JENSEN TYPE INEQUALITIES AND RADIAL NULL SETS

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Abstract. We extend Jensen's formula to obtain an upper estimate of $\log|f(0)|$ for analytic functions in the unit disk $\mathbf{D}$ that are subject to a growth restriction. Suppose we have a closed subset E of the unit circle and f in addition is continuous in the union of the open disk and E. We obtain a formula that gives an upper estimate of of $\log|f(0)|$ in terms of the values of f on E and the so-called k-entropy of E. When the set E is taken to be the whole unit circle, we get the classical Jensen's inequality. Our formula is then applied to the study of radial null sets. 2000 Mathematics Subject Classification: 30H05, 30E25, 46E15.

1 Growth Spaces

In what follows, k denotes an increasing twice differentiable function that maps $[0,1)$ onto $[0,\infty)$ and satisfies

$$\int_0^1 k(r)dr < \infty \tag{1}$$

$$(1-r)k'(r) \text{is non-decreasing} \tag{2}$$

$$\frac{k(1-t/2)}{k(1-t)} \le C \quad (0 < t < \frac{1}{2}) \tag{3}$$

$A^{<k>}$ denotes the Banach space of analytic functions f in $\mathbf{D}$ with the norm

$$||f||_{<k>} = \sup\{|f(z)|\exp(-k(|z|)) : z \in \mathbf{D}\} < \infty.$$

$UBA^{<k>}$ denotes the unit ball of $A^{<k>}$; it consists of f satisfying

$$\log|f(z)| \le k(|z|) \quad (z \in \mathbf{D}.)$$

In the special case that $k(r) = \lambda_\alpha(r) = \alpha \log \frac{1}{1-r}$ for $\alpha > 0$, we write $A^{-\alpha}$ for $A^{<k>}$.

2 Two Problems

(A) Find good (upper and lower) estimates for the quantity

$$\mathcal{J}(\mathcal{Z}, k) = \sup\{\log|f(0)| : f \in UBA^{<k>}, f\,|_{\mathcal{Z}} = 0\}$$

where $\mathcal{Z} = \{a_n\} \subset \mathbf{D}$ is a given sequence.
(B) Find good estimates for

$$\mathcal{J}(E, \varphi, k) = \sup\{\log|f(0)| : f \in UBA^{<k>} \cap C(\mathbf{D} \cup E), |f|\,|_E = \varphi\}$$

where $E \subset \partial\mathbf{D}$ is a closed set and φ is a non-negative continuous function on E.
Note that for $k \equiv 0$, ($A^{<0>} = H^\infty$) both problems have exact solutions:

$$\mathcal{J}(\mathcal{Z}, 0) = -\sum_n \log\frac{1}{|a_n|}$$

$$\mathcal{J}(E, \varphi, 0) = \int_E \log\varphi(\zeta)dm(\zeta)$$

where dm is the normalized Lebesgue measure on $\partial\mathbf{D}$. (Here, we assume $0 \le \varphi(\zeta) \le 1$ on E.)

3 Results for $A^{-\alpha}$

Although the main thrust of this paper is problem (B), we give here for the sake of comparison the following result on problem (A) for $A^{-\alpha}$ (see [2] for the proof.)
We define the logarithmic entropy of a finite set $E \subset \partial\mathbf{D}$ as

$$\hat{\kappa}(E) = \sum_n |I_n| \log\frac{e}{|I_n|}$$

where $\{I_n\}$ are the complementary arcs of E and $|\bullet|$ denotes normalized Lebesgue measure.
For a finite set $S \subset \mathbf{D}$ not containing 0, we define

$$T(S) = \sum\{\log\frac{1}{|z|} : z \in S\}$$

and the radial projection of S :

$$PrS = \{\frac{z}{|z|} : z \in S\}.$$

Then we have

$$\mathcal{J}(\mathcal{Z}, \lambda_\alpha) \le \inf_{S\subset\mathcal{Z}} \{\alpha[\hat{\kappa}(PrS) + \log\hat{\kappa}(PrS)] - T(s) + \alpha\log^+ T(s)\} + C_\alpha$$

and

$$\mathcal{J}(\mathcal{Z}, \lambda_\alpha) \ge \inf_{S\subset\mathcal{Z}} \{\alpha[\hat{\kappa}(PrS) - \log\hat{\kappa}(PrS)] - T(s)\} - C_\alpha$$

where $C_\alpha > 0$ depends only on α, and the infima are taken over all finite subsets S of $\mathcal{Z}$.

