A Survey of Linear Extremal Problems in Analytic Function Spaces

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Abstract. The purpose of this survey paper is to recall the major benchmarks of the theory of linear extremal problems in Hardy spaces and to outline the current status and open problems remaining in Bergman spaces. We focus on the model extremal problem of maximizing the norm of the linear functional associated with integration against a polynomial of finite degree, and discuss known solutions of particular cases of that problem. We examine duality and its application in both Hardy and Bergman spaces. Finally, we discuss some recent progress on the finiteness of the Blaschke product of the extremal solution in Bergman spaces.

1. Introduction and Historical Remarks

Solving extremal problems has been one of the major stimuli for progress in complex analysis, starting with the Schwarz lemma in the late 19th century, followed by work on coefficients of bounded analytic functions by C. Carathéodory and L. Fejér, Landau, Szasz, and others. At the end of the First World War, F. Riesz considered a best approximation problem in the Hardy space H^1, and in 1926, Szasz associated this problem with a dual problem in H^∞. This duality was rediscovered by Geronimus and, in a more general framework, by Krein in 1938. Extremal problems in multiply connected domains were studied by Grunsky (1940), Heins (1940), Robinson (1943), Goluzin (1946), and Ahlfors (1947). Macintyre and Rogosinski (1950) gave a detailed survey of results related to extremal problems involving coefficients of functions in all Hardy classes. Systematic use of duality in linear extremal problems for analytic functions started with S. Ya. Khavinson (1949) and independently Rogosinski and Shapiro (1953). Further studies were undertaken by Bonsall, Royden, Read, Adamyan, Arov, Krein, Walsh, among others. For a full account of the history of the development of extremal problems and references, see [**18**, pp. 51–57].

Work on Bergman spaces began with Ryabych in the early 1960s, who started the investigation of the existence and regularity of solutions ([**21, 22**]). In 1991, Osipenko and Stessin ([**19**]) solved an explicit optimization problem in Bergman spaces involving linear combinations of the value of a function and its derivative

2000 *Mathematics Subject Classification.* Primary: 30H05, Secondary:30E05 .

Both authors would like thank the Institut Mittag-Leffler (Djursholm, Sweden) for support while working on this project. The second author also gratefully acknowledges partial support from the National Science Foundation under the grant DMS-0855597.

at a particular point in the disk. The theory of contractive divisors in Bergman spaces, initiated by Hedenmalm ([**7**]), followed by Duren, D. Khavinson, Shapiro and Sundberg ([**2, 3**]), prompted a burst of activity in Bergman spaces which gave insight into the structure of the z-invariant subspaces of Bergman spaces. These developments are recorded in two books on Bergman spaces ([**4, 8**]). In 1997, D. Khavinson and Stessin made a deeper study of linear extremal problems in Bergman spaces ([**9**]). Ferguson gave a simpler proof of Ryabych's regularity results and generalized them in 2009 and 2010 ([**5**]). The paper [**26**] contains a nice discussion of results on extremal problems in Bergman spaces.

The purpose of this survey is to recall the major benchmarks of the theory in Hardy spaces and to outline the current status of developments in Bergman spaces as well as the obstacles that still remain there. The plan of the paper is as follows: we begin in Section 2 by defining Bergman spaces, state a model extremal problem, and investigate the existence and uniqueness of extremals. In Section 3, we give examples and known solutions of that extremal problem in special cases. Section 4 discusses the Duality Theorem, and in Section 5, we apply duality to see how to get the solutions in Hardy spaces. In Section 6, we tackle the Bergman space case, discuss the difficulties and examine the connection with partial differential equations. In Section 7, we give a proof of a new result that the Blaschke product of the extremal solution for Bergman spaces A^p for p close to 2 is finite.

2. A Model Extremal Problem in Bergman Spaces

Let us begin by examining a model extremal problem in the Bergman space.

Definition 2.1. For $0 < p < \infty$, define the Bergman space as

$$A^p = \left\{ f \text{ analytic in } \mathbb{D} : \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{\frac{1}{p}} =: \|f\|_{A^p} < \infty \right\},$$

where $dA(z) = \frac{1}{\pi} dxdy$ denotes normalized area measure in the unit disk $\mathbb{D}$, $z = x + iy$.

Consider the following **model extremal problem:** Fix $1 \leq p < \infty$. Given a non-zero polynomial

$$\omega(z) = \sum_{k=0}^{N} a_k z^k,$$

describe the extremal solutions of the problem:

$$\lambda_p := \sup \left\{ Re \left(\int_{\mathbb{D}} f(z)\overline{\omega(z)}\, dA(z) \right) : \|f\|_{A^p} \leq 1 \right\} \tag{2.1}$$

or, equivalently,

$$\sup \left\{ Re \left(\sum_{0}^{N} c_k f^{(k)}(0) \right),\, c_k = \frac{a_k}{(k+1)!}, \|f\|_{A^p} \leq 1 \right\}. \tag{2.2}$$

Solving Problem (2.1) is equivalent to solving the following problem:

$$\inf \left\{ \|F\|_{A^p} : \int_{\mathbb{D}} F\overline{\omega} dA = 1 \right\}, \tag{2.3}$$

since it is easily checked that F^* is a solution to (2.3) if and only if f^* is a solution to (2.1), where $F^* = f^*/\lambda_p$. Let us examine questions of existence and uniqueness.

