# On Error Formulas for Multivariate Polynomial Interpolation

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**Abstract.** In this paper we prove that the existence of an error formula of a form suggested in [2] leads to some very specific restrictions on an ideal basis that can be used in such formulas. As an application, we provide a negative answer to one version of the question posed by Carl de Boor (cf. [2]) regarding the existence of certain minimal error formulas for multivariate interpolation.

## §1. Introduction

The various form of "error formulas" for multivariate interpolation is a popular subject of discussion in the literature (cf. [2]-[5] and [8]-[14], [17], and [18]). In particular, a possible algebraic nature of such formulas was suggested in [2], [10], [11], [12], [17] and [18].

In this paper we prove that the existence of error formula of the form suggested in [2] leads to some very specific restrictions on the basis of the ideal that can be used in such formulas. As an application, we supply a (very) negative answer to one version of the question posed by Carl de Boor (cf. [2]) regarding the existence of certain minimal error formulas for multivariate interpolation. We will need some notation.

We write $\mathbb{F}$ for the real or complex field, and $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, x_2, ..., x_d]$ for polynomials of $d$ variables. The symbols $\mathbb{F}[x], \mathbb{F}[x,y], \mathbb{F}[x,y,z]$ denote the polynomials of one, two and three variables respectably. An element $f \in \mathbb{F}[\mathbf{x}]$ is written as a finite sum $\sum \hat{f}(k_1, ..., k_d) x_1^{k_1} x_2^{k_2} .... x_d^{k_d}$ or in the multiindex notations $\sum \hat{f}(\mathbf{k})\mathbf{x}^{\mathbf{k}}$ with $\hat{f}(\mathbf{k}) \in \mathbb{F}$. For a polynomial $f \in \mathbb{F}[\mathbf{x}]$, we use non-standard but convenient notation

$$f(D) := \sum \hat{f}(k_1, ..., k_d) \frac{1}{k_1! k_2! ... k_n!} \frac{\partial^{k_1 + ... + k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} .... \partial x_d^{k_d}} \tag{1}$$

to denote the differential operator on $\mathbb{F}[\mathbf{x}]$. The space of polynomials of degree less than $n$ is denoted by $\mathbb{F}_{<n}[\mathbf{x}]$, The set of polynomials of degree

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$n$ is $\mathbb{F}_n[\mathbf{x}]$, while the space of homogeneous polynomials of degree $n$ is denoted as $\mathbb{F}_{[n]}[\mathbf{x}]$. Finally the set of monomials of degree $n$ is $M_n[\mathbf{x}]$, and $M[\mathbf{x}]$ is the set of all monomials in $\mathbb{F}[\mathbf{x}]$.

Every polynomial $f \in \mathbb{F}[\mathbf{x}]$ can be written (uniquely) as a finite sum $f = \sum f^{[k]}$ with $f^{[k]} \in \mathbb{F}_{[k]}[\mathbf{x}]$ being a homogeneous component of $f$. The non-zero homogeneous component that corresponds to the largest $k$ is called the leading form of the polynomial $f$, and is denoted by $\mathrm{Lf}(f)$. Hence, the leading form of a polynomial $f \in \mathbb{F}[\mathbf{x}]$ is the unique homogeneous polynomial $\mathrm{Lf}(f)$ such that $\deg(f - \mathrm{Lf}(f)) < \deg f$. Similarly, the non-zero homogeneous component that correspond to the least $k$ is the least form of $f$, and is denoted by $\mathrm{lf}(f)$. For $f = 0$ we set $\mathrm{Lf}(f) = \mathrm{lf}(f) = 0$. For every ideal $J \subset \mathbb{F}[\mathbf{x}]$, we use $Z(J)$ to denote the associated variety

$$Z(J) = \{\mathbf{z} \in \mathbb{F}^d : f(\mathbf{z}) = 0, \forall f \in J\}.$$

The ideal $J$ is called zero-dimensional (cf. [7]) if

$$\dim(\mathbb{F}[\mathbf{x}]/J) < \infty,$$

which implies (and for $\mathbb{F} = \mathbb{C}$ is equivalent to) the condition that the set $Z(J)$ is finite.

