On Real Solutions for Systems of Polynomial Equations

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August 17, 2007

Standard exercise in Calculus uses the intermediate value theorem to prove that every real polynomial of odd degree has at least one real zero. In this note we give a simple proof of a (hopefully original) observation that generalizes the above statement to polynomials of several variables. Namely, we prove that every ideal of polynomials of odd codimension has a common real zero.

Let $\mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, ..., x_d]$ denotes the algebra of polynomials in d variables with real coefficients. For an ideal $I \subset \mathbb{R}[\mathbf{x}]$ we consider the factor-algebra

$$A := \mathbb{R}[\mathbf{x}]/I = \{[f],\ f \in \mathbb{R}[\mathbf{x}]\}.$$

consisting of cosets

$$[f] := \{g \in \mathbb{R}[\mathbf{x}] : (g - f) \in I\}$$

with a naturally defined operation of addition and multiplication.

The codimension of I is defined as $\dim(\mathbb{R}[\mathbf{x}]/I)$.

If $d = 1$ and a univariate polynomial $p \in \mathbb{R}[\mathbf{x}]$ has an odd degree, say $\deg p = 2n + 1$ then the ideal

$$I := \langle p \rangle := \{fp,\ f \in \mathbb{R}[\mathbf{x}]\ \}$$

complements the space $\mathbb{R}_{<2n+1}[\mathbf{x}]$ of polynomials of degree less than $2n+1$, i.e.,

$$\mathbb{R}_{<2n+1}[\mathbf{x}] \oplus I = \mathbb{R}[\mathbf{x}].$$

Hence $\mathbb{R}[\mathbf{x}]/I$ is isomorphic to $\mathbb{R}_{<2n-1}[\mathbf{x}]$ and $\dim(\mathbb{R}[\mathbf{x}]/I)$ is odd. Every zero of p is a zero of every polynomial in I, hence

$$V(I) := \{x \in \mathbb{R} : q(x) = 0 \text{ for all } q \in I\} \neq \varnothing.$$

We will now extend this last statement to a multivariate setting:

Theorem 1 *Let $I \subset \mathbb{R}[\mathbf{x}]$ be an ideal of odd codimension. Then*

$$V(I) := \{\mathbf{x} \in \mathbb{R}^d : q(\mathbf{x}) = 0 \textit{ for all } q \in I\} \neq \varnothing.$$

Proof. The argument relies on the well-known properties (cf. [1], pp. 51–55) of endomorphisms

$$M_i([f]) := [x_i][f] = [x_i f],\ i = 1, ..., d,$$

on the factor-algebra $A := \mathbb{R}[\mathbf{x}]/I$.

If $f = \sum c(m_1, ..., m_d) x_1^{m_1} ... x_d^{m_d} \in I$ then $[f] = 0$ and

$$f(M_1, ..., M_d) := \sum c(m_1, ..., m_d) M_1^{m_1} ... M_d^{m_d} = 0$$

since for every $g \in \mathbb{R}[\mathbf{x}]$, $f(M_1, ..., M_d)([g]) = [fg] = [f][g] = 0$.

Assume that endomorphisms $M_1, ..., M_d$ have a common eigenvector $[h] \neq 0$, i.e., there exist numbers $\lambda_1, ..., \lambda_d$ such that $M_i([h]) = \lambda_i [h]$ for each $i = 1, ..., d$. Then for every $f \in I$

$$0 = f(M_1, ..., M_d)([h]) = (\sum c(m_1, ..., m_d) \lambda_1^{m_1} ... \lambda_d^{m_d})[h]$$

and since $[h] \neq 0$, it follows that $f(\lambda_1, ..., \lambda_d) = 0$. Thus $(\lambda_1, ..., \lambda_d) \in V(I)$. It remains to show that $M_1, ..., M_d$ have a common eigenvector if I has an odd codimension. Once again it is well known (cf. [2], Lemma 4) and the proof presented below is purely for convenience.

We want to prove that any sequence $L_1, ..., L_d$ of pairwise commuting endomorphisms on an odd-dimensional $\mathbb{R}$-vector space G have a common eigenvector. The proof is by induction in d. If $d = 1$ then the characteristic polynomial of L_1 has an odd degree and thus has a real root that corresponds to an eigenvector of L_1. Assume that the statement is true for any sequence of $d - 1$ commuting endomorphisms. Let H be a subspace of G of minimal odd dimension, invariant with respect to $L_1, ..., L_d$. Let $\tilde{L}_1, ..., \tilde{L}_d$ be the restrictions of $L_1, ..., L_d$ to H and let $h \in H$ be a non-zero eigenvector for $\tilde{L}_d$ corresponding to an eigenvalue λ. Consider the spaces

$$H_1 := \ker(\tilde{L}_d - \lambda I),\ H_2 := \mathrm{ran}\ (\tilde{L}_d - \lambda I).$$

These two spaces are invariant with respect to $L_1, ..., L_d$ and one of the two has an odd dimension since $\dim H_1 + \dim H_2 = \dim H$. Since $H_1 \neq \{0\}$, $\dim H_2 < \dim H$ and from minimality of H it follows that H_1 has and odd dimension hence $H_1 = H$. Thus L_d is a multiple of identity on H and any eigenvector in H, common to $L_1, ..., L_{d-1}$ is also an eigenvector of L_d. ∎

Example 2 *Consider the system of three quadratic equations*

$$\begin{cases} p_0(x, y) := x^2 - (a_0 + b_0 x + c_0 y) = 0 \\ p_1(x, y) := xy - (a_1 + b_1 x + c_1 y) = 0 \\ p_2(x, y) := y^2 - (a_2 + b_2 x + c_2 y) = 0 \end{cases} \tag{1}$$

The ideal I generated by p_0, p_1, p_2 complements the space $\mathbb{R}_{<2}[x, y] = span\{1, x, y\}$ if and only if (see [3])

$$\begin{aligned} a_0 &= -c_0 c_2 + c_0 b_1 - c_1 b_0 + c_1^2 \\ a_1 &= -b_1 c_1 + c_0 b_2 \\ a_2 &= -b_2 b_0 + b_2 c_1 - b_1 c_2 + b_1^2 \end{aligned} \tag{2}$$

Therefore, assuming (2) the codimension of I is odd and the system of equations (1) has a real solution.

References

[1] Cox, David, John Little, and Donal O'Shea, Using Algebraic Geometry, Grad. Texts in Math., Springer-Verlag, New York, 1998.

[2] Derksen, Harm , The Fundamental Theorem of Algebra and Linear Algebra, The American Mathematical Monthly, Vol. 110, No. 7 (2003), pp. 620–623

[3] Shekhtman, Boris, Ideal Projectors onto Planes, Approximation Theory XI: Gatlinburg 2004, C. K. Chui, M. Neamptu and L.L.Schumaker (eds.), Nashboro Press (2005), pp. 395-404.