For $1 \le p < \infty$, if $\{F_n\}$ is a sequence of A^p functions approaching the infimum in (2.3), then their A^p norms are bounded, and thus, thinking of these functions as linear functionals on A^q for $1/p + 1/q = 1$, by the weak* compactness of bounded sets in A^p, there exists a function $F^* \in A^p$ and a subsequence F_{n_k} of F_n such that F_{n_k} approaches F^* weak*. In particular,

$$1 = \int_{\mathbb{D}} F_{n_k}\overline{\omega} dA \to \int_{\mathbb{D}} F^*\overline{\omega} dA,$$

and therefore

$$\int_{\mathbb{D}} F^*\overline{\omega} dA = 1,$$

and of course, $F_{n_k} \to F^*$ pointwise. Finally, by Fatou's theorem, $\|F^*\|_{A^p} \le \liminf \|F_{n_k}\|_{A^p}$, and therefore

$$\|F^*\|_{A^p} = \inf\left\{ \|F\|_{A^p} : \int_{\mathbb{D}} F\overline{\omega} dA = 1 \right\},$$

as desired.

If $p = 1$, the argument is similar but slightly more delicate, since to use weak* compactness, we must think of A^1 as a subset of the set of complex measures on $\overline{\mathbb{D}}$. In this case, for a sequence $\{F_n\}$ of A^1 functions approaching the infimum in (2.3), the measures $F_n dA$ form a bounded sequence of measures on the disk, and therefore, by weak* compactness of bounded measures on $\overline{\mathbb{D}}$, there exists a measure $d\mu^*$ such that some subsequence $F_{n_k} dA$ approaches $d\mu^*$ weak*, that is

$$\int_{\mathbb{D}} F_{n_k} f dA \to \int_{\mathbb{D}} f d\mu,$$

for every f continuous in $\overline{\mathbb{D}}$. We now appeal to a version of the F&M Riesz theorem for A^1 proved by H. Shapiro ([**23, 24**]), which can be stated as follows.

THEOREM 2.2. *Let Ω be any bounded open set with smooth boundary, and let $M(\overline{\Omega})$ be the Banach space of bounded complex measures on $\overline{\Omega}$, and suppose $f_n \in A^1(\Omega)$ is a sequence of functions such that $f_n dA \to d\mu$ weak*, for some $\mu \in M(\overline{\Omega})$. Then there exists $f \in A^1(\Omega)$ such that $d\mu = f dA$.*

Here, $A^1(\Omega)$ is naturally defined as the space of integrable analytic functions in the domain Ω. Note that in the original statement of this theorem, the domain Ω is allowed to have non-smooth boundary points, and then the limit measure is of the form $f d\mu + d\nu$, where ν a singular measure supported on these non smooth boundary points. See pp. 75 – 76 of [**24**] for details.

Now, getting back to the proof of existence, we see that by Theorem 2.2 applied to $\mathbb{D}$, the measure μ^* in question is absolutely continuous, and therefore, there exists a function F^* such that the measures $F_{n_k} dA$ approach $F^* dA$ weak*. The rest of the argument is the same as for $p > 1$, since ω is continuous.

Finally, the Bergman spaces A^p are strictly convex for all $1 \le p < \infty$, (see, for example, [**4**, pp. 28–29]), which implies that there can only be one element of minimal norm satisfying $\int_{\mathbb{D}} F\overline{\omega} dA = 1$. Therefore the solution to Problem 2.3 and therefore to Problem 2.1 is unique. Note that for $1 < p < \infty$, the argument showing existence and uniqueness of an extremal for the model problem considered here immediately extends to $\omega \in A^q$, for $1/p + 1/q = 1$.

The main thrust of this work is to study the smoothness properties of the extremal functions. It is always expected that the solution of a "nice" extremal problem is much better than the generic function in the space. In particular, for ω a polynomial as in the above Model Problem (2.1), it turns out that the extremal solution f^* is a bounded analytic function, continuous in the closed disk. However, obtaining this regularity already involves significant technical difficulty and requires the use of deep results from nonlinear partial differential equations ([**22, 5, 9**]).

Let us now turn to some examples of known solutions to the Model Problem for particular cases.

3. Examples of known solutions

For some particular polynomials or for $p = 2$, the solutions to Model Problem 2.1 are known explicitly and are discussed briefly here.

EXAMPLE 3.1. The simplest example is that of A^2. In that case, by the Cauchy-Schwarz inequality, the supremum

$$\sup\left\{Re\left(\int_{\mathbb{D}} f(z)\overline{\omega(z)}\, dA(z)\right) : \|f\|_{A^p} \le 1\right\}$$

is attained by the function $f^* = \omega/\|\omega\|_{A^2}$, for any polynomial ω, and in fact, for any $\omega \in A^2$. Henceforth, we will assume that $p \neq 2$.

EXAMPLE 3.2. If $\omega = z^N$, then Ryabych showed ([**21**]) that the extremal function for

$$\sup\left\{Re\, f^{(N)}(0) : \|f\|_{A^p} \le 1\right\}$$

is $f^*(z) = \left\{\frac{Np+1}{2}\right\}^{1/p} z^N$.