Likewise, with every set $Z \subset \mathbb{F}^d$ we associate an ideal

$$J(Z) := \{f \in \mathbb{F}[\mathbf{x}] : f(\mathbf{z}) = 0, \forall \mathbf{z} \in Z\}.$$

It is easy to see (cf. [6]) that $J \subset J(Z(J))$. An ideal $J$ is called a radical ideal if $J(Z(J)) = J$. Equivalently (cf. [6]) an ideal $J$ is a radical ideal if an only if $f^m \in J$ for some integer $m$ implies $f \in J$.

For a subset $B \subset \mathbb{F}[\mathbf{x}]$ we use $\langle B \rangle$ to denote the ideal generated by $B$. The set $B$ is called the basis of the ideal $\langle B \rangle$. By the Hilbert basis theorem, for any ideal $J \subset \mathbb{F}[\mathbf{x}]$ there exist *finite* basis $B$ such that $J = \langle B \rangle$. There are several notions of minimal bases. For clarity we will call a basis $B$ *unshortenable* if $\langle B \rangle \neq \langle B_0 \rangle$ for any proper subset $B_0 \subset B$. A basis $B$ is called a *minimal basis* for an ideal $J = \langle B \rangle \subset \mathbb{F}[\mathbf{x}]$ if $\#B_0 < \#B$ implies that $B_0$ is not a basis for $J$. For an ideal $J \subset \mathbb{F}[\mathbf{x}]$ we set

$$\mathfrak{m}(J) := \#B, \text{ with } B \text{ being a minimal basis for } J.$$

A basis $B$ for an ideal $J \in \mathbb{F}[\mathbf{x}]$ is called an $H$ -basis if for every $f \in J$ there exist $\{g_{b,f} \in \mathbb{F}[\mathbf{x}], b \in B\}$ such that

$$f = \sum_{b \in B} g_{b,f} b \text{ and } \deg g_{b,f} + \deg b \leq \deg f, \qquad \text{for all } b \in B.$$

**Definition 1.** *(Birkhoff, [1]). Let $E$ be a subspace of $\mathbb{F}[\mathbf{x}]$. A projector $P$ from $\mathbb{F}[\mathbf{x}]$ onto $E$ is called ideal if $\ker P$ is an ideal in $\mathbb{F}[\mathbf{x}]$.*

The following characterization of ideal projectors is due to de Boor (cf. [2]).

**Theorem 1.** *A linear mapping* $P : \mathbb{F}[\mathbf{x}] \to \mathbb{F}[\mathbf{x}]$ *is an ideal projector if and only if the equality*

$$P(fg) = P(fPg) \tag{1.0}$$

*holds for all* $f, g \in \mathbb{F}[\mathbf{x}]$.

The standard example of an ideal projector is a *Lagrange* projector, i.e., a projector $P$ for which $Pf$ is the unique element in its range that agrees with $f$ at a certain finite set $Z$ in $\mathbb{F}^d$. For its kernel consists of exactly those polynomials that vanish on $Z$, i.e., it is the zero-dimensional radical ideal whose variety is $Z$.

Let $P$ be an ideal projector onto $\mathbb{F}_{<n}[\mathbf{x}]$. In [2] and [4] Carl de Boor asked for the existence of an error formula of the following form:

$$f(\mathbf{x}) - Pf(\mathbf{x}) = \sum_{b \in B} b(\mathbf{x}) \mu_{b,\mathbf{x}}(H_b(D)f), \tag{1.1}$$

where $B$ is a (minimal) basis for the ideal $\ker P$, $H_b$ is a homogeneous polynomial satisfying the "orthogonality conditions"

$$H_b(D)c = \delta_{b,c} \text{ for } b, c \in B \tag{1.2}$$

and $\mu_{b,\mathbf{x}}$ is a linear functional on $\mathbb{F}[\mathbf{x}]$ that depends on $b$ and $\mathbf{x}$, but not on the function $f$.

For $d = 1$, such formulas exist (cf. [2] and [16]), and the minimal basis $B$ consists of one (unique) monic polynomial of degree $n$ that generates the ideal $\ker P$.

In this paper we will show that (1.1) and (1.2) implie that the sets

$$\{H_b : b \in B\} \text{ and } \{\mathrm{Lf}(b) : b \in B\}$$

form (dual) linear bases for the linear space $\mathbb{F}_{[n]}[\mathbf{x}]$ of homogeneous polynomials of degree $n$. In particular, this implies that the cardinally of $B$,

$$\#B = N(n) := \binom{n+d-1}{d-1}, \tag{1.3}$$

which is the number of monomials of degree $n$ in $\mathbb{F}[\mathbf{x}]$. Since (as we will show in Section 3) for all Lagrange projectors $P$ onto $\mathbb{F}_{<n}[\mathbf{x}]$ , there exists a basis $B$ such that $\langle B\rangle = \ker P$ and $\#B = d$, and since

$$\binom{n+d-1}{d-1} > d \tag{1.4}$$

for $d > 1$, hence for these projectors (1.1) and (1.2) cannot be valid with minimal $B$.