COROLLARY 3.1 *For a sequence $\mathcal{Z}$ such that 0 is not in $\mathcal{Z}$, define*

$$D^+(\mathcal{Z}) = \inf\{m : \inf_{S\subset\mathcal{Z}} (m\hat{\kappa}(PrS) - T(s)) > -\infty\}.$$

Then $D^+(\mathcal{Z}) \le \alpha$ is necessary and $D^+(\mathcal{Z}) < \alpha$ is sufficient for $\mathcal{Z}$ to be an $A^{-\alpha}$ zero set.

Note that for other spaces $A^{<k>}$ such that k has faster than logarithmic growth, a similar description of zero sets is not known.

4 Problem (B) for $A^{<k>}$

THEOREM 4.1

$$\begin{aligned}\mathcal{J}(E, \varphi, k) &\le \int_E \max\{\log\varphi(\zeta), \log p\} dm(\zeta) - (\log p)\frac{\alpha}{1-\alpha}(1 - |E|) \\ &+ (\frac{L}{\alpha})^{\log_2 C} Entr_k(E)\end{aligned}$$

where $0 < p \le 1$, $0 < \alpha \le \frac{1}{2}$ are arbitrary, C is the constant in (3), L is an absolute constant, and $Entr_k(E)$ is the k-entropy of E, defined as follows:

$$Entr_k(E) = \sum_n \int_{1-|I_n|}^{1} k(t)dt$$

where $\{I_n\}$ are the complementary arcs of E.

Special cases: (1) $E = \partial\mathbf{D}$. Letting $p \to 0^+$, we get

$$\mathcal{J}(\partial\mathbf{D}, \varphi, k) \leq \int_{\partial\mathbf{D}} \log\varphi(\zeta) dm(\zeta)$$

which is the classical Jensen's inequality (in fact, equality.)
(2) If $0 \leq \varphi(\zeta) \leq 1$ on E and $p = \max_{\zeta\in E}\varphi(\zeta)$, we obtain

$$\mathcal{J}(E, \varphi, k) \leq (\log p)\frac{|E| - \alpha}{1 - \alpha} + (\frac{L}{\alpha})^{\log_2 C} Entr_k(E).$$

Choosing $\alpha = |E|/2$, we get

$$\mathcal{J}(E, \varphi, k) \leq \frac{1}{2}(\log p)|E| + (\frac{2L}{|E|})^{\log_2 C} Entr_k(E).$$

(3) If $p = 1$ and $\alpha = \frac{1}{2}$, then

$$\mathcal{J}(E, \varphi, k) \leq \int_E \log^+ \varphi(\zeta) dm(\zeta) + (2L)^{\log_2 C} Entr_k(E).$$

Proof: Write

$$\partial\mathbf{D} - E = \bigcup_n I_n$$

where the I_n are open disjoint arcs on the unit circle. Call a_n and b_n the endpoints of I_n. Let $0 < \alpha \leq \frac{1}{2}$. Let γ_n be the open arc of the circle inside the unit disk passing through a_n and b_n and forming an angle of $\pi\alpha$ (we will think of it as the normalized angle α) with the arc I_n. Let $\Gamma = \bigcup_n \gamma_n$. $\Gamma \cup E$ forms the boundary of an open subset Ω of the unit disk containing the origin. For the proof, we construct three functions U_1, U_2, and V as follows.
Step 1: Construction of U_1 and U_2.
Define

$$U_1(z) = \int_E Re(\frac{\zeta + z}{\zeta - z}) dm(\zeta).$$

U_1 is the harmonic measure of E with respect to $\mathbf{D}$.

LEMMA 4.1

$$\lim_{r\to 1^-} U_1(r\zeta) = \chi_E(\zeta) \text{ a.e. on } \partial\mathbf{D}$$

where χ_E is the characteristic function of E. In addition, $U_1(z) \leq \alpha$ for $z \in \Gamma$.