EXAMPLE 3.3. If we would like to consider Example 2 in a more general setting by estimating values of a function at a point β instead of at the origin, we can allow ω to be a simple rational function, namely, $\omega(z) = (1 - \overline{\beta}z)^{-2}$, for $|\beta| < 1$, the Bergman kernel at β. In that case, Problem (2.1) becomes

$$\sup\{Re\, f(\beta) : \|f\|_{A^p} \le 1\}.$$

This problem was studied by Ryabych in [**21**] and also by Vukotić in [**25**], and the extremal solution has the form

$$f^*(z) = (1 - |\beta|^2)^{2/p}(1 - \overline{\beta}z)^{-4/p}.$$

EXAMPLE 3.4. In 1991, Osipenko and Stessin ([**19**]) considered the case of a simple linear polynomial, $\omega(z) = a_0 + a_1 z$. Problem (2.1) then becomes

$$\sup\left\{Re(c_0 f(0) + c_1 f'(0)) : \|f\|_{A^p} \le 1\right\}.$$

They showed that

$$f^*(z) = C\frac{z-\beta}{1-\overline{\beta}z}(1-\overline{\alpha}z)^{2/p}$$

where $|\beta| \le 1$, $|\alpha| \le 1$, and C is a constant. They also wrote down equations relating c_0 and c_1 to β and α. In addition, they considered the case when 0 is replaced by an arbitrary point $\zeta \in \mathbb{D}$, i.e., when ω is a linear combination of the Bergman kernel and its derivative. Their results were already technically quite difficult, and there doesn't seem to be much hope of generalizing their approach, which was based on "guessing" the exact form of the extremal, to higher degree polynomials.

EXAMPLE 3.5. Problems of the type considered in Examples 1 through 4 are connected to what are often called Carathéodory-Fejér type problems. An important example is the problem of finding, for given $|\beta_j| < 1$ for $j = 1, \dots, m$:

$$\inf\{\|f\|_{A^p} : f^{(N)}(0) = 1, f(0) = \cdots = f^{(N-1)}(0) = f(\beta_1) = \cdots = f^{(k_m)}(\beta_m) = 0\}. \tag{3.1}$$

This problem is equivalent to the problem of finding, for given $|\beta_j| < 1$ for $j = 1, \dots, m$:

$$\sup\{Re\, f^{(N)}(0) : \|f\|_{A^p} \le 1, f(0) = \cdots = f^{(N-1)}(0) = f(\beta_1) = \cdots = f^{(k_m)}(\beta_m) = 0\}, \tag{3.2}$$

whose extremal solution is by definition a "contractive divisor" in A^p. Contractive divisors in Bergman spaces were discovered and studied in the 1990s, first by Hedenmalm ([**7**]) for $p = 2$, and then by Duren, Khavinson, Shapiro, and Sundberg ([**2, 3**]) for all p, and later by MacGregor and Stessin ([**12**]). Contractive divisors turned out to be intimately connected to the theory of z invariant subspaces in Bergman spaces. For more on this subject, see [**4, 8**].

The extremal functions are

$$f^*(z) = Cz^N \prod_1^m \left\{ \frac{z - \beta_j}{1 - \overline{\beta_j} z} \right\}^{k_j} R(z)^{2/p},$$

where R is a rational function with poles at ∞ and at $\{1/\bar{\beta}_j\}_{j=1}^m$ of degree less than or equal to $2N + \sum_{j=1}^m k_j$. The problem is equivalent to problem (2.1) with ω being a specific linear combination of the Bergman kernel $(1 - \overline{\zeta} z)^{-2}$ and its $\frac{\partial}{\partial \overline{\zeta}}$ derivatives at the β_j.

All of these particular examples hint at a more general theory and a simple form of the extremal solutions. Let us now examine this question and turn to the tools that helped establish the qualitative form of the solutions in spaces simpler than the Bergman spaces.

4. Duality

In the late 1940s and early 1950s, the systematic use of duality to solve extremal problems in complex function theory became prevalent. The simplest form of a duality statement is the following.

THEOREM 4.1 (see, e.g., [**18**] p. 2). *Let X be a normed linear space, and let X^* be the space of bounded linear functionals on X. Suppose $E \subset X$ is a subspace of X and l_0 is a given linear functional in X^*. Then*

$$\sup\{|l_0(f)| : f \in E, \|f\| \le 1\} = \inf\left\{\|l_0 - l\| : l \in E^\perp\right\} = \|l_0 - l^*\|$$

for some $l^* \in E^\perp$, where $E^\perp$ is the annihilator of E in X^*, defined to be the set $\{l \in X^* : l(f) = 0\ \forall f \in E\}$.

Duality thus pairs a linear extremal problem with a problem of best approximation. This pairing allows one to gain a deeper understanding of each one in turn, as we shall see. The short proof follows immediately from the Hahn-Banach theorem and is included to emphasize how the best approximation in the dual problem appears.