In the last section we discuss a stronger possibility, that a minimal bases for the kernel of an ideal projector $P$ onto $\mathbb{F}_{<n}[\mathbf{x}]$ admits an error formula of type (1.1), (1.2) if and only if $P$ is the Taylor projector.

We will need an analog of Theorem 1 for the projector $P' := I - P$.

**Theorem 2.** *A linear mapping $P$ on $\mathbb{F}[\mathbf{x}]$ is an ideal projector if and only if $P' = I - P$ satisfies*

$$P'(fg) = fP'g + P'(fPg), \qquad \text{all } f, g \in \mathbb{F}[\mathbf{x}]. \tag{1.5}$$

**Proof:** We have $P'(fg) = fg - P(fg)$ and

$$fP'g + P'(fPg) = f(g - Pg) + fPg - P(fPg) = fg - P(fPg).$$

Hence (1.5) is equivalent to (1.0). $\square$

## §2. The Bases for Error Formulas

We will start with a simple observation.

**Lemma 1.** *Let $P$ be an ideal projector onto $\mathbb{F}_{<n}[\mathbf{x}]$, and let (1.1) holds with $\langle B\rangle = \ker P$ and for every $b \in B$, $H_b$ is a homogeneous polynomial satisfying (1.2). Then*

*1) $b(\mathbf{x})\mu_{b,\mathbf{x}}(1) = b(\mathbf{x})$ for all $b \in B$.*

*2) The set $B$ is $\mathbb{F}$-linearly independent.*

*3) $\deg H_b \geq n$ for all $b \in B$.*

**Proof:** Since $b \in \ker P$, hence

$$b(\mathbf{x}) = (b - Pb)(\mathbf{x}) = b(\mathbf{x})\mu_{b,\mathbf{x}}(1)$$

by (1.1) and (1.2), which proves 1). To prove 2), assume that

$$\sum_{b\in B} \alpha_b b = 0 \text{ for some } \alpha_b \in \mathbb{F}.$$

Fix a $b^* \in B$. Then by linearity of $\mu_{b,\mathbf{x}}$ , by 1) and (1.2) we have

$$0 = H_{b^*}(D)(\sum_{b\in B} \alpha_b b) = \alpha_{b^*} b^* \Longrightarrow \alpha_{b^*} = 0.$$

Now, suppose that

$$m := \min\{\deg H_b : b \in B\} < n$$

and $H^* \in \{H_b : b \in B\}$ be such that $\deg H^* = m$. Then

$$0 \neq H^*(D)H^* \in \mathbb{F} \text{ and } \alpha_b := H_b(D)H^* \in \mathbb{F}, \quad \text{for all } b \in B.$$

Since $H^* \in \mathbb{F}_{<n}[\mathbf{x}]$,

$$0 = H^*(\mathbf{x}) - PH^*(\mathbf{x}) = \sum_{b \in B} b(\mathbf{x})\mu_{b,\mathbf{x}}(H_b(D)H^*) = \sum_{b \in B} \alpha_b b(\mathbf{x})$$

which contradicts 2) thus proves 3). □

We now proceed with the main theorem of this section.

**Theorem 3.** *Let $P$ be an ideal projector onto $\mathbb{F}_{<n}[\mathbf{x}]$ and let (1.1) holds with $\langle B\rangle = \ker P$ and homogeneous polynomials $H_b$ satisfying (1.2). Then the sets*

$$\{H_b : b \in B\} \text{ and } \{Lf(b) : b \in B\}$$

*form (dual) linear bases for the linear space $\mathbb{F}_{[n]}[\mathbf{x}]$ of homogeneous polynomials of degree $n$. In particular $\#B = N(n)$.*

