})$  to (1) lifts as a meromorphic function on  $X \times X^*$  for every  $v \in C(\partial\Omega)$  that is the restriction of a polynomial  $R(z, \bar{z})$ , then the inverse  $\varphi^{-1}: \mathbb{D} \rightarrow \Omega$  is rational (i.e.  $\Omega$  is a quadrature domain).*

For our last result, we need to introduce some more notation. Suppose that  $u(z, \bar{z})$  is a harmonic function in a domain  $G \subset \mathbb{C}$ . Let  $\gamma: [0, 1] \rightarrow G$  be a closed piecewise smooth curve and define the *period* of  $u(z, \bar{z})$  relative to  $\gamma$  by

$$\text{per}(u; \gamma) := \int_{\gamma} *du, \quad (5)$$

where  $*$  is the (Hodge) star operator; i.e.  $*du = -u_y dx + u_x dy$ . (Thus, a local harmonic conjugate of  $u(z, \bar{z})$  is obtained by  $v(z, \bar{z}) := \int_{z_0}^z *du$  for  $z$  in some small disk centered at  $z_0$ .) Observe that  $*du$  is a closed 1-form, and hence the period with respect to a curve  $\gamma$  only depends on the homotopy class of  $\gamma$ . We shall say that  $u(z, \bar{z})$  is *period free* if  $\text{per}(u; \gamma) = 0$  for every closed piecewise smooth curve  $\gamma$  in  $G$ . (Thus, if  $u$  is period free in  $G$ , then  $u$  has a harmonic conjugate in  $G$ ). *The last result we formulate is the following.*

**THEOREM 6** *Let  $\Omega$  be a simply connected domain in the plane with smooth boundary. Assume that a Riemann map  $\varphi: \Omega \rightarrow \mathbb{D}$  is algebraic and that, for every  $v \in C(\partial\Omega)$  that is the restriction of a polynomial  $R(z, \bar{z})$ , there is a discrete subset  $A \subset \mathbb{C}$  (possibly depending on  $v$ ) such that the solution  $u(z, \bar{z})$  to (1) extends as a period-free harmonic function in  $\mathbb{C} \setminus A$ . Then,  $\Omega$  is a disk.*

We remark that the conclusion of Theorem 6 is not true without the assumption that the Riemann map is algebraic. For instance, as mentioned earlier, if  $\Omega$  is an ellipse, then every solution to the Dirichlet problem (1) with polynomial data is a polynomial, and hence extends to  $\mathbb{C}$  as a period-free harmonic function (see e.g., [12]). There are also other domains  $\Omega$  (with nonalgebraic Riemann maps, of course) for which all solutions to the Dirichlet problem with polynomial data extend as period-free harmonic functions to  $\mathbb{C} \setminus A$  for some discrete set  $A$  (see [6]).

The ideas behind the proofs of Theorems 5 and 6 rest on an idea of Hansen and Shapiro [13] of placing “rectangles”  $p_0 = (a, c)$ ,  $q_0 = (b, c)$ ,  $p_1 = (b, d)$ ,  $q_1 = (a, d)$  on

the complexified boundary  $V$  of  $\Omega$  in  $\mathbb{C}^2$ . Here,  $a \neq b, c \neq d$  are complex numbers and if the (algebraic) equation of  $\partial\Omega$  is given by  $P(z, \bar{z}) = 0$  in  $\mathbb{R}^2$ , where  $P$  is an irreducible polynomial, then  $V := \{(z, w) \in \mathbb{C}^2: P(z, w) = 0\}$  represents its complexification in  $\mathbb{C}^2$ , i.e.,  $V \cap \mathbb{R}^2 = \partial\Omega$ . These ideas are closely related to the notion of a closed lightning bolt in  $\mathbb{R}^n$  introduced by Arnold and Kolmogorov to study Hilbert's 13th problem on expressing a function in  $n$  variables as a superposition of functions of fewer variables. We shall refer to [14] for the history of the problem, detailed discussions and relevant references. Here we just very briefly sketch how this notion applies to our situation and the proof of Theorem 5.