PROOF. First note that for any $l \in E^{\perp}$ and for any $f \in E$ with $\|f\| \leq 1$, we have

$$|l_0(f)| = |(l_0 - l)(f)| \leq \|l_0 - l\|.$$

Therefore,

$$\|l_0\|_E = \sup \{|l_0(f)| : f \in E, \|f\| \leq 1\} \leq \inf \left\{\|l_0 - l\| : l \in E^{\perp}\right\}.$$

Now by the Hahn-Banach theorem, there exists a linear functional L on X such that $L|_E = l_0$ and $\|L\|_X = \|l_0\|_E$. Therefore $(l_0 - L) \in E^{\perp}$, and

$$\|l_0\|_E \leq \inf \left\{\|l_0 - l\| : l \in E^{\perp}\right\} \leq \|l_0 - (l_0 - L)\|_X = \|L\|_X,$$

and therefore the chain of inequalities is actually a chain of equalities. Putting $l^* = l_0 - L$ gives the desired result. □

One immediate interesting note is that the best approximation l^* therefore always exists, even in this general setting. Whether the supremum is actually achieved and gives rise to an extremal $f^* \in E$ is not always the case and depends on the spaces and linear functionals being considered. Finally, in order to be able to use the duality theorem, we need to have information about the annihilator spaces. To see how this works, let us, in the following section, apply duality to our model problem, but in Hardy spaces. (See, for example, [**17, 1**] for details.)

5. Duality applied to H^p-theory

Although Bergman spaces are defined in a natural way, the functions in those spaces are quite complicated, and their structure is not entirely understood. For example, Bergman space functions need not have any radial limits on the unit circle. On the other hand, the Hardy spaces are classical spaces whose structure is quite well understood, although their definition is in some sense less natural. We begin this section by defining the Hardy spaces.

DEFINITION 5.1. For $0 < p < \infty$, define the Hardy space as

$$H^p := \left\{ f \text{ analytic in } \mathbb{D} : \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta =: \|f\|_{H^p}^p < \infty \right\}.$$

The Hardy space of bounded analytic functions is defined by

$$H^{\infty} = \left\{ f \text{analytic in } \mathbb{D} : \|f\|_{\infty} := \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.$$

It is well-known that Hardy space functions f have radial limits $\lim_{r \to 1^-} f(re^{i\theta}) =: f(e^{i\theta})$ for almost every $\theta \in \mathbb{T}$, and that $\|f\|_{H^p} = \|f(e^{i\theta})\|_{L^p}$, where L^p is the usual space of measurable and p integrable functions on the circle. For $\omega(z) = \sum_{k=0}^{N} a_k z^k$, let us now consider our model problem in the *Hardy space*, for $1 \leq p < \infty$, of finding

$$\lambda := \sup \left\{ Re \int_0^{2\pi} f(e^{i\theta}) \overline{\omega(e^{i\theta})} \frac{d\theta}{2\pi}, \|f\|_{H^p} \leq 1 \right\}. \tag{5.1}$$

In order to apply the Duality Theorem 4.1 to this setting, we will consider the space $E = H^p$ as a subspace of the space $X = L^p(\mathbb{T})$. Then the dual space X^* is L^q, where $1/p + 1/q = 1$, and the annihilator space $E^{\perp}$ is the set of L^q functions g such that

$$\int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta = 0$$

for every $f \in H^p$. In particular,

$$\int_0^{2\pi} e^{in\theta}\overline{g(e^{i\theta})}d\theta = 0$$

for every integer $n \geq 0$, and therefore, by the classical F & M Riesz theorem, (see, for example, [**1**, p. 41]), $\overline{g} \in H^q$ and vanishes at the origin. (We will write the space of such functions as H^q_0.) Therefore, Problem (5.1) above can be written, in its dual form, as:

$$\lambda = \min\left\{\|\overline{\omega} - \phi^*\|_{L^q(\mathbb{T})}, \phi^* \in H^q_0\right\}.$$

By the general duality theorem, we know the best approximation ϕ^* exists, and one can also show that ϕ^* is unique whenever f^* exists. It turns out that the extremal f^* for the supremum problem exists and is unique for $p > 1$, but for $p = 1$ in general this is not the case: in fact, the existence of f^* here for $p = 1$ follows from the continuity of ω (as in the A^1 case discussed in Section 2). See [**18**], Sections 4 and 6, for more details on questions of existence and uniqueness in the Hardy space case.

Now given these extremals f^* and ϕ^*, notice that we have the following chain of inequalities, which must actually be equalities:

$$\begin{aligned}
\lambda = Re\int_0^{2\pi} f^*(e^{i\theta})\,\overline{\omega(e^{i\theta})}\frac{d\theta}{2\pi} &= Re\int_0^{2\pi} f^*(e^{i\theta})\,(\overline{\omega(e^{i\theta})} - \phi^*(e^{i\theta}))\frac{d\theta}{2\pi}\\
&\leq \int_0^{2\pi} |f^*(e^{i\theta})||\,\overline{\omega(e^{i\theta})} - \phi^*(e^{i\theta})|\frac{d\theta}{2\pi}\\
&\leq \|f\|_{H^p}\|\overline{\omega} - \phi^*\|_{L^q}\\
&\leq \|\overline{\omega} - \phi^*\|_{L^q}\\
&= \lambda.
\end{aligned}$$

