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Some Simple Open Problems on Interpolation of Individual Functions

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Abstract

We discuss some simple problems of interpolating individual functions. In particular the problem of simultaneous interpolation of several functions and the problem of Newton interpolation.

1 Introduction

A well-known Faber theorem asserts that given any sequence P_n of projections from $C_{[0,1]}$ onto the space of polynomials $\mathcal{P}_n$ of degree $n-1$, there exists a continuous function f such that $P_n f$ does not converge to f in the uniform norm.

In particular let $\Delta_n : 0 \leq t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)} \leq 1$ be a sequence of partitions of the interval $[0,1]$. Let $L(\Delta_n) : C_{[0,1]} \to \mathcal{P}_n$ be the usual Lagrange interpolation operator. Then there exists $f \in C$ such that $L(\Delta_n)f \not\to f$.

Also let μ be a positive Borel measure on $[0,1]$. Let $p_k(\mu)$ be a sequence of orthonormal polynomials with respect to μ. The Faber theorem implies that for any sequence of positive Borel measures μ_n there exists a function $f \in C_{[0,1]}$ such that the functions

$$F_n(\mu_n)f := \sum_{k=0}^{n-1} \left(\int p_k(\mu_n) f d\mu_n \right) p_k(\mu_n) \not\to f.$$

An elegant observation of Marcinkiewicz tells us that the situation changes dramatically if the function f is given is advance.

Theorem. *Given a function $f \in C_{[0,1]}$ there exists a sequence of interpolation points Δ_n such that $L(\Delta_n)f \to f$.*

Proof. Let $b_n(f) \in \mathcal{P}_n$ be polynomials of best uniform approximation to f. Then $f - b_n(f)$ alternates sign at least $n+1$ times, hence is zero (by the intermediate value theorem) at least n times. Pick these n points to be Δ_n. Then $L(\Delta_n)f = b_n$ and $L(\Delta_n)f \to f$. ■

My general question is:

Given a sequence of "random" projections $P_n : C_{[0,1]} \to \mathcal{P}_n$ and a "random" function f, what are the chances that $P_n f \to f$?

The answer must be either 1 or 0. But which one?

2 Open Problems

What happens if at least one of the arguments in the proof of the Marcinkiewicz theorem does not apply? As far as I know nothing is known. My interest in this area was started by E. Levin who asked me the following:

Problem 1. *Given two functions $f, g \in C_{[0,1]}$, does there exist a sequence of interpolation points $\Delta_n \subset [0,1]$ such that $L(\Delta_n)f \to f$ and $L(\Delta_n)g \to g$?*

I conjecture that the answer is "yes". Some numerical experiments support this conjecture. Of course there are obvious generalizations of this problem. What happens if we have a fixed number of functions and/or change $\mathcal{P}_n$ by an arbitrary subspace $E_n \subset C(K)$?

If we do not demand that the interpolation property holds for our projections then the answer is affirmative.

Proposition 1. *Let X be a Banach space, $E_n \subset X$ be an n-dimensional subspace of X. Let $f_1, \ldots, f_m \in X$ be m fixed elements in X. Then there is* *a projection* $P_n : X \to E_n$ *such that*

$$\|f_k - P_n f_k\| \le (\sqrt{m}+2)\,\mathrm{dist}\,(f_k, E_n) \quad \text{for all} \quad k = 1, \ldots, m.$$

Proof. Consider an $(n+m)$-dimensional subspace

$$X_{n+m} := \mathrm{span}\,\{f_1, \ldots, f_m\} \oplus E_n$$

of X. Then $E_n \subset X_{n+m}$ is a subspace of codimension m. Hence (cf. [2]) there exists a projection $\tilde{P}_n : X_{n+m} \to E_n$ such that $\|\tilde{P}_n\| \le \sqrt{m}+1$. Let P_n be an arbitrary extension of P_n onto X. Then

$$\|f_k - P_n f_k\| = \|f_k - \tilde{P}_n f_k\| \le (\sqrt{m}+2)\,\mathrm{dist}\,(f_k, E_n).$$

■

Remark. With a bit more effort the estimate can be improved to $\|f_k - Pf_k\| \le \sqrt{m}\,\mathrm{dist}\,(f_k, E_n)$.

