On Simultaneous Block-Diagonalization of Cyclic Commuting Matrices

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Abstract

We study simultaneous block-diagonalization of cyclic d-tuples of commuting matrices. Some application to ideal projectors are also presented. In particular we extend Hans Stetter's theorem characterizing Lagrange projectors.

1 Introduction

Let V be a finite-dimensional space over complex field $\mathbb{C}$ and let $\mathbf{L} := (L_1, ..., L_d)$ be a d-tuple of pairwise commuting operators on V. Every polynomial $p(x_1, ..., x_d) = \sum c_{k_1,...k_d} x_1^{k_1} ... x_d^{k_d} \in \mathbb{C}[x_1, ..., x_d]$ defines an operator

$$p(\mathbf{L}) := \sum c_{k_1,...k_d} L_1^{k_1} ... L_d^{k_d} \tag{1.1}$$

on V. A d-tuple $\mathbf{L}$ is called **cyclic** if there exists a vector $v_0 \in V$ such that

$$\{p(\mathbf{L})v_0, p \in \mathbb{C}[x_1, ..., x_d]\} = V. \tag{1.2}$$

A vector v_0 satisfying (1.2) is called a cyclic vector for $\mathbf{L}$.

A vector $v \in V$ is called a common eigenvector for $\mathbf{L}$ if for all $j = 1, ..., d$ there exist $\lambda_j \in \mathbb{C}$ such that $L_j v = \lambda_j v$. The d-tuple $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_d) \in \mathbb{C}^d$ is called an eigentuple of $\mathbf{L}$. The set of all eigentuples of $\mathbf{L}$ is denoted by $\sigma(\mathbf{L})$.

In case $d = 1$, an operator L is cyclic if and only if L is **1-regular**, i.e., every eigenspace of L is at most one-dimensional. For $d > 1$ this is false in both directions as the following example (already used in [4] for different purposes) demonstrates:

Example 1.1 *First consider* $\mathbf{L} = (L_1, L_2)$ *on* $\mathbb{C}^3$ *given by*

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{1.3}$$

This is a cyclic d-tuple, $\sigma(\mathbf{L}) = \{(0,0)\}$ *and vectors* $(0,1,0)$ *and* $(0,0,1)$ *are common eigenvectors for* $\mathbf{L}$*. On the other hand* $\mathbf{L}^t = (L_1^t, L_2^t)$ *is given by*

$$L_1^t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ L_2^t = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.4}$$

is not cyclic, yet the only common eigenspace is one-dimensional, spanned by the vector $v = (1,0,0)$*.*

Observe that L is 1-regular means that the Jordan form of L does not contain two Jordan blocks with the same eigenvalue. In other words, the number of Jordan blocks in the Jordan form of L is precisely the same as the number of distinct eigenvalues of L: $\#\sigma(L)$.

The main result of this paper (Theorem 2.6) shows that a cyclic d-tuple $\mathbf{L}$ of d commuting operators is simultaneously block-diagonalizable into $\#\sigma(\mathbf{L})$ blocks and $\#\sigma(\mathbf{L})$ is the largest number of blocks in any simultaneous block-diagonalization of $\mathbf{L}$. The converse is still false. Indeed, the pair $\mathbf{L} = (L_1, L_2)$ in the preceding example can be decomposed into exactly as many blocks as the pair $\mathbf{L}^t = (L_1^t, L_2^t)$.

Definition 1.2 *Let* $\mathbf{L} := (L_1, ..., L_d)$ *be a d-tuple of operators on* V*. A direct sum decomposition*

$$V = V_1 \oplus V_2 \oplus ... \oplus V_t \tag{1.5}$$

is $\mathbf{L}$*-invariant if each subspace* V_k*,* $k = 1, ..., t$ *is an invariant subspace for each of the operators* L_j*,* $j = 1, ..., d$*.*

Letting $L_{j,k} := L_j \mid_{V_k}$ denote the restriction of L_j onto V_k we write

$$\mathbf{L}_k = \mathbf{L} \mid_{V_k} := (L_{1,k}, ..., L_{d,k}). \tag{1.6}$$

The simultaneous block-diagonalization of $\mathbf{L}$ into t blocks amounts to nothing more then the $\mathbf{L}$-invariant decomposition (1.5) of V: Indeed, for an appropriately chosen bases, the matrix $\tilde{L}_j$ of L_j can be written in a block-diagonal form

$$\tilde{L}_j = \begin{bmatrix} \tilde{L}_{j,1} & 0 & \cdots & 0 \\ 0 & \tilde{L}_{j,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{L}_{j,t} \end{bmatrix},$$

where each $\tilde{L}_{j,k}$ is a square matrix of size $\dim V_k$ representing operator $L_{j,k}$ on V_k. We write $\mathbf{L} = diag(\mathbf{L}_k)$.