Therefore, because we have equality in Hölder's inequality above, we must have that

$$\lambda^q|f^*(e^{i\theta})|^p = |\overline{\omega(e^{i\theta})} - \phi^*(e^{i\theta})|^q$$

almost everywhere on the circle. Taking q-th roots gives that

$$\lambda|f^*(e^{i\theta})|^{p/q} = \lambda|f^*(e^{i\theta})|^{p-1} = |\overline{\omega(e^{i\theta})} - \phi^*(e^{i\theta})|$$

almost everywhere on $\mathbb{T}$. In addition, for equality above, we must have that $\|f\|_{H^p} = 1$ and

$$f^*(e^{i\theta})(\overline{\omega(e^{i\theta})} - \phi^*(e^{i\theta})) \geq 0$$

a.e. on the circle. Putting these together gives an equivalent characterization of extremality of f^*, ϕ^* as:

$$\lambda\frac{|f^*(e^{i\theta})|^p}{f^*(e^{i\theta})} = \overline{\omega(e^{i\theta})} - \phi^*(e^{i\theta})\,a.e.\text{ on }\ \mathbb{T}. \tag{5.2}$$

Equivalently, the function $\lambda\frac{|f^*|^p}{f^*} - \overline{\omega}$ annihilates H^p.

Now notice that the function $f^*(z)(\overline{\omega(z)} - \phi^*(z))$ is well-behaved inside the disk, in a neighborhood of the unit circle, since $f^* \in H^p$ and $\phi^* \in H^q$, and for $|z| = 1$, $\overline{\omega(z)} = \overline{a_0} + \sum_{k=1}^N \frac{\overline{a_k}}{z^k}$. Therefore the product $f^*(\overline{\omega} - \phi^*)$ can be thought of as a function that is in the Hardy class H^1 in an annulus inside the unit disk. Moreover, this function is positive (hence real) on the circle. Thus, one can apply the Schwarz reflection principle (see, for example, [**16**, pp. 183-185]) to extend the product to $\mathbb{C}\setminus\{0\}$. Moreover, at $z=0$, the singularity is given by the behavior of $\overline{\omega}$, and hence is a pole of order N, and therefore by reflection a similar behavior occurs at ∞. Hence $f^*(\overline{\omega}-\phi^*)$ is a rational function on the Riemann sphere with two poles, each of order N at 0 and ∞. Moreover, since $f^*(\overline{\omega}-\phi^*)$ is positive on the unit circle, by the argument principle, the increment of the argument of the product is 0 on the circle, and hence the product has $2N$ zeros, which by the symmetry required by the reflection principle, must be divided equally and symmetrically inside and outside the disk. (Zeros on the circle come with even multiplicity.) Hence,

$$f^*(z)(\overline{\omega(z)}-\phi^*(z)) = Cz^{-N}\prod_{j=1}^{N}(z-\alpha_j)(1-\overline{\alpha_j}z), \quad |\alpha_j|\le 1$$

for some constant C. In particular, f^* has no singular part, and the zeros of f^* are among the α_j such that $|\alpha_j|<1$, and therefore the Blaschke factor of f^* is

$$\prod_{j=1}^{k}\frac{z-\alpha_j}{1-\overline{\alpha_j}z},$$

where $k\le N$. Finally, the outer part of f^* is determined by the equation on the boundary that requires

$$\begin{aligned}
\lambda|f^*(z)|^p &= f^*(z)(\overline{\omega(z)}-\phi^*(z))\\
&= Cz^{-N}\prod_{1}^{N}(z-\alpha_j)(1-\overline{\alpha_j}z)\\
&= C\prod_{1}^{N}\left[\frac{z-\alpha_j}{z}\right](1-\overline{\alpha_j}z)\\
&= C\prod_{1}^{N}(1-\alpha_j\overline{z})(1-\overline{\alpha_j}z)\\
&= C\prod_{1}^{N}|1-\overline{\alpha_j}z|^2.
\end{aligned}$$

Therefore the extremal f^* is of the form

$$f^*(z) = C\prod_{1}^{k}\frac{z-\alpha_j}{1-\overline{\alpha_j}z}\prod_{1}^{N}(1-\overline{\alpha_j}z)^{2/p}, \; k\le N, \tag{5.3}$$

giving a qualitative solution to the extremal problem.

Let us now turn to a discussion of this problem in Bergman spaces.

6. Extremal problems in Bergman spaces

Recall the problem of finding and describing the extremal solutions of

$$\lambda_p := \sup\left\{Re\left(\int_{\mathbb{D}} f(z)\overline{\omega(z)}\,dA(z)\right) : \|f\|_{A^p}\le 1\right\}, \tag{6.1}$$

where ω is a polynomial of degree N. As seen in Section 3, for each $1<p<\infty$, the solution f^* exists and is unique. For $p=2$, the solution is $f^*=\omega/\|\omega\|$ by the Cauchy-Schwarz inequality. For $1\le p<\infty$, and even for any $\omega\in H^q, q:$ $1/p+1/q=1$, Ryabych proved ([**22**]) that the solution is in H^p, and, in particular, has boundary values at almost every point of the unit circle $\mathbb{T}$. T. Ferguson ([**5**]) recently refined that proof.

