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On the Strong Form of the Faber Theorem

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ABSTRACT. Let H_n^∞ be the space of polynomials of degree at most $n-1$ on the unit circle with the uniform norm. We show that the dual spaces $(H_n^\infty)^*$ cannot be embedded uniformly in any $\mathcal{L}_1$-space. This result implies (but is not equivalent to) the famous theorem of Faber.

Let H_n be the space of polynomials of degree at most $n-1$ and let H_n^p be the n-dimensional Banach space H_n considered as a subspace of $L_p(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. We use $d(X,Y)$ to denote the Banach-Mazur distance, $\pi_1(T)$ to denote the absolutely summing norm of an operator T and $\gamma_\infty(T)$ for the L_∞-factorization norm of T. (cf [4]).

Theorem 1. *There exists a constant $C>0$ such that, if E_n is an n-dimensional subspace of an $\mathcal{L}_1$-space, then*

$$d((H_n^\infty)^*, E_n) \geq C \log\ n.$$

Proof. Let X be an $\mathcal{L}_1$-space. Let $E_n \subset X$ be an n-dimensional subspace. Let

$$T : (H_n^\infty)^* \hookrightarrow X$$

be such that $T(H_n^\infty)^* = E_n$. We want to prove that

$$\|T\|\|T^{-1}\| \geq C \log\ n.$$

Assume without loss of generality that $\|T^{-1}\| = 1$. Then (cf [3]) there are vectors $x_1^*, \ldots, x_n^* \in X^*$ such that

$$T^* x_j^* = e^{ij\theta}$$

and $\|x_j^*\| \leq 1$.

Consider the following operators:

$$J : H_n^\infty \to H_n^1;\ Je^{ik\theta} = e^{ik\theta},$$
$$S : H_n^1 \to \ell_1^{(n)};\ Se^{ik\theta} = \frac{1}{k+1} e_k,$$

where $e_1, \ldots, e_n$ is the canonical basis in $\ell_1^{(n)}$,

$$A : \ell_1^{(n)} \to X^*;\ Ae_k = x_k^*.$$

We illustrate these operators on the diagram

$$X^* \xrightarrow{T^*} H_n^\infty \xrightarrow{J} H_n^1 \xrightarrow{S} \ell_1^{(n)}, \quad A : \ell_1^{(n)} \to X^*$$

By the Pietsch Factorization Theorem (cf [4])

$$\pi_1(J) = \|J\| = 1.$$

By the Hardy inequality (cf [1]) there exists a constant $c > 0$ independent of n, such that

$$\|S\| \leq c$$

Finally, we have

$$\|A(\Sigma a_j e_j)\| = \|\Sigma a_j x_j^*\| \leq \Sigma |a_j| = \|\Sigma a_j e_j\|$$

and hence

$$\|A\| \leq 1.$$

Since A maps an $\mathcal{L}_1$-space into an $\mathcal{L}_\infty$-space, we have

$$\gamma_\infty(A) = \|A\| = 1.$$

Next observe that

$$tr(ASJT^*) = tr(T^*ASJ) = \sum_{k=0}^{n} \frac{1}{k+1} \geq \log\ n.$$

By trace-duality (cf [4])

$$\log n \leq tr(ASJT^*) \leq \pi_1(SJT^*)\gamma_\infty(A)$$
$$\leq \|S\|\pi_1(J)\|T^*\|\|A\| \leq c\|T\|.$$

Thus

$$\|T\| = \|T^*\| \geq \frac{1}{c} \log\ n.\ \Box$$

Theorem 2. *The results of the Theorem 1 remain true if we replace the space H_n^∞ by any of the following spaces:*

(1) *span* $[e^{i\lambda_k\theta}]_{k=1}^n \subset C(\mathbb{T})$ *where* λ_k *are arbitrary distinct integers,*
(2) *span* $[\cos j\theta]_{j=1}^n \subset C[0,\pi]$
(3) *span* $[t^k]_{k=1}^n \subset C[a,b]$.

Proof. The proof is the same as that of Theorem 1 if we replace the use of the Hardy inequality by the generalized Hardy inequality (cf [2]) for the trigonometric polynomials and by the Sidon inequality (cf [6]) for the cosine polynomials. The isometry between the algebraic and cosine polynomials proves the remaining case. □

Corollary 1 (Faber Theorem). *Let $P_n : C(\mathbb{T}) \to H_n$ be an arbitrary sequence of projections onto the trigonometric polynomials. Then $\|P_n\| \geq C\log n$.*

Proof. Let $P_n : C(\mathbb{T}) \to H_n^\infty$. Then $P_n^* : (H_n^\infty)^* \to C^*(\mathbb{T})$. Moreover, $P_n^*(H_n^\infty)^*$ is an n-dimensional subspace of $C^*(\mathbb{T})$ and $\|(P_n^*)^{-1}\| = 1$. Since $C^*(\mathbb{T})$ is an $\mathcal{L}_1$-space, we conclude

$$C\log n \leq d((H_n^\infty)^*, P_n^*(H_n^\infty)^*) \leq \|P_n^*\|\|(P_n^*)^{-1}\| = \|P_n\|. \quad \square$$

Remark 1. The result of Theorem 1 is sharp.

Proof. Let Q_n be the Fourier projections from $C(\mathbb{T})$ onto H_n. Then $\|Q_n\| \sim \log n$ and

$$d((H_n^\infty)^*, Q_n^*(H_n^\infty)^*) = d((H_n^\infty)^*, H_n^1) \leq \|Q_n\|. \quad \square$$

We will now mention two examples to conclude that Theorem 1 does not follow from the Faber Theorem.

Example 1. Let $A(\mathbb{T})$ be the disk-algebra. Then, for any sequence of projections P_n from $A(\mathbb{T})$ onto H_n, we have (cf [6]) $\|P_n\| \geq C\log n$. Despite this fact $(H_n^\infty)^*$ is uniformly isomorphic to the subspaces of $A^n(\mathbb{T})$. Indeed it was shown by Bourgain and Pelczynski (cf [5]) that H_n^∞ are uniformly isomorphic to subspaces V_n of $A(\mathbb{T})$ which are uniformly complemented. Using the reasoning of Corollary 1, we get the desired conclusion. □

It may appear that the reason for this example is in the fact that $A(\mathbb{T})$ is not an $\mathcal{L}_\infty$-space and the relative projectional constant of H_n^∞ in $A(\mathbb{T})$ is not isomorphic-invariant.

Therefore

Example 2. Let $V_n \subset C(\mathbb{T})$ be such that

$$d(V_n, \ell_2^{(n)}) = 1.$$

It is well-known (cf [4]) that for any sequence of projections P_n from $C(\mathbb{T})$ onto V_n, we have $\|P_n\| \geq \frac{1}{2}\sqrt{n}$. Yet V_n^* are isometric to $\ell_2^{(n)}$ and hence (by Dvoretzky's Theorem) V_n^* are uniformly isomorphic to subspaces of $C^*(\mathbb{T})$ which is an $\mathcal{L}_1$-space. □

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