Thus our main theorem shows that for a cyclic d-tuple $\mathbf{L}$ of commuting operators on V, the maximal number t in an $\mathbf{L}$-invariant decomposition (1.5) of V is exactly the same as $\#\sigma(\mathbf{L})$.

In Section 3 we apply this theorem to study decompositions of ideal projectors and associated multiplication operators. In particular, we extend Stetter's characterization of Lagrange projectors (cf. [10], [2]) to general ideal projectors.

We use the rest of this section to recall a few well-known facts from commutative algebra (cf. [5], [6]):

For every ideal $J \subset \mathbb{C}[x_1, ..., x_d]$ we use $Z(J)$ to define the associated variety

$$Z(J) := \{\mathbf{z} \in \mathbb{C}^d : p(\mathbf{z}) = 0, \text{ for all } p \in J \}$$

in $\mathbb{C}^d$. The ideal J is called **zero-dimensional** if $\dim \mathbb{C}[x_1, ..., x_d]/J < \infty$. An ideal J is zero-dimensional if and only if the set $Z(J)$ is finite. In fact $\#Z(J) \leq \dim \mathbb{C}[x_1, ..., x_d]/J$. An ideal J is **radical** if $f^m \in J$ for some integer m implies $f \in J$. A zero-dimensional ideal J is radical if and only if $\#Z(J) = \dim \mathbb{C}[x_1, ..., x_d]/J$. An ideal $J \subset \mathbb{C}[x_1, ..., x_d]$ is called **primary** if $fg \in J$ implies $f \in J$ or $g^m \in J$ for some integer m. A zero-dimensional ideal is primary if and only if the variety $Z(J)$ consists of a single point in $\mathbb{C}^d$. The Lasker-Noether theorem, applied to zero-dimensional ideals, states that every zero-dimensional ideal has a unique **minimal primary decomposition**, that is

$$J = \cap_{k=1}^{\#Z(J)} J_k$$

where each J_k is a primary zero-dimensional ideal and $Z(J_k) \cap Z(J_n) = \varnothing$ for $n \neq k$.

2 Cyclic d-tuples.

Let V be a finite-dimensional space over the complex field $\mathbb{C}$ and let $\mathbf{L} := (L_1, ..., L_d)$ be a d-tuple of pairwise commuting operators on V. A d-tuple $\mathbf{L}$ defines an ideal

$$J_{\mathbf{L}} := \{p \in \mathbb{C}[x_1, ..., x_d] : p(\mathbf{L}) = 0\} \subset \mathbb{C}[x_1, ..., x_d]. \tag{2.1}$$

Proposition 2.1 *The ideal $J_{\mathbf{L}}$ is zero-dimensional, hence $Z(J_{\mathbf{L}})$ is finite.*

Proof. Let $\mathcal{L}(V)$ be the space of all linear operators on V. Since V is finite dimensional, so is $\mathcal{L}(V)$. Let $\phi_{\mathbf{L}} : \mathbb{C}[x_1, ..., x_d] \to \mathcal{L}(V)$ be a mapping defined by

$$\phi_{\mathbf{L}}(p) := p(\mathbf{L}) \in \mathcal{L}(V).$$

Since $\ker \phi_{\mathbf{L}} = J_{\mathbf{L}}$, the factorization

$$\begin{tikzcd}
\mathbb{C}[x_1, ..., x_d] \arrow[rr, "\phi_{\mathbf{L}}"] \arrow[dr, "\alpha"] & & \mathcal{L}(V) \\
& \mathbb{C}[x_1, ..., x_d]/J_{\mathbf{L}} \arrow[ur, "\beta"] &
\end{tikzcd}$$

induces an injection β into a finite-dimensional space. Thus $\dim \mathbb{C}[x_1, ..., x_d]/J_{\mathbf{L}} < \infty$ and hence $\# Z(J_{\mathbf{L}}) < \infty$. ∎

The following proposition collects a few simple and well-known properties of commuting d-tuples of operators. The proofs are given purely for convenience.