Moreover, deeper and more technical results in [**9**] imply that for $1 < p < \infty$, the solution f^* of (2.1), with ω a polynomial, is continuous in the closed disk, in fact is Lip $(\gamma, \overline{\mathbb{D}})$ for some γ that depends on p ([**9**]). These results support the philosophy that extremal functions are much better than the generic functions of the space where the problem is set. The most that is known regarding the extremal solution to date is the following theorem of D. Khavinson and M. Stessin ([**9**]):

THEOREM 6.1. *Let ω be a polynomial of degree N and let $1 < p < \infty$. The extremal solution f^* to*

$$\lambda_p := \sup\left\{Re\left(\int_{\mathbb{D}} f(z)\overline{\omega(z)}\,dA(z)\right) : \|f\|_{A^p} \le 1\right\}, \tag{6.2}$$

is in $Lip(\gamma, \overline{\mathbb{D}})$ and has the form

$$f^*(z) = C\prod_{j=1}^{\infty}\frac{\beta_j - z}{1-\overline{\beta_j}z}\frac{|\beta_j|}{\beta_j}\prod_{j=1}^{N}(1-\overline{\alpha_j}z)^{2/p},$$

where C, $|\beta_j| < 1$, and $|\alpha_j| \le 1$ are constants, and the β_j can only accumulate to those values of α_l that lie on the circle.

One of the missing pieces of the theory of extremal problems in Bergman spaces versus that in Hardy spaces is information about the Blaschke product. We thus state the following:

CONJECTURE 6.1. The Blaschke product of the extremal solution has at most N factors.

In the next section, we will show some progress in this direction, but the full conjecture is still open.

Let us now examine some key steps in the proof of Theorem 6.1 and see how the theory of PDEs comes into play. We will discuss here only the case $p > 1$, since for $p = 1$, these methods break down. The validity of Theorem 6.1 for $p = 1$ remains unknown.

Recall that we have already shown that there exists a unique solution f^* to Problem (6.1). In order to apply duality, we consider A^p as a subspace of $L^p(\mathbb{D})$, and identify the annihilator of A^p by appealing to Khavin's lemma (see [**24**, Lemma 4.2 p. 26] and [**6**]).

LEMMA 6.2. *The annihilator of $A^p(\mathbb{D})$ inside $L^p(\mathbb{D})$ can be described by*

$$(A^p)^{\perp} \cong \left\{\frac{\partial v}{\partial \bar z} : v \in W_0^{1,q}(\mathbb{D})\right\},$$

where $W_0^{1,q}(\mathbb{D})$ is the Sobolev space of functions vanishing on $\mathbb{T}$ and with gradients in $L^q(\mathbb{D})$, with $1/p + 1/q = 1$ and $\frac{\partial v}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

By following the same reasoning as in Section 5, and setting, for the sake of brevity, $\lambda := \lambda_p$, we then obtain that

$$\lambda\frac{|f^*|^p}{f^*} - \overline{\omega} = \frac{\partial u^*}{\partial \bar z} \text{ a. e. in } \mathbb{D}, \tag{6.3}$$

where $u^* \in W_0^{1,q}$. Now set $v(z) := u^*(z) + \overline{\Omega(z)}$, $\Omega(z) := \int_0^z \omega(\zeta)d\zeta$, so that

$$\frac{\partial v}{\partial \bar{z}} = \lambda \frac{|f^*|^p}{f^*}.$$

Then v solves the *nonlinear* boundary value problem:

$$\begin{aligned} \frac{\partial}{\partial z}(|v_{\bar{z}}|^{q-2} v_{\bar{z}}) &= 0 \text{ in } \mathbb{D}; \\ v &= \overline{\Omega} \text{ on } \mathbb{T}. \end{aligned} \tag{6.4}$$

By results of Ch. Morrey, O. Ladyzhenskaya, and N. Uraltseva (see the discussion in [**9**] and [**13, 14, 15, 10**]), the unique solution v of (6.4) belongs to $C^{1+\beta}(\overline{\mathbb{D}}), \beta = \beta(q)$. Since

$$f^* = \lambda^{q-1} \frac{|v_{\bar{z}}|^q}{v_{\bar{z}}},$$

we get that $f^* \in \text{Lip}(\gamma, \overline{\mathbb{D}}), \gamma = \gamma(p)$. (See [**9**] for details.)

Remark 6.3. For values of p in any compact subset of $(1, \infty)$, the corresponding extremals f^* can all be taken to be $\text{Lip}(\gamma, \overline{\mathbb{D}})$ for the same γ. This will turn out to be key later (see Section 7) when estimating the number of zeros of extremal functions.

Note that Conjecture 6.1 is equivalent to the following:

Conjecture 6.2. The solution of the BVP (6.4)

$$\begin{aligned} \frac{\partial}{\partial z}(|v_{\bar{z}}|^{q-2} v_{\bar{z}}) &= 0 \text{ in } \mathbb{D}; \\ v &= \overline{\Omega} \text{ on } \mathbb{T}. \end{aligned} \tag{6.5}$$

has at most N critical points in $\mathbb{D}$, that is, points where $\frac{\partial v}{\partial \bar{z}} = 0$.