**Proof:** Let $M_n[\mathbf{x}]$ be the set of monomials of degree $n$. For every $w \in M_n[\mathbf{x}]$, let

$$u_w := w - Pw \in \ker P. \tag{2.1}$$

Since $\operatorname{ran} P = \mathbb{F}_{<n}[\mathbf{x}]$, hence polynomials $\{u_w, w \in M_n[\mathbf{x}]\}$ are linearly independent polynomials of degree $n$ and

$$\dim \operatorname{span}\{u_w, w \in M_n[\mathbf{x}]\} = N(n). \tag{2.2}$$

Now let $B$ satisfy the assumptions of the theorem. From $\mathbb{F}_{<n}[\mathbf{x}] \cap \langle B\rangle = \{0\}$ we conclude that $\deg b \geq n$ and, from the lemma above, $\deg H_b \geq n$ for every $b \in B$, which implies

$$c_{b,w} := H_b(D)(u_w) \in \mathbb{F}. \tag{2.3}$$

Let

$$\mathcal{H}_n := \{H_b : b \in B, \deg H_b = n\} \text{ and } \mathcal{B}_n := \{b \in B : H_b \in \mathcal{H}_n\}. \tag{2.4}$$

Since $H \in \mathbb{F}_{[m]}[\mathbf{x}]$, $m > n$ implies $H(D)w = 0$, for all $w \in \mathbb{F}_{[n]}[\mathbf{x}]$, hence (1.1) implies

$$P'u_w = u_w = \sum_{b \in \mathcal{B}_n} b(\mathbf{x})\mu_{b,\mathbf{x}}(H_b(D)u_w) = \sum_{b \in \mathcal{B}_n} c_{b,w} b(\mathbf{x}).$$

and from (2.2) and (2.3) we conclude

$$\operatorname{span}\{u_w : w \in M_n[\mathbf{x}]\} \subset \operatorname{span}\{b : b \in \mathcal{B}_n\},$$

and thus by (2.2)

$$N(n) \leq \dim \operatorname{span}\{b : b \in \mathcal{B}_n\} = \#\mathcal{B}_n, \tag{2.5}$$

where the last equality is by the Lemma 1.

Once again from the second statement in Lemma 1, it follows that $\#\mathcal{B}_n \leq \dim \mathbb{F}_{[n]}[\mathbf{x}] = N(n)$. Hence $\{H_b : b \in \mathcal{B}_n\}$ is a basis for $\mathbb{F}_{[n]}[\mathbf{x}]$. Now, suppose that $\tilde{b} \in \mathcal{B}_n$ is such that $\deg \tilde{b} > n$. Then for some

$$f = \sum_{b \in \mathcal{B}_n} c_b H_b \in \mathbb{F}_{[n]}[\mathbf{x}],\ c_b \in \mathbb{F}, \tag{2.6}$$

$f(D)\tilde{b}$ is not a constant. On the other hand by (1.2),

$$(\sum_{b \in \mathcal{B}_n} c_b H_b(D)\tilde{b}) = c_{\tilde{b}} \in \mathbb{F}$$

is a constant, which gives the contradiction. In other words, for every $b \in \mathcal{B}_n$ we have $\deg b = n$, which proves that the sets

$$\{H_b : b \in \mathcal{B}_n\} \text{ and } \{\mathrm{Lf}(b) : b \in \mathcal{B}_n\}$$

form linear bases for the linear space $\mathbb{F}_{[n]}[\mathbf{x}]$.

It remains to show that $B \backslash \mathcal{B}_n = \emptyset$. Indeed if not, then some $\tilde{b} \in B \backslash \mathcal{B}_n$ has $\deg \tilde{b} > n$ and once again we have (2.7) for some $f$ satisfying (2.6). On the other hand, from (1.2) we have $\sum_{b \in \mathcal{B}_n} c_b H_b(D)\tilde{b} = 0$ which gives the desired contradiction. □

Let $w := (w_1, w_2, ..., w_{N(n)})$ be a fixed ordering of monomials in $M_n[\mathbf{x}]$.

**Corollary 1.** *Let $P$ be an ideal projector onto $\mathbb{F}_{<n}[\mathbf{x}]$ that admits the error formula (2.1), (2.2) for some bases $B$. Then there exists an $N(n) \times N(n)$ invertible scalar matrix $F_P$ such that the elements of $B$ form a vector $F_P^T(w - Pw)$ and the polynomials in $\{H_b : b \in B\}$ can be written as a vector $F_P^{-1}w$.*

**Corollary 2.** *Let $P$ be an ideal projector onto $\mathbb{F}_{<n}[\mathbf{x}]$ that admits the error formula (2.1), (2.2) for some basis $B$. Then $B$ is an $H$ -basis for* $\ker P$.