Proposition 2.2 *Let* $\mathbf{L}$ *be a d-tuple of pairwise commuting operators on* V*. Then*

(i) $\mathbf{L}$ *has a common eigenvector, i.e.,* $\sigma(\mathbf{L}) \neq \varnothing$*.*

(ii) If $v \in V$ *is a common eigenvector for* $\mathbf{L}$ *that corresponds to an eigentuple* $\boldsymbol{\lambda}$ *then* $p(\mathbf{L})v = p(\boldsymbol{\lambda})v$

(iii) If $\lambda \in \sigma(\mathbf{L})$ *and* $p \in J_{\mathbf{L}}$ *then* $p(\lambda) = 0$*, i.e.,* $\sigma(\mathbf{L}) \subset Z(J_{\mathbf{L}})$*.*

(iv) If $\mathbf{L}$ *is cyclic and* v_0 *is a cyclic vector for* $\mathbf{L}$ *then*

$$J_{\mathbf{L}} = \{p \in \mathbb{C}[x_1, ..., x_d] : p(\mathbf{L})v_0 = 0\}. \tag{2.3}$$

Proof. (i) By induction. For $d = 1$ the statement is obvious. Let $(\lambda_1, ..., \lambda_{d-1}) \in \mathbb{C}^{d-1}$ be an eigentuple for $(L_1, ..., L_{d-1})$. Then the subspace $H := \cap_{j=1}^{d-1} \ker(L_j - \lambda_j I) \subset V$ is non-zero and invariant with respect to L_d. Indeed If $h \in H$ then $(L_j - \lambda_j I)L_d h = L_d(L_j - \lambda_j I)h = 0$, hence $L_d h \in H$ and any eigenvector of L_d in H is a common eigenvector for $(L_1, ..., L_d)$.

(ii) Follows from applying $p(\mathbf{L})$ in the form (1.1) to v.

(iii) Let $v \in V$ is a common eigenvector for $\mathbf{L}$ that corresponds to an eigentuple $\boldsymbol{\lambda} \in \sigma(\mathbf{L})$. For any $p \in J_{\mathbf{L}}$ we have $0 = p(\mathbf{L})v = p(\boldsymbol{\lambda})v$ and, since $v \neq 0$, $p(\boldsymbol{\lambda}) = 0$.

(iv) assume that $p(\mathbf{L})v_0 = 0$ and $v \in V$. Then, by cyclicity, there exists a polynomial $q \in \mathbb{C}[x_1, ..., x_d]$ such that $v = q(L)v_0$. We have $p(\mathbf{L})v = p(\mathbf{L})q(\mathbf{L})v_0 = q(\mathbf{L})p(\mathbf{L})v_0 = 0$. Hence $p(\mathbf{L})v = 0$ for all $v \in V$ and $p \in J_{\mathbf{L}}$. ∎

Remark 2.3 *In the Theorem 2.6 below we will show that* $\sigma(\mathbf{L})$ *is actually equal to* $Z(J_{\mathbf{L}})$*.*

The next lemma is the key to our analysis of $\mathbf{L}$-invariant decomposition of V:

Lemma 2.4 *Let* $\mathbf{L} := (L_1, ..., L_d)$ *be a cyclic d-tuple of pairwise commuting operators on* V*. Let* $V = V_1 \oplus V_2$ *be an* L*-invariant decomposition of* V*. Then* $\mathbf{L}_k := \mathbf{L} \,|V_k$ *(*$k = 1, 2$*) is cyclic on* V_k *and* $\sigma(\mathbf{L}_1) \cap \sigma(\mathbf{L}_2) = \varnothing$*. In other words common eigenvectors for* $\mathbf{L}$ *in* V_1 *and* V_2 *correspond to different eigentuples.*