A complex "lightning bolt" is a finite set of points (vertices)  $p_0, q_0, p_1, \dots, p_n, q_n$  in  $\mathbb{C}^2$  such that each complex line connecting  $p_j$  to  $q_j$  or  $q_j$  to  $p_{j+1}$  is either "horizontal" or "vertical", i.e., has either its first or second coordinate fixed. A lightning bolt is said to be irreducible if it does not contain a lightning bolt with smaller number of vertices still connecting the first and last vertex ( $p_0$  and  $q_n$ ). A lightning bolt is closed if  $p_0 = q_n$ . Every closed lightning bolt, as is easily seen, has an even number of vertices and supports a finite measure  $\mu$  consisting of charges with alternating signs at the vertices, i.e.,

$$\mu := \sum_{j=0}^n \delta_{p_j} - \sum_{j=0}^n \delta_{q_j},$$

where  $\delta_{p_j}$  (respectively,  $\delta_{q_j}$ ) denotes a unit point mass at the point  $p_j$  (respectively,  $q_j$ ). The measure  $\mu$  is an annihilating measure for all holomorphic functions in  $\mathbb{C}^2$  representable in the form  $f(z) + g(w)$ . Therefore, if a variety  $V$ , representing the complexified boundary of the domain  $\Omega$  in  $\mathbb{C}^2$ , i.e.  $V \cap \mathbb{R}^2 = \partial\Omega$ , supports a closed lightning bolt, there exists a vast set of functions, holomorphic in a neighborhood of  $V$  (even polynomials!), that cannot be approximated by sums of (holomorphic) functions of one variable,  $f(z) + g(w)$ . The crux in the proofs of Theorems 5 and 6 is a construction that produces on the variety  $V$ , a connected component of the complexified boundary of the domain  $\Omega$ , a closed irreducible lightning bolt that carries a measure annihilating all functions  $f(z) + g(w)$ , with  $f, g$ , holomorphic in a neighborhood of  $V$ . In fact, already the existence of a closed lightning bolt on  $V$  would be sufficient, but of course, it is not hard to see that every closed lightning bolt contains an irreducible lightning bolt. Essentially, the technical subtlety of the construction reduces to the following; since  $V$  represents a Riemann surface of degree at least 2, we could, starting at any noncritical point  $p$  of  $V$  construct a lightning bolt by simply going on a horizontal  $\{z = z_0\}$ , or vertical  $\{w = w_0\}$  line from  $p$  until we hit  $V$  again and then proceed at each step changing the "type" of the line emanating from a newly obtained vertex to the opposite from the type of the complex line on which we have arrived at the vertex, of course, avoiding critical values and critical points of  $V$ , a finite set. The difficulty is to show that the process will terminate rather than produce a lightning bolt with infinitely many vertices running away to infinity. To achieve this we have to resort to the specific construction of a rather special family of grids of points obtained as orbits of a special finite subgroup of the monodromy group with two generators, to prevent an associated lightning bolt "running away" to infinity.

Let us conclude this article by sketching a proof of the following simplified version of Theorem 5. If  $p$  is a polynomial, we shall denote by  $p^*$  the polynomial obtained from  $p$



by conjugating all the coefficients and use similar notation for rational functions as well.