Proposition 2. *Given two functions* $f_1, f_2 \in C_{[0,1]}$, *there exists a sequence of positive measures* μ_n *on* $[0,1]$ *such that*

$$\sum_{k=0}^{n-1} \left(\int p_k(\mu_n) f_j d\mu_n \right) p_k(\mu_n) \to f_j, \quad j = 1, 2.$$

Proof. Consider a complex valued function

$$g(t) = f_1(t) + i f_2(t).$$

Let $\tilde{\mathcal{P}}_n \subset \tilde{C}_{[0,1]}$ be a subspace of complex polynomials of the space of complex-valued continuous functions on $[0,1]$. Let $b_n \in \mathcal{P}_n$ be the polynomial of best approximation to g. Then (by the Hahn-Banach theorem) there exists a measure ν_n such that $\|\nu_n\| = 1$

$$\int p d\nu_n = 0 \quad \text{for all} \quad p \in \mathcal{P}_n$$

and

$$\int (g - b_n) d\nu_n = \|g - b_n\|.$$

Hence the modulus of $g - b_n$ is constant on the support of ν_n and $d\nu_n = \arg(g - b_n) d\mu_n$ where μ_n is a positive measure.

Letting $g_1 := g - b_n$ we have

$$\int g_1 \cdot p d\mu_n = \|g_1\| \int p \operatorname{sign}(g_1) d\mu_n = \|g_1\| \int p d\nu_n = 0$$

and

$$\sum_{k=0}^{n-1} \left(\int g_1 \cdot p_k(\mu_n) d\mu_n \right) p_k(\mu_n) = 0.$$

Thus

$$\sum_{k=0}^{n-1} \left(\int g \cdot p_k(\mu_n) d\mu_n \right) p_k(\mu_n) = b_n.$$

But $p_k(\mu_n)$ are real-valued polynomials, therefore

$$\sum_{k=0}^{n-1} \left(\int f_1 p_k(\mu_n) d\mu_n \right) p_k(\mu_n) = \operatorname{Re} b_n$$

and

$$\sum \left(\int f_2 p_k(\mu_n) d\mu_n \right) p_k(\mu_n) = \operatorname{Im} b_n.$$

■

Remark. Perhaps one way to solve Problem 1 is to interpolate at the zeros of $p_{n+1}(\mu_n)$ where μ_n is from Proposition 2.

Problem 2. *Given three functions $f_1, f_2, f_3 \in C_{[0,1]}$ do there exist measures μ_n on $[0,1]$ such that*

$$\sum_{k=0}^{n-1} \left(\int f_j p_k(\mu_n) d\mu_n \right) p_k(\mu_n) \to f_j, \quad j = 1, 2, 3?$$

Problem 3. *Let $K \subset [0,1]$ be an arbitrary closed subset. Let $f \in C(K)$. Does there exist a sequence of interpolation points Δ_n such that $L(\Delta_n)f \to f$?*

Remark. In this case the intermediate value theorem does not apply. Using an argument identical to the one in Proposition 2, one can show the existence of weighted orthogonal projections $F_n(\mu_n)$ with $F_n(\mu_n)f \to f$.

We now turn to the Newton Interpolation. Let $\Delta = \{t_1, t_2, t_3, \ldots\}$ be an ordered countable collection of distinct points in $[0,1]$. Let

$$L_n(\Delta) := L(\Delta_n) \quad \text{with} \quad \Delta_n := \{t_1, t_2, \ldots, t_n\}.$$

This is the Newton interpolation in the special case of the Lagrange interpolation where $\Delta_n \subset \Delta_{n+1}$, i.e., at each step we add just one extra point.

Problem 4. *Given a function $f \in C_{[0,1]}$ does there exist $\Delta \subset [0,1]$ such that $L_n(\Delta)f \to f$?*

I conjecture that the answer is "no" in general. Here is one small fact to support this conjecture.

Let T be the unit circle, i.e., $T = \{z : |z| = 1\}$. Let $\tilde{\mathcal{P}}_n$ be the set of complex polynomials of degree $n-1$. Let $f(z) = \overline{z}$.

Proposition 3. *For every set Δ*

$$\limsup \|L_n(\Delta)f\| = \infty.$$

Proof. It is easy to check that for $\Delta_n = \{z_1, \dots, z_{n-1}\}$

$$L_n(\Delta)f = \overline{z} - \overline{z}\frac{\Pi(z_j - z)}{\Pi z_j}.$$

Hence $|f - L_n(\Delta)f| = |\prod_{j=1}^{n-1}(z_j - z)|$. By [3] we have $\limsup |\prod_{j=1}^{n-1}(z_j - z)| = \infty$.