Proof: The first statement is clear from the definition of U_1 as harmonic measure. Notice that

$$U_1(z) \leq W_n(z) = \int_{\partial \mathbf{D} - I_n} Re(\frac{\zeta + z}{\zeta - z}) dm(\zeta)$$

for every n. W_n is the harmonic measure of $\partial \mathbf{D} - I_n$ and has a few nice geometric properties. In particular, $W_n(z)$ is constant on any circle passing through a_n and b_n. In fact it is not hard to see that if z is a point in the disk, and if we consider the circle C_n passing through a_n, b_n, and z, then $W_n(z) = \alpha_n(z)$, where $\alpha_n(z)$ is the (normalized) angle between the arc I_n and the circle C_n. Therefore, for any $z \in \Gamma$, $z \in \gamma_n$ for some n, and so $U_1(z) \leq W_n(z) \leq \alpha$.$\square$
Now let $0 < p \leq 1$ and define

$$U_2(z) = \int_E Re(\frac{\zeta + z}{\zeta - z}) \max\{\log \varphi(\zeta), \log p\} dm(\zeta).$$

Notice that U_2 is harmonic in $\mathbf{D}$, and

$$U_2(z) \geq (\log p) U_1(z) \geq (\log p)\alpha$$

for $z \in \Gamma$, by Lemma 4.1.
Step 2: Construction of V. First let $K(s) = k(1 - e^{-s})$ for $s > 0$ and extend K so that $K(s) = 0$ for $s < 0$. Now define a function

$$S(z) = \sum_n \max(S_n^1(z)(1 - \alpha_n(z)), S_n^2(z)(1 - \alpha_n(z)))$$

where

$$S_n^1(z) = K(\log |\frac{2(z - a_n)}{(z - b_n)(b_n - a_n)}|)$$

and

$$S_n^2(z) = K(\log |\frac{2(z - b_n)}{(z - a_n)(b_n - a_n)}|).$$

Notice that

$$\log |\frac{2(z - a_n)}{(z - b_n)(b_n - a_n)}|$$

is harmonic in $\mathbf{D}$ and since $(1 - r)k'(r)$ is non-decreasing, $S_n^1(z)$ is subharmonic in $\mathbf{D}$. Moreover, the level curves of S_n^1 are orthogonal to the level curves of $(1-\alpha_n(z))$. Since the product of two subharmonic functions whose gradients are orthogonal is subharmonic, we conclude that $S_n^1(z)(1-\alpha_n(z))$ is subharmonic in $\mathbf{D}$. A similar argument shows that

$$S_n^2(z)(1-\alpha_n(z))$$

is subharmonic in $\mathbf{D}$. Therefore the maximum $S(z)$ of those two functions is subharmonic in $\mathbf{D}$.

LEMMA 4.2

$$\int_{\partial\mathbf{D}} S(\zeta)dm(\zeta) \leq Entr_k(E).$$

Proof:

$$\begin{aligned}\int_{\partial\mathbf{D}} S(\zeta)dm(\zeta) &= \int_{\partial\mathbf{D}} \sum_n \max(S_n^1(\zeta)(1-\alpha_n(\zeta)), S_n^2(\zeta)(1-\alpha_n(\zeta)))\\ &= \sum_n \int_{I_n} \max(S_n^1(\zeta), S_n^2(\zeta))dm(\zeta).\end{aligned}$$

Let's study that last integral. If ζ is closer to a_n than to b_n, for example, then the integrand becomes

$$\begin{aligned}K(\log \frac{2|\zeta-b_n|}{|\zeta-a_n||b_n-a_n|}) &\leq K(\log \frac{2}{|\zeta-a_n|})\\ &\leq K(\log \frac{\pi}{t-\theta_n})\end{aligned}$$

where $\zeta = e^{it}$, $a_n = e^{i\theta_n}$. A similar estimate holds when ζ is closer to $b_n = e^{i\psi_n}$. Therefore we get:

$$\begin{aligned}\int_{\partial\mathbf{D}} S(\zeta)dm(\zeta) &\leq \sum_n \frac{1}{2\pi}(\int_{\theta_n}^{\frac{\theta_n+\psi_n}{2}} K(\log \frac{\pi}{t-\theta_n})dt\\ &+ \int_{\frac{\theta_n+\psi_n}{2}}^{\psi_n} K(\log \frac{\pi}{\psi_n-t})dt)\\ &\leq \sum_n \int_{1-|I_n|}^{1} k(r)dr\\ &= Entr_k(E).\Box\end{aligned}$$

Assume $Entr_k(E)$ is finite and define the following harmonic function

$$V(z) = \int_{\partial \mathbf{D}} Re(\frac{\zeta + z}{\zeta - z}) S(\zeta) dm(\zeta).$$

By the maximum principle, $S(z) \leq V(z)$ for $z \in \mathbf{D}$.