J. Lewis ([**11**]) has proved a real-valued version of this result for q-Laplacians.

Now let us examine the outer part F of the extremal. We would like to show that $|F|^p = |f^*|^p$ on $\mathbb{T}$ is a positive trigonometric polynomial of degree less than or equal to N. First notice that the Hardy-Littlewood theorem ([**1**, Theorem 5.1]) implies that since $f^* \in \text{Lip}(\gamma, \overline{\mathbb{D}})$, $(f^*)' \in A^{p_1}$, for some $p_1 > 1$. By duality, we get the so-called *orthogonality relationships*

$$\int_{\mathbb{D}} \left\{ \lambda \frac{|f^*|^p}{f^*} - \overline{\omega} \right\} g dA = 0$$

for all $g \in A^1$. Note that, since ω is of degree N, $\int_{\mathbb{D}} \overline{\omega} z^{N+k} g dA = 0$ for any integer $k \geq 1$ and any $g \in A^1$. Now, Green's formula gives for every integer $k \geq 1$:

$$\begin{aligned} \int_{\mathbb{T}} |f^*|^p z^{N+k} d\theta = i \int_{\mathbb{T}} |f^*|^p z^{N+k+1} d\bar{z} &= p \int_{\mathbb{D}} \frac{|f^*|^p}{f^*} (f^*)' z^{N+k+1} dA \\ &+ 2 \int_{\mathbb{D}} \frac{|f^*|^p}{f^*} (N+k+1) z^{N+k} f^* dA. \end{aligned} \tag{6.6}$$

Each of these last two terms is 0, by the orthogonality relationships, and therefore, for each $k \geq 1$,

$$\int_{\mathbb{T}} |f^*|^p z^{N+k} d\theta = 0,$$

showing that the outer part of f^* is the $2/p$-th root of a polynomial of degree at most N. We therefore get the desired form of the extremal f^* as in Theorem 6.1.

The problem of showing that the Blaschke factor has at most N terms, or, even, indeed, is finite, still remains. We will discuss this more in the next section. Notice, though, that instead of Problem (6.1), we can consider the problem with ω a rational function, to get point evaluations at points other than the origin. More specifically, given $\omega(z) := \sum_{k=1}^{N} \frac{a_k}{(1-\overline{w_k}z)^2}$, $|w_k| < 1$, a linear combination of Bergman reproducing kernels, our problem becomes that of finding the extremal solutions to

$$\sup\left\{ Re\left(\sum_{1}^{N} a_k f(w_k)\right), \ |w_k| < 1, \|f\|_{A^p} \le 1 \right\}. \tag{6.7}$$

Then the results of Khavinson and Stessin ([**9**]), analogous to Theorem 6.1, imply that

$$f^*(z) = CB(z) \prod_{j=1}^{2N-2} (1-\overline{\alpha_j}z)^{2/p} \prod_{1}^{N} (1-\overline{w_j}z)^{-4/p},$$

where C is a constant, $|\alpha_j| \le 1$, $j = 1, \cdots, 2N-2$, are constants and the zeros of the Blaschke product B may only accumulate to those α_j that lie on $\mathbb{T}$.

We now turn to a discussion of what more can be said about the Blaschke factor, at least for values of p close to 2.

7. A continuity approach

In an attempt to shed some light on Conjecture 6.1, we begin with the following lemma.

Lemma 7.1. *For $1 \le p < \infty$, $\lambda(p) := \lambda_p$ is a decreasing function of p.*

Proof. First note that one can easily use Jensen's inequality to show that the norms $\|f\|_p$ are increasing. Letting $1 \le p < q < \infty$, let f_p^* be the extremal solution to Problem 2.1 for p and f_q^* for q. Then $\|f_q^*\|_p \le \|f_q^*\|_q = 1$, and therefore f_q^* is a competitor for the extremal problem 2.1 for p. Therefore

$$Re\left(\int_{\mathbb{D}} f_q^*(z)\overline{\omega(z)}\, dA(z)\right) \le Re\left(\int_{\mathbb{D}} f_p^*(z)\overline{\omega(z)}\, dA(z)\right) = \lambda(p),$$

or $\lambda(q) \le \lambda(p)$, as desired. □

Lemma 7.2. *Let $1 < p < \infty$, and suppose that $p_n \to p$ and that $f_{p_n}^*(z) \to f(z)$ uniformly in $\overline{\mathbb{D}}$. Then $f = f_p^*$, that is, f is the extremal function for the Problem* (2.1) *for p.*

Proof. Since $f_{p_n}^*(z) \to f(z)$ uniformly in $\overline{\mathbb{D}}$, we have that

$$\int_{\mathbb{D}} f_{p_n}^*(z)\overline{\omega(z)}\, dA(z) \to \int_{\mathbb{D}} f(z)\overline{\omega(z)}\, dA(z).$$

Note that since $|f_{p_n}^*| \to |f|$ and since $p_n \to p$, $|f_{p_n}^*|^{p_n} \to |f|^p$, and therefore by Fatou's lemma and since $\|f_{p_n}^*\|_{p_n} = 1$, $\|f\|_p \le 1$. Therefore, f is a competitor for the extremal problem (2.1) for p.