Remark. This proposition is of no practical interest since f can not be approximated by polynomials at all. But it is peculiar that the Newton interpolants are unbounded on the same function f independent of Δ.

The appearance of lim sup in Proposition 3 is not accidental.

Proposition 4. *Given any function $f \in C_{[0,1]}$ there exists $\Delta \subset [0,1]$ such* *that*

$$\liminf \|f - L_n(\Delta)f\| = 0.$$

Proof. Let b_n be the best approximation of f. We appeal to the theorem of Kadec [1] according to which there exists a subset $N_1 \subset N$ and sequences of points $z_n = \{0 \le s_1^{(n)} < \cdots < s_n^{(n)} \le 1\}$ such that $b_n(s_j^{(n)}) = f(s_j^{(n)})$ and

$$\sup_{t\in[0,1]} \operatorname{dist}(t, \tau_n) \le \frac{1}{\sqrt{n}} \quad \text{for} \quad n \in N_1.$$

Since the operators $L(\tau_n)$ depend continuously on the τ_n we can find ε_n such that for every collection $\Delta_n(\varepsilon_n) = \{t_1^{(n)}, \ldots, t_n^{(n)}\}$ with

$$t_j^{(n)} \in U(s_j^{(n)}, \varepsilon_n) := \{t : |s_j^{(n)} - t| \le \varepsilon_n\}$$

we have

$$\|f - L(\Delta_n(\varepsilon_n))f\| \le 2 \operatorname{dist}(f, \mathcal{P}_n),$$

and $U(s_j^{(n)}, \varepsilon_n) \cap U(s_k^{(n)}, \varepsilon_n) = \emptyset$ for $j \ne k$. We now proceed with the construction. Let $\Delta_0 = \{s_0^0\}$. There exists n such that

$$\operatorname{dist}(s_0^0, \Delta_{n_1}) < \varepsilon_0.$$

We reorder Δ_{n_1} so that $s_0^{(n_1)} \in U(s_0^0, \varepsilon_0)$. Next, there exists n_2 such that for some reordering of Δ_{n_2} we have $|s_k^{n_1} - s_k^{n_2}| < \varepsilon_{n_1}$ and $s_0^{n_2} \subset U(s_0^0, \varepsilon_0)$.

Continuing this way we obtain a collection of sets

$$U(s_0^0, \varepsilon_0)$$

$$U(s_0^{(n_1)}, \varepsilon_{n_1}),\ \ U(s_1^{(n_1)}, \varepsilon_n) \cdots U(s_{n_1}^{(n_1)}, \varepsilon_{n_1})$$

$$U(s_0^{(n_2)}, \varepsilon_{n_2}),\ \ U(s_1^{(n_2)}, \varepsilon_{n_2}) \cdots U(s_{n_1}^{(n_2)}, \varepsilon_{n_2}) \cdots U(s_{n_2}^{(n_2)}, \varepsilon_{n_2})$$

so that every row consists of non-intersecting sets and every column has the nested property

$$U(s_j^{(n_k)}, \varepsilon_{n_k}) \subset U(s_j^{n_{k-1}}, \varepsilon_{n_{k-1}}).$$

Hence there exist points $t_j \in \bigcap_k U(s_j^{(n_k)}, \varepsilon_{n_{k-1}})$. Let $\Delta = \{t_1, t_2, \ldots\}$. Then $|t_j - s_j^{n_j}| \le \varepsilon_{n_j}$ and we have $\|f - L_{n_j}(\Delta)f\| \le 2\,\mathrm{dist}\,(f, \mathcal{P}_{n_j})$. ∎

Remark. Unfortunately the proof of Proposition 4 utilizes the same idea as the proof of the Marcinkiewicz theorem and adds little to the understanding of Problem 4.

Problem 5. *Given a function* $f \in C_{[0,1]}$, *does there exist one measure* μ *such that*

$$\sum_{k=0}^{n-1} \left(\int f p_k(\mu) d\mu \right) p_k(\mu)$$

converges to f?

Remark. The problems in this paper and the avenues to the solutions of these problems were formed during the numerous conversations with my friends. Specifically I would like to mention Bruce Chalmers, Eli Levin, Doron Lubinsky, Ed Saff, Herbert Stahl, and Villi Totik.

References

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