**Proof:** It is known (cf. [10]) that a basis $B$ is an $H$-basis if and only if $\langle \mathrm{Lf}(b), b \in B\rangle = \langle \mathrm{Lf}(f), f \in \langle B\rangle\rangle$. By Theorem 3, if $B$ admits an error formula, then

$$\langle \mathrm{Lf}(b), b \in B\rangle = \langle M_n[\mathbf{x}]\rangle\,.$$

Suppose that for some non-zero $f \in \langle B\rangle$, we have $\mathrm{Lf}(f) \notin \langle M_n[\mathbf{x}]\rangle$. Then $\deg f < n$ and hence $f \in \langle B\rangle \cap \mathbb{F}_{<n}[\mathbf{x}] = \ker P \cap \operatorname{ran} P$ which is a contradiction. □

## §3. Computation of $\mathfrak{m}(J)$

To fulfill the promise made in the introduction, it remains to observe that for Lagrange projectors onto $\mathbb{F}_{<n}[\mathbf{x}]$ there exists a basis $B$ such that $\langle B\rangle = \ker P$ and $\#B = d$. The idea of the proof is very simple. Assume that $Z(\ker P) = \{\mathbf{x}^{(j)}, j = 1, ..., \dim \mathbb{F}_{<n}[\mathbf{x}]\}$ and that the last coordinates $x_d^{(j)}$ of the $\mathbf{x}^{(j)}$ are all distinct. Let

$$p_d(\mathbf{x}) = \prod_{j=1}^{\dim \mathbb{F}_{<n}[\mathbf{x}]} (x_d - x_d^{(j)}),$$

and for $k = 1, ..., d-1$, let $p_k(\mathbf{x}) \in \operatorname{span}\{1, x_d, ..., x_d^{\dim \mathbb{F}_{<n}[\mathbf{x}]}\}$ be polynomials that interpolate $x_k$ at the points $\{\mathbf{x}^{(j)}, j = 1, ..., \dim \mathbb{F}_{<n}[\mathbf{x}]\}$. Then the polynomials $\{p_d(\mathbf{x}), x_k - p_k(\mathbf{x}), k = 1, ..., d-1\}$ form a basis for the ideal $\ker P$ of cardinality $d$. The general case is reduced to the above argument by change of variables. Actually, it yields a bit more:

**Theorem 4.** *Let $J \subset \mathbb{F}[\mathbf{x}] = \mathbb{F}^d[x_1, x_2, ..., x_d]$ be a zero-dimensional radical ideal. Then $\mathfrak{m}(J) = d$.*

**Proof:** Let $\mathfrak{H}_d$ be the hyperplane

$$\mathfrak{H}_d := \{(z_1, ..., z_d) \in \mathbb{F}^d : z_d = 0\}.$$

With every two distinct points $u, v \in Z(J) \subset \mathbb{F}$ we associate the unique hyperplane $\mathfrak{H}_{u,v} \subset \mathbb{F}$ orthogonal to the non-zero vector $(u - v) \in \mathbb{F}$. Since the set $Z(J)$ is finite, there are only finitely many such hyperplanes, and thus there exists a vector $\mathbf{y} = (y_1, ..., y_{d-1}, 1) \in \mathbb{F}$ such that the inner product

$$< \mathbf{u} - \mathbf{v}, \mathbf{y} > \neq 0 \text{ for all distinct } u, v \in Z(J) \text{ and } y_i \neq 0, \forall i = 1, ..., d. \tag{3.1}$$

We introduce the (linear) polynomial $g \in \mathbb{F}_{\leq 1}[\mathbf{x}]$ defined by

$$g(\mathbf{x}) = < \mathbf{x}, \mathbf{y} > \tag{3.2}$$

and consider the linear subspace $L \subset \mathbb{F}[\mathbf{x}]$ defined by

$$L := \operatorname{span}\{g^k, k = 0, ..., \#Z(J) - 1\}. \tag{3.3}$$

Thus

$$\dim L = \#Z(J) = \dim(\mathbb{F}[\mathbf{x}]/J). \tag{3.4}$$

We claim that

$$L \oplus J = \mathbb{F}[\mathbf{x}]. \tag{3.5}$$

In view of (3.4), it suffices to prove that $L \cap J = \{0\}$. Indeed if $f \in J \cap L$, then the polynomial

$$f = \sum_{j=0}^{\#Z(J)-1} a_j g^j(u) = \sum_{j=0}^{\#Z(J)-1} a_j z^j$$

equals to zero for $\#Z(J)$ *distinct* values of

$$z =< \mathbf{u}, \mathbf{y} >, \mathbf{u} \in Z(J).$$

Hence $f = 0$.