Now if f is not the solution to the extremal problem (2.1), then there exists $g = f_p^* \in A^p$ such that $\|g\|_p = 1$, g is continuous in $\overline{\mathbb{D}}$ (by Theorem 6.1), and, for some $\varepsilon > 0$,

$$Re\left(\int_{\mathbb{D}} g\overline{\omega}\, dA\right) > Re\left(\int_{\mathbb{D}} f\overline{\omega}\, dA\right) + \varepsilon.$$

Therefore, there exists N such that for $n \geq N$,

$$Re\left(\int_{\mathbb{D}} g\overline{\omega}\, dA\right) > Re\left(\int_{\mathbb{D}} f_n\overline{\omega}\, dA\right) + \varepsilon/2. \tag{7.1}$$

Now g is continuous in the closed unit disk, and therefore for $p_n \to p$, the functions $|g|^{p_n}$ are bounded above by some constant, and therefore (by the bounded convergence theorem),

$$\gamma_n^{p_n} := \int_{\mathbb{D}} |g|^{p_n}\, dA \to \int_{\mathbb{D}} |g|^p\, dA = 1.$$

Note that the functions g/γ_n have norm 1 in A^{p_n}.

Since $\gamma_n \to 1$ and by (7.1), there exists M such that for $m \geq M$ and for $n \geq N$,

$$\frac{1}{\gamma_m} Re\left(\int_{\mathbb{D}} g\overline{\omega}\, dA\right) > Re\left(\int_{\mathbb{D}} f_n^*\overline{\omega}\, dA\right) + \varepsilon/4.$$

Choosing a large enough $n = m$ satisfying this inequality leads to a contradiction of the extremality of f_n^*.

Therefore, we must indeed have that $g = f_p^*$, as desired. □

Note that the hypothesis in Lemma 7.2 that the functions $f_{p_n}^*$ converge uniformly to f in the closed disk is stronger than what is really necessary for the proof: what is required is that the measures $f_{p_n}^* dA$ converge weakly to $f dA$.

Corollary 7.3. For $1 < p < \infty$, $\lambda(p)$ is a continuous function of p.

Proof. If $p_n \to p$, then, by the remark after the statement of Lemma 6.2, the functions $f_{p_n}^*$ are all in $\text{Lip}(\gamma, \overline{\mathbb{D}})$ for the some γ, and therefore form a uniformly bounded and equicontinuous family. Therefore, by the Arzela-Ascoli theorem, there exists a subsequence $f_{p_{n_k}}^*$ that converges uniformly in $\overline{\mathbb{D}}$ to some function f. By Lemma 7.2, $f = f_p^*$. Therefore, by the bounded convergence theorem,

$$\int_{\mathbb{D}} f_{p_{n_k}}^* \overline{\omega}\, dA(z) \to \int_{\mathbb{D}} f_p^* \overline{\omega}\, dA(z).$$

Taking real parts, we get that $\lambda(p_{n_k}) \to \lambda(p)$. But since the function λ is monotone, $\lambda(p_n) \to \lambda(p)$. □

Theorem 7.4. *If ω has no zeros on the boundary of the disk, then there exists $\mathbb{D}elta > 0$ such that if $|p-2| < \mathbb{D}elta$, the extremal function f_p^* has at most N zeros in $\overline{\mathbb{D}}$.*

Proof. If $p = 2$, then we know the solution is $f_2^* = \omega$, and ω has at most N zeros in the unit disk, because it is a polynomial of degree N.

First note that the extremal functions cannot have zeros that accumulate inside the unit disk (otherwise they would be identically zero) and therefore the zeros of the extremals can only accumulate to the boundary of the disk.

Let us first show that there exists $\mathbb{D}elta$ such that for p in a $\mathbb{D}elta$ neighborhood of 2, f_p^* has a finite number of zeros. Suppose not. Then there exists a sequence $p_n \to 2$ such that $f_{p_n}^*$ have infinitely many zeros in a compact neighborhood of the boundary of the disk. These zeros must have an accumulation point, and by the previous remark, this accumulation point must be on the boundary of the disk.

As in the proof of Corollary 7.3 and using Theorem 6.1, by passing to a subsequence if necessary, the $f_{p_n}^*$ converge uniformly in $\overline{\mathbb{D}}$ to ω by Lemma 7.2. But then

ω must vanish at the accumulation point on the boundary of the disk, which is a contradiction. Therefore there exists some neighborhood of p for which f_p^* have finitely many zeros. Moreover by Hurwitz' theorem, there exists some neighborhood of 2 such that for all p in that neighborhood, all the f_p^* have the same number of zeros as ω inside $\mathbb{D}$, that is, at most N. $\square$

REMARK 7.5. This argument works for a more general ω, as long as we know ω has no zeros on the boundary and has at most N zeros in the disk. In particular, we have the following corollary.

COROLLARY 7.6. For p sufficiently close to 2, the extremal solution to

$$sup\left\{Re\left(\sum_{k=1}^{N} a_k f(w_k)\right),\ |w_k|<1, \|f\|_{A^p}\le 1\right\} \tag{7.2}$$

has at most $2N-2$ zeros in the unit disk, provided that $\sum_{k=1}^{N}\frac{a_k}{(1-\bar{w}_k z)^2}$ has no zeros on the unit circle.

The analogue to Conjecture 6.1 is thus that Corollary 7.6 holds for all $p>1$.

The question remains how to deal with the case when ω has zeros on the circle. It is then natural to consider a sequence of functions ω_n without zeros on the circle that converge to ω. In addition, for a fixed p, one can then easily show that the extremals f^*_{p,ω_n} converge to $f_{p,\omega}$. However, the challenge remains to obtain uniform estimates on both p and n that would allow for a fixed neighborhood of $p=2$ that does not depend on n.

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