<u>LEMMA</u> 4.3 *V has the following properties.*

$$\begin{aligned} V(0) &\leq Entr_k(E) \\ V(\zeta) &= 0 \textit{ for } \zeta \in E \\ V(z) &\geq (\frac{L}{\alpha})^{-\log_2 C} k(|z|) \textit{ for some absolute constant } L \textit{ and for } z \in \Gamma. \end{aligned}$$

Proof: The first two properties are immediate from the definition of V and by lemma 4.2. For the third, let us examine the behavior of V on Γ. First of all, it is geometrically clear that for $z \in \Gamma$, there exists an absolute constant L such that if $z \in \gamma_n$,

$$0 < \frac{\min(|z - a_n|, |z - b_n|)}{1 - |z|} \leq \frac{L}{\alpha} < \infty.$$

Therefore for $z \in \gamma_n$, (let's say z is closer to a_n than to b_n)

$$\begin{aligned} V(z) &\geq S(z) \\ &\geq K(\log \frac{2|z - b_n|}{|z - a_n||b_n - a_n|}) \\ &\geq K(\log \frac{1}{|z - a_n|}) \\ &\geq K(\log \frac{\alpha}{L(1 - |z|)}) \\ &= k(1 - \frac{L}{\alpha}(1 - |z|)) \\ &\geq C^{[\log_2 \frac{L}{\alpha}]+1} k(1 - (1 - |z|)) \\ &\qquad \text{(by property (3) of } k) \\ &\geq C^{\log_2 \frac{2L}{\alpha}} k(|z|) \\ &= (\frac{2L}{\alpha})^{\log_2 C} k(|z|). \end{aligned}$$

By relabeling L, we get the statement of the lemma.□
Step 3: Construction of H and application of the maximum principle. Finally, let us define

$$H(z) = U_2(z) - (\log p)\frac{\alpha}{1-\alpha}(1 - U_1(z)) + (\frac{L}{\alpha})^{-\log_2 C} V(z).$$

H is harmonic in the disk. Moreover, for $\zeta \in E$,

$$H(\zeta) \geq \log \varphi(\zeta) = \log |f(\zeta)|.$$

On the other hand, if $z \in \Gamma$,

$$H(z) \geq (\log p)\alpha - (\log p)\frac{\alpha}{1-\alpha}(1-\alpha) + k(|z|) = k(|z|) \geq \log |f(z)|.$$

Recall that Ω is the part of the unit disk that is bounded by $\Gamma \cup E$. We therefore have a harmonic function H whose values on the boundary of Ω dominate the boundary values of $\log |f(z)|$, a function that is subharmonic in Ω. By the maximum principle, we can conclude that

$$\begin{aligned} \log |f(0)| &\leq H(0) \\ &\leq \int_E \max\{\log \varphi(\zeta), \log p\} dm(\zeta) - (\log p)\frac{\alpha}{1-\alpha}(1 - |E|) \\ &+ (\frac{L}{\alpha})^{\log_2 C} Entr_k(E) \end{aligned}$$

as desired.□

COROLLARY 4.1 *If $f \in A^{<k>}$ and*

$$\lim_{r \to 1^-} f(r\zeta) = 0$$

uniformly in $\zeta \in E$, and if $|E| > 0$ and $Entr_k(E) < \infty$, then $f \equiv 0$.

Remark: It is well-known (Lusin-Privalov theorem) that there are non-zero analytic functions that have zero radial limits on a set E of full Lebesgue measure [4]. (However, this cannot happen if $|E| > 0$ and at the same time E is of the second Baire category.) More specifically, no matter how slowly $k(r)$ tends to $+\infty$, there is always a non-zero f in $A^{<k>}$ such that

$\lim_{r \to 1^-} f(r\zeta) = 0$ a.e. [3]. By Egoroff's theorem, f may have uniform radial limits 0 on a closed set E whose measure is arbitrarily close to the full measure of $\partial\mathbf{D}$. Corollary 4.1 shows that such uniform radial null sets E with $|E| > 0$ must have infinite k-entropy. A similar phenomenon was discovered by S. V. Hruščev (see [1], p. 278-305) in connection with the Khinchin-Ostrowski property.

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