**PROPOSITION 7** *Let  $\Omega$  be a smoothly bounded Jordan domain in  $\mathbb{C}$  such that the Riemann map  $\varphi: \Omega \rightarrow \mathbb{D}$  is rational, i.e.  $\varphi(z) = p(z)/q(z)$  where  $p, q$  are irreducible polynomials. Assume, in addition, that the complexified variety  $W := \{(z, w): Q(z, w) := p(z)p^*(w) - q(z)q^*(w) = 0\}$  is irreducible, and hence coincides with the complexified boundary  $V$  of  $\partial\Omega$ . Suppose that the solution  $u$  of (1) is rational for every  $v \in C(\partial\Omega)$  that is the restriction of a rational function whose polar variety does not meet  $\partial\Omega$ . Then,  $\Omega$  is a disk.*

**Remark 8** The assumption that the Riemann map  $\varphi$  is rational means, of course, that the Riemann surface  $X$  in Theorem 5 is just the Riemann sphere and  $\pi$  is the identity. The conclusion in Theorem 5, under the additional assumption that  $\varphi$  is rational, implies that  $\Omega$  is a disk. For, if both  $\varphi$  and  $\varphi^{-1}$  are rational, then  $\varphi$  is a linear fractional transformation and, hence,  $\Omega$  is a disk. The crucial hypothesis that  $W = V$  in the above proposition simplifies matters significantly. Unfortunately, it is often not the case for domains  $\Omega$  with algebraic boundaries, and the proofs then become much less transparent and more technical [1].

*Sketch of Proof of Proposition 7* Suppose that  $\Omega$  is not a disk, i.e., the degree  $m$  of the rational function  $\varphi$  is at least 2. Construct a family of closed lightning bolts with four vertices (Hansen–Shapiro “rectangles”) on the variety  $W$  which, according to the hypothesis in our simplified version, coincides with  $V$ , the complexified boundary  $\partial\Omega$ , as follows. Let  $\zeta$  be a point in the complex plane which is a noncritical value for the rational function  $\varphi$ , i.e. the set  $\{\varphi^{-1}(\zeta)\}$  consists of  $m$  distinct points. We can also assume that neither  $\{\varphi^{-1}(\zeta)\}$  nor  $\{(1/\varphi^*)^{-1}(\zeta)\}$  contain  $\infty$ . Choose  $a \neq b$  in  $\{\varphi^{-1}(\zeta)\}$  and  $c \neq d$  in  $\{(1/\varphi^*)^{-1}(\zeta)\}$ . We have  $\varphi(a) = \varphi(b) = 1/\varphi^*(c) = 1/\varphi^*(d)$ . Moreover, for  $(z, w)$  such that  $z$  is not a pole of  $\varphi$  and  $w$  is not a pole of  $1/\varphi^*$ ,  $Q(z, w) = 0$  is equivalent to  $\varphi(z) = 1/\varphi^*(w)$ , and hence  $M := \{A = (a, c), B = (a, d), C = (b, c), D = (b, d)\}$  is a “rectangle” on the variety  $V = W = \{(z, w): Q(z, w) = 0\}$ . Moreover, there is a whole continuum of such rectangles on  $V$ . Therefore, for every rational harmonic function  $u$  representable in  $\mathbb{C}^2$  by  $u(z) = f(z) + g(w)$ , where  $f, g$  are rational functions of one variable, we can always find a closed rectangle  $M$  with vertices  $A, B, C, D$  on  $V$  such that all vertices stay away from the poles of either  $f$  or  $g$  on  $V$ . Then, as is readily checked,  $u(A) + u(C) = u(B) + u(D)$  holds for all such  $u$ . Thus, taking the data  $v$  for the Dirichlet problem (1) to be a polynomial in  $z$  and  $w$  such that  $v(A) = v(B) = v(C) = 0$  while  $v(D) = 1$  (one can easily see that  $v$  can be chosen to be a quadratic polynomial), we arrive at a contradiction. Hence, the degree of  $\varphi$  must be 1 and  $\Omega$  is a disk. The proposition is now proved. ■

Finally, we note that it follows from the argument above that the conclusion of the proposition already holds if one only assumes that the solution  $u$  of (1) is rational for every data  $v$  that is the restriction to  $\partial\Omega$  of a quadratic polynomial.

## Acknowledgements

The authors Bell, Ebenfelt, and Khavinson were partially supported by the NSF grants DMS-0305958, DMS-0401215, and DMS-0139008 respectively.

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