Let $Q$ be the ideal projector from $\mathbb{F}[\mathbf{x}]$ onto $L$ determined by the decomposition (3.5). Thus $\ker Q = J$ and $Q$ is a Lagrange projector. We claim that the set $B$ of $d$ polynomials

$$B = \{g^{\#Z(J)} - Q(g^{\#Z(J)}) \text{ and } x_j - Q(x_j),\ j = 1, ..., d-1\} \tag{3.6}$$

is a basis for the ideal $J$, i.e. $J = \langle B \rangle$.

Clearly, $\langle B \rangle \subset J$. Since $Q' = I - Q$ is a projector onto $J$, it suffices to prove that

$$Q'f \in \langle B \rangle \text{ for every } f \in \mathbb{F}^d[\mathbf{x}]. \tag{3.7}$$

We will do so in several steps. Let $A_j$ be the subalgebra of $\mathbb{F}[\mathbf{x}]$ generated by $\{g, x_1, ..., x_j\}$.

*Step 1:* $Q'f \in \langle B \rangle$ for every $f \in A_0$.

Since $A_0$ is a subalgebra generated by one polynomial $g$, we have to prove that $Q'g^m \in \langle B \rangle$ for all integers $m$. It is obviously so for $m \leq \#Z(J)$. We now proceed by induction. Assume that $m > \#Z(J)$ and

$$Q'g^k \in \langle B \rangle \text{ for all } k \leq m.$$

Then by (1.5),

$$Q'g^{m+1} = g \cdot Q'g^m + Q'(g \cdot Qg^m),$$

where the first term is in $\langle B \rangle$ by inductive assumption ($Q'g^m \in \langle B \rangle$) and the second terms belongs to $\langle B \rangle$, since $g \cdot Qg^m$ contains only scalar multiples of $g \cdot g^k$ for $k \leq \#Z(J)$.

*Step 2:* $Q'f \in \langle B \rangle$ for every $f \in A_j$ with $j = 0, 1, ..., d-1$.

Assume that the result is proven for a fixed $j \leq d-2$. We will use induction on $k$ to prove that $Q'(x_{j+1}^k \cdot A_j) \subset \langle B \rangle$ for all integers $k$ . Let $f \in A_j$. Using (1.5) once more, we have

$$Q'(x_j^{k+1} \cdot f) = x_j^k \cdot f \cdot Q'(x_j) + Q'(x_j^k \cdot f \cdot Q(x_j)).$$

Again, the first term is in $\langle B \rangle$, since, by (3.6), $Q'(x_j)$ is. The second term belongs to $\langle B \rangle$ since $f \cdot Q(x_j) \in L \cdot A_j$, and by the inductive assumption

$Q'(x_j^k \cdot f \cdot Q(x_j)) \in \langle B \rangle$. Thus we proved that for the algebra $A_{d-1}$ generated by $g$ and $x_2, ..., x_d$,

$$Q'(A_{d-1}) \subset \langle B \rangle \,.$$

*Step 3:* $Q'f \in \langle B \rangle$ for every $f \in A_d$ .

It is left to prove that $Q'(x_d^k \cdot A_{d-1}) \subset \langle B \rangle$. Observe that by the choice of the vector $\mathbf{y}$ and by (3.2),

$$x_d = g - \sum_{k=1}^{d-1} y_k x_k$$

and $x_d^k \cdot A_{d-1} = (g - \sum_{k=1}^{d-1} y_k x_k)^k \cdot A_{d-1} \subset A_{d-1}$. Since $A_d = \mathbb{F}[\mathbf{x}]$, the last step proves (3.7) and the inequality $\mathfrak{m}(J) \le d$ with it.

