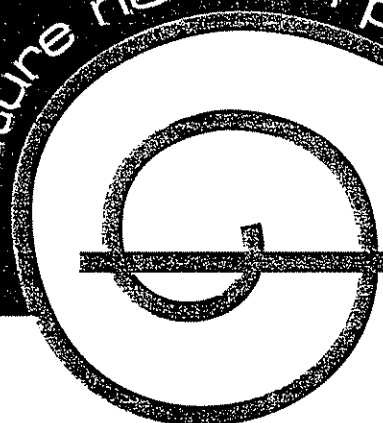


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stochastic processes and
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On the Strong Form of the Faber Theorem

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ABSTRACT. Let H_n^∞ be the space of polynomials of degree at most $n - 1$ on the unit circle with the uniform norm. We show that the dual spaces $(H_n^\infty)^*$ cannot be embedded uniformly in any \mathcal{L}_1 -space. This result implies (but is not equivalent to) the famous theorem of Faber.

Let H_n be the space of polynomials of degree at most $n - 1$ and let H_n^p be the n -dimensional Banach space H_n considered as a subspace of $L_p(\mathbb{T})$, where \mathbb{T} is the unit circle. We use $d(X, Y)$ to denote the Banach-Mazur distance, $\pi_1(T)$ to denote the absolutely summing norm of an operator T and $\gamma_\infty(T)$ for the L_∞ -factorization norm of T . (cf [4]).

Theorem 1. *There exists a constant $C > 0$ such that, if E_n is an n -dimensional subspace of an \mathcal{L}_1 -space, then*

$$d((H_n^\infty)^*, E_n) \geq C \log n.$$

Proof. Let X be an \mathcal{L}_1 -space. Let $E_n \subset X$ be an n -dimensional subspace. Let

$$T : (H_n^\infty)^* \hookrightarrow X$$

be such that $T(H_n^\infty)^* = E_n$. We want to prove that

$$\|T\| \|T^{-1}\| \geq C \log n.$$

Theorem 2. *The results of the Theorem 1 remain true if we replace the space H_n^∞ by any of the following spaces:*

- (1) $\text{span} [e^{i\lambda_k \theta}]_{k=1}^n \subset C(\mathbb{T})$ where λ_k are arbitrary distinct integers,
- (2) $\text{span} [\cos j\theta]_{j=1}^n \subset C[0, \pi]$
- (3) $\text{span} [t^k]_{k=1}^n \subset C[a, b]$.

Proof. The proof is the same as that of Theorem 1 if we replace the use of the Hardy inequality by the generalized Hardy inequality (cf [2]) for the trigonometric polynomials and by the Sidon inequality (cf [6]) for the cosine polynomials. The isometry between the algebraic and cosine polynomials proves the remaining case. \square

Corollary 1 (Faber Theorem). *Let $P_n : C(\mathbb{T}) \rightarrow H_n$ be an arbitrary sequence of projections onto the trigonometric polynomials. Then $\|P_n\| \geq C \log n$.*

Proof. Let $P_n : C(\mathbb{T}) \rightarrow H_n^\infty$. Then $P_n^* : (H_n^\infty)^* \rightarrow C^*(\mathbb{T})$. Moreover, $P_n^*(H_n^\infty)^*$ is an n -dimensional subspace of $C^*(\mathbb{T})$ and $\|(P_n^*)^{-1}\| = 1$. Since $C^*(\mathbb{T})$ is an \mathcal{L}_1 -space, we conclude

$$C \log n \leq d((H_n^\infty)^*, P_n^*(H_n^\infty)^*) \leq \|P_n^*\| \|(P_n^*)^{-1}\| = \|P_n\|. \quad \square$$

Remark 1. The result of Theorem 1 is sharp.

Proof. Let Q_n be the Fourier projections from $C(\mathbb{T})$ onto H_n . Then $\|Q_n\| \sim \log n$ and

$$d((H_n^\infty)^*, Q_n^*(H_n^\infty)^*) = d((H_n^\infty)^*, H_n^1) \leq \|Q_n\|. \quad \square$$

We will now mention two examples to conclude that Theorem 1 does not follow from the Faber Theorem.

Example 1. Let $A(\mathbb{T})$ be the disk-algebra. Then, for any sequence of projections P_n from $A(\mathbb{T})$ onto H_n , we have (cf [6]) $\|P_n\| \geq C \log n$. Despite this fact $(H_n^\infty)^*$ is uniformly isomorphic to the subspaces of $A^n(\mathbb{T})$. Indeed it was shown by Bourgain and Pelczynski (cf [5]) that H_n^∞ are uniformly isomorphic to subspaces V_n of $A(\mathbb{T})$ which are uniformly complemented. Using the reasoning of Corollary 1, we get the desired conclusion. \square

It may appear that the reason for this example is in the fact that $A(\mathbb{T})$ is not an \mathcal{L}_∞ -space and the relative projectional constant of H_n^∞ in $A(\mathbb{T})$ is not isomorphic-invariant.

Therefore

Example 2. Let $V_n \subset C(\mathbb{T})$ be such that

$$d(V_n, \ell_2^{(n)}) = 1.$$

It is well-known (cf [4]) that for any sequence of projections P_n from $C(\mathbb{T})$ onto V_n , we have $\|P_n\| \geq \frac{1}{2} \sqrt{n}$. Yet V_n^* are isometric to $\ell_2^{(n)}$ and hence (by Dvoretzky's Theorem) V_n^* are uniformly isomorphic to subspaces of $C^*(\mathbb{T})$ which is an \mathcal{L}_1 -space. \square

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