Proof. Let v_0 be a cyclic vector for $\mathbf{L}$ and let $v_0 = v_0' + v_0''$ with $v_0' \in V_1$ and $v_0'' \in V_2$. Then clearly v_0' is a cyclic vector for $\mathbf{L}_1$ and v_0'' is a cyclic vector for $\mathbf{L}_2$. Let $p \in \mathbb{C}[x_1, ..., x_d]$ be such that $p(\mathbf{L})v_0 = v_0'$. Then

$$v_0' = p(\mathbf{L})v_0 = p(\mathbf{L})(v_0' + v_0'') = p(\mathbf{L})v_0' + p(\mathbf{L})v_0'',$$

hence

$$(1 - p)(\mathbf{L})v_0' = p(\mathbf{L})v_0''.$$

Since $V_1 \cap V_2 = \{0\}$ and V_1 and V_2 are invariant with respect to $\mathbf{L}$, it follows that $(1-p)(\mathbf{L})v_0' = p(\mathbf{L})v_0'' = 0$. But $(1-p)(\mathbf{L})v_0' = (1-p)(\mathbf{L}_1)v_0' = 0$ and $p(\mathbf{L})v_0'' = p(\mathbf{L}_2)v_0'' = 0$. Hence, by Proposition 2.2 (iv), $(1-p) \in J_{\mathbf{L}_1}$ and $p \in J_{\mathbf{L}_2}$. Now, if $\boldsymbol{\lambda} \in \sigma(\mathbf{L}_1) \cap \sigma(\mathbf{L}_2)$ then, by Proposition 2.2 (iii), $p(\boldsymbol{\lambda}) = 0$ and $(1-p)(\boldsymbol{\lambda}) = 0$ which is clearly not possible. ■

Remark 2.5 *The converse does not hold. Since the operators* $\mathbf{L}^t = (L_1^t, L_2^t)$ *from Example 1.1 have (vacuously) the following property: For any decomposition of* $\mathbb{C}^3 = V_1 \oplus V_2$ *into* $\mathbf{L}$*-invariant subspaces,* V_1 *and* V_2 *cannot each have an eigenvector that correspond to the same eigenvalue.*

Theorem 2.6 *Let* $\mathbf{L} := (L_1, ..., L_d)$ *be a cyclic d-tuple of pairwise commuting operators on* V*. Let*

$$V = V_1 \oplus V_2 \oplus ... \oplus V_t \tag{2.4}$$

be an $\mathbf{L}$*-invariant decomposition of* V*. Then*

(i) $t \leq \#\sigma(\mathbf{L}) \leq \#Z(J_{\mathbf{L}})$.

(ii) There exists an $\mathbf{L}$*-invariant decomposition of* V*:*

$$V = V_1 \oplus V_2 \oplus ... \oplus V_m \tag{2.5}$$

with $m = \#Z(J_{\mathbf{L}})$.

(iii) In particular, $\sigma(\mathbf{L}) = Z(J_{\mathbf{L}})$ *and* $m = \#\sigma(\mathbf{L}) = \#Z(J_{\mathbf{L}})$ *is a maximal number of subspaces in any* $\mathbf{L}$*-invariant decomposition of* V*.*

(iv) The decomposition (2.5) with $m = \#Z(J_{\mathbf{L}})$ *is unique and* $\sigma(\mathbf{L}_k)$*, where* $\mathbf{L}_k := \mathbf{L} \mid V_k$*, is a singleton.*

Proof. The first inequality in (i) follows from Lemma 2.4 by pigeonhole principle, the second from Proposition 2.2 (iii).

To prove the second statement of the theorem, let $Z(J_{\mathbf{L}}) = \{\mathbf{z}_1, ..., \mathbf{z}_m\} \subset \mathbb{C}^d$ and let

$$J_{\mathbf{L}} = \cap_{k=1}^m J_k \tag{2.6}$$

be the primary decomposition of $J_{\mathbf{L}}$, i.e., each J_k is a primary ideal with $Z(J_k) = \{\mathbf{z}_k\}$. We use $J^{(k)}$ to denote the ideal $(\cap_{s\neq k} J_s)$. Let v_0 be a cyclic vector for $\mathbf{L}$ and define

$$V_k = \{p(\mathbf{L})v_0, p \in J^{(s)}\}. \tag{2.7}$$

We first claim each V_k is an invariant subspace for each L_j. Indeed $L_j p(\mathbf{L})v_0 = (x_j p)(\mathbf{L})v_0$ and if $p \in J^{(s)}$, so is $x_j p$ thus showing that $L_j p(\mathbf{L})v_0 \in V_k$.