To prove the reverse inequality, suppose that $J = \langle b_1, ..., b_{d-1} \rangle$. Then by a well-known theorem from algebraic geometry (cf. [6], Proposition 5., p.460) we have $\dim Z(J) = d - (d-1) = 1$, which contradicts the assumption that $Z(J)$ is finite. □

Since $P'(M_n[\mathbf{x}])$ is a basis for the ideal $\ker P$, for any ideal projector $P$ onto $\mathbb{F}_{<n}[\mathbf{x}]$, it follows that

$$d \le \mathfrak{m}(\ker P) \le N(n) := \binom{n+d-1}{d-1}.$$

The previous theorem shows that the lower bound is attained on all Lagrange projectors (radical ideals). We will now show that the upper bound is attained for Taylor projectors onto $\mathbb{F}_{<n}[\mathbf{x}]$, as the error formula requires.

**Theorem 5.** *Let* $B := M_n[\mathbf{x}]$*. Then the ideal* $J := \langle B \rangle$ *is an algebraic complement of* $\mathbb{F}_{<n}[\mathbf{x}]$*, and*

$$\mathfrak{m}(\langle B \rangle) = N(n) = \binom{n+d-1}{d-1}.$$

**Proof:** Observe that $q \in \langle B \rangle$ if an only if $q = \sum_{k \ge n} q^{[k]}$ and $p \in \mathbb{F}_{<n}[\mathbf{x}]$ if and only if $p = \sum_{k<n} p^{[k]}$. Since every $f \in \mathbb{F}[\mathbf{x}]$ can be written uniquely as

$$f = (\sum_{k<n} f^{[k]}) + (\sum_{k \ge n} f^{[k]}),$$

it follows that $\langle B \rangle$ is an algebraic complement of $\mathbb{F}_{<n}[\mathbf{x}]$.

By way of contradiction, assume that $\langle B \rangle = \big\langle b_1, ..., b_{N(n)-1} \big\rangle$. Then for every $\mathbf{x}^\lambda \in M_n[\mathbf{x}]$ with $|\lambda| = n$,

$$\mathbf{x}^\lambda = \sum_{k=1}^{N(n)-1} a_k \cdot b_k$$

for some polynomials $a_k \in \mathbb{F}[\mathbf{x}]$. Since $n$ is “the least degree” for each $b_k \in \langle B \rangle$, it follows that

$$\mathbf{x}^\lambda = \sum_{k=1}^{N(n)-1} a_k^{[0]} \cdot b_k^{[n]}.$$

But $a_k^{[0]} \in \mathbb{F}$ and thus

$$\operatorname{span}\{\mathbf{x}^\lambda : |\lambda| = n\} \subset \operatorname{span}\{b_1^{[n]}, ..., b_{N(n)-1}^{[n]}\}. \tag{3.8}$$

Since $\{\mathbf{x}^\lambda : |\lambda| = n\}$ is a linearly independent set of polynomials, the space on the left has dimension $N(n)$, and the space on the right has dimension at most $N(n) - 1$, contradicting the embedding (3.8). □

**Conjecture 1.** *For $d > 1$, the Taylor projectors are the only ideal projectors onto $\mathbb{F}_{<n}[\mathbf{x}]$ that admit the error formula (1.1), (1.2) with minimal $B$.*

In partial support of this statement, let us mention that the conjecture is true in the bivariate case.

**Theorem 6.** *Let $P$ be a bivariate ideal projector onto $\mathbb{F}_{<n}[x, y]$ that admits the error formula (2.1), (2.2) for minimal bases $B$. Then $P$ is a bivariate Taylor projector.*

**Proof:** It was shown in [15] that Taylor projectors are the only bivariate ideal projectors satisfying

$$\mathfrak{m}(\ker P) = \binom{n+d-1}{d-1}. \quad \square$$

## §4. Final Remarks

1) The existence of error formulas satisfying (1.1), (1.2), without any ”minimality” assumptions is in itself problematic. It is entirely possible that the Chung-Yao projectors considered in [4] are the only Lagrange projectors that admit such formulas. The Theorem 7 of [17] contains a mistake. Contrary to its claim, it is not-known if the ideal projector $P$ from $\mathbb{F}[x, y]$ onto $\mathbb{F}_{<2}[x, y]$ defined by

$$Px^2 = 0, \ Pxy = 0, \ Py^2 = x$$

admits an error formula.

2) All the error formulas constructed in [4], [17] and [18], of the type discussed in this paper, happen to have two peculiar properties: The basis $B$ used in these formulas are unshortenable and each element of the basis is a product of linear factors. One wonders if one or both of these properties are indeed necessary for the formulas.

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