Now if $k \neq l$ and $v \in V_k \cap V_l$ we have $v = p(\mathbf{L})v_0 = q(\mathbf{L})v_0$ with $p \in (\cap_{s\neq k} J_s)$ and $q \in (\cap_{s\neq l} J_s)$. We have $(p(\mathbf{L}) - q(\mathbf{L}))v_0 = 0$ implying (by proposition 2.2 (iv)) $(p(\mathbf{L}) - q(\mathbf{L})) = 0$ hence $p - q \in J_{\mathbf{L}} = \cap_{k=1}^m J_k$. In particular $p - q \in J^{\{k)}$ and since $p \in J^{\{k)}$, so is q. Thus $q \in J^{\{k)} \cap J^{(l)} = J_{\mathbf{L}}$ and, by definition of $J_{\mathbf{L}}$, $q(\mathbf{L}) = 0$ implying $v = q(\mathbf{L})v_0 = 0$. This shows that $V_k \cap V_l = \{0\}$. It remains to prove that

$$V_1 + V_2 + ... + V_m = V \tag{2.8}$$

Let $h_k \in \mathbb{C}[x_1, ..., x_d]$ be such that

$$h_k(\mathbf{z}_s) = \delta_{k,s}. \tag{2.9}$$

Since h_k is equal to zero for each point of $Z(J^{(k)})$, by Hilbert's Nullstellensatz, there exists an integer n such that $h_k^n \in J^{(k)}$ for all k. From (2.9)

$$(1 - \sum_{k=1}^{m} h_k^n)(z_s) = 0 \text{ for all } s = 1, ..., m$$

and again, by the Nullstellensatz, there exists an integer l so that $(1-\sum\limits_{k=1}^{m} h_k^n)^l \in J_{\mathbf{L}}$, thus $(1 - \sum\limits_{k=1}^{m} h_k^n)^l(\mathbf{L}) = 0$. Expanding $(1 - \sum\limits_{k=1}^{m} h_k^n)^l$ we obtain

$$(1 - \sum_{k=1}^{m} h_k^n)^l = 1 - \sum_{k=1}^{m} h_k^n p_k \in J_{\mathbf{L}}$$

for some polynomials $p_k \in \mathbb{C}[x_1, ..., x_d]$ (p_k are polynomials in h_k^n). Hence $I = \sum\limits_{k=1}^{m} h_k^n p_k(\mathbf{L})$ and every $v \in V$ has a decomposition $v = \sum\limits_{k=1}^{m} h_k^n p_k(\mathbf{L})v$. Since $v = f(\mathbf{L})v_0$ for some $f \in \mathbb{C}[x_1, ..., x_d]$, we obtain

$$v = \sum_{k=1}^{m} h_k^n p_k(\mathbf{L}) f(\mathbf{L}) v_0 = \sum_{k=1}^{m} (h_k^n p_k f)(\mathbf{L}) v_0.$$

Since $h_k^n \in J^{(k)}$, it follows that $(h_k^n p_k f) \in J^{(k)}$ and $(h_k^n p_k f)(\mathbf{L})v_0 \in V_k$ thus proving (2.8) and (ii).

(iii) follows immediately from parts (i) and (ii) of the theorem.

To prove (iv), suppose that

$$V = U_1 \oplus U_2 \oplus ... \oplus U_m \tag{2.10}$$

is an $\mathbf{L}$-invariant decomposition with $m = \#Z(J_{\mathbf{L}})$ and let $\mathbf{L}_k$ be the restriction of $\mathbf{L}$ to U_k. The ideal $J_{\mathbf{L}_k}$ is primary, for otherwise the primary decomposition of this ideal would lead to an $\mathbf{L}_k$-invariant (hence $\mathbf{L}$-invariant) decomposition of U_k and thus decomposition of V into more then m subspaces. This would contradict part (i) of the theorem. It is now easy to check that

$$J_{\mathbf{L}} = \cap_{k=1}^{m} J_{\mathbf{L}_k} \tag{2.11}$$

is the primary decomposition of $J_{\mathbf{L}}$, hence coincites with decomposition (2.6). Without loss of generality, let $J_{\mathbf{L}_k} = J_k$ for all $k = 1, ..., m$. Let

$$v_0 = u_0^{(1)} + ... + u_0^{(m)} \tag{2.12}$$

be the decomposition of the cyclic vector v_0 according to (2.9). For every $v \in V$ there exists a polynomial $p \in \mathbb{C}[x_1, ..., x_d]$ such that

$$v = p(\mathbf{L})v_0 = p(\mathbf{L})u_0^{(1)} + ... + p(\mathbf{L})u_0^{(m)} = p(\mathbf{L}_1)u_0^{(1)} + ... + p(\mathbf{L}_m)u_0^{(m)}. \quad (2.13)$$

It follows from (2.10) that $v \in U_k$ if and only if

$$p(\mathbf{L}_s)u_0^{(s)} = 0 \text{ for all } s \neq k. \quad (2.14)$$

But, clearly, $u_0^{(k)}$ is a cyclic vector for $\mathbf{L}_k$, hence (2.14) is equivalent to $p \in \cap_{s \neq k}(J_{\mathbf{L}_k}) = J^{(k)}$. Thus $p(\mathbf{L})v_0 \in U_k$ if and only if $p \in J^{(k)}$ and $U_k = V_k$. ■

Let us finish this section with another observation on cyclic commuting d-tuples:

For a d-tuple $\mathbf{L} := (L_1, ..., L_d)$ of pairwise commuting operators on V define $\mathfrak{C}(\mathbf{L})$ to be the set of all operators that commute with every operator $L_1, ..., L_d$. In case $d = 1$, an operator L is cyclic if and only if every operator in $\mathfrak{C}(L)$ is a polynomial in L:

$$\mathfrak{C}(L) = \{p(L), \ p \in \mathbb{C}[x_1]\}.$$

Theorem 2.7 *Let $\mathbf{L} := (L_1, ..., L_d)$ be a cyclic d-tuple of pairwise commuting operators on V. If $T \in \mathfrak{C}(\mathbf{L})$ then $T = q(A)$ for some $q \in \mathbb{C}[x_1, ..., x_d]$. The converse does not hold. The d-tuple $\mathbf{L}^t = (L_1^t, L_2^t)$ defined in (1.4) is not cyclic, yet*

$$\mathfrak{C}(\mathbf{L}^t) = \{p(\mathbf{L}^t), \ p \in \mathbb{C}[x_1, x_2]\}.$$

Proof. Assume that $\mathbf{L}$ is cyclic and let v_0 be a cyclic vector for $\mathbf{L}$. If $T \in \mathfrak{C}(\mathbf{L})$, let $q \in \mathbb{C}[x_1, ..., x_d]$ be a polynomial such that $Tv_0 = q(\mathbf{L})v_0$. We claim that $T = q(\mathbf{L})$. Indeed, let

$$\{v_j = f_j(\mathbf{L})v_0, \ j = 1, ..., N\}$$

be a basis for V. Then

$$q(\mathbf{L})v_j = q(\mathbf{L})f_j(\mathbf{L})v_0 = f_j(\mathbf{L})q(\mathbf{L})v_0 = f_j(\mathbf{L})Tv_0 = Tf_j(\mathbf{L})v_0 = Tv_j$$

for every $j = 1, ..., N$, which shows that $q(\mathbf{L}) = T$.

As to the converse, let T commutes with L_1^t and L_2^t from example ??. Then T^t commutes with L_1 and L_2. By the first part of the theorem, there exists a polynomial $q \in \mathbb{C}[x_1, x_2]$ such that $T^t = q(L_1, L_2)$. Hence $T = q(L_1^t, L_2^t)$. ■

3 Decomposition of Ideal projectors

In this section we use Theorem 2.6 to extend Stetter's characterization of Lagrange projectors(cf. [10], [2]) to general ideal projectors acting in the space $\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, ..., x_d]$ of polynomials in d variables.

Definition 3.1 *(cf. [1]) A linear idempotent map P on $\mathbb{C}[\mathbf{x}]$ is called an* ***ideal projector*** *if* $\ker P$ *is an ideal in* $\mathbb{C}[\mathbf{x}]$.

Theorem 3.2 *(de Boor [2]) A linear operator P on $\mathbb{C}[\mathbf{x}]$ is an ideal projector if and only if*

$$P(fg) = P(fPg) \tag{3.1}$$

for all $f, g \in \mathbb{C}[\mathbf{x}]$.

A standard example of an ideal projector onto an N-dimensional subspace $V \subset \mathbb{C}[\mathbf{x}]$ is a **Lagrange** projector, i.e., a linear projector P for which Pf is the unique element in V such that $f(\mathbf{z}_k) = Pf(\mathbf{z}_k)$, $j = 1, ..., N$ for some set $\{\mathbf{z}_1, ..., \mathbf{z}_N\}$ of N distinct points in $\mathbb{C}^d$. In this case the ideal $\ker P$ is a radical ideal, its associated variety

$$Z(\ker P) := \{\mathbf{z} \in \mathbb{C}^d : f(\mathbf{z}) = 0,\ \forall f \in \ker P\} = \{\mathbf{z}_1, ..., \mathbf{z}_N\}.$$

The minimal primary decomposition for the ideal $\ker P$ is

$$\ker P = J_1 \cap J_2 \cap ... \cap J_N$$

where each J_j is a maximal ideal $J_j = \{f \in \mathbb{C}[\mathbf{x}] : f(\mathbf{z}_j) = 0\}$.

Every ideal projector P onto V generates a d-tuple $\mathbf{M}_P = (M_1, ..., M_d)$of d **multiplication operators** on V defined by

$$M_j(v) := P(x_j v)$$

for every $v \in V$. The d-tuple $\mathbf{M}_P$ is a cyclic d-tuple of pairwise commuting operators on V (cf. [2]) and

$$\{p(M_1, ..., M_d)v_0,\ p \in \mathbb{C}[\mathbf{x}]\} = V$$

with $v_0 := P1 \in V$. Some insight into the relation between P and $\mathbf{M}_P$ is shed by a beautiful observation of Stetter [10] (cf. also [2], [4], [6]):

Theorem 3.3 *The ideal projector P is a Lagrange projector if and only if $M_1, ..., M_d$ are simultaneously diagonalizable, i.e., there exists a basis $\{v_1, \ldots, v_N\}$ in V consisting of common eigenvectors of M_j such that:*

$$M_j v_k = z_{j,k} v_k,\ j = 1, ..., d,\ k = 1, ..., N$$

for some $z_{j,k} \in \mathbb{C}$. In this case the projector P interpolates at sites $\mathbf{z}_k := (z_{j,k}, j = 1, ..., d\ ) \in \mathbb{C}^d$ and the eigenvectors v_k are the fundamental polynomials of Lagrange interpolation, i.e., $v_k(\mathbf{z}_s) = 0$ if $k \neq s$.

Normalizing v_k in the above theorem so that $v_k(\mathbf{z}_s) = \delta_{k,s}$ we can write the projector P as

$$P = \sum_{k=1}^{N} P_k$$

where each P_k is a one-dimensional Lagrange projector defined by $P_k f = f(\mathbf{z}_s)v_k$ satisfying the orthogonality relations:

$$P_k P_s = 0 \text{ if } k \neq s.$$

As an immediate application of the Theorem 2.6, we obtain the following generalization of the Stetter's theorem to arbitrary ideal projectors:

Theorem 3.4 *Let P be an ideal projector onto the N-dimensional subspace V. Let*

$$\ker P = J_1 \cap J_2 \cap ... \cap J_m, \ \ m \leq N$$

be the minimal primary decomposition of $\ker P$. *Then*

(i) $\mathbf{M}_P$ *has a unique (up to order of blocks) block diagonalization* $\mathbf{M}_P = diag(\mathbf{M}_k)$ *consisting of m blocs and m is a maximal number of blocks in any block-diagonalization of* $\mathbf{M}_P$.

(ii) Each block $\mathbf{M}_k$ *defines a distinct prime ideal*

$$I_k = \{p \in \mathbb{C}[\mathbf{x}] : p(\mathbf{M}_k) = 0\}$$

and

$$\ker P = I_1 \cap J_2 \cap ... \cap I_m$$

is the minimal primary decomposition of the $\ker P$.

Remark 3.5 *If P is an ideal projector,* $\mathbf{M}_P = (M_1, ..., M_d)$ *and the operators M_j are simultaneously diagonalizable, then the number of blocks $m = N$ is clearly maximal, hence we obtain the Stetter's theorem.*

Let us illustrate this theorem on a simple example:

Example 3.6 *Let P be an ideal projector from $\mathbb{C}[x, y]$ onto its subspace $V := span\{1, x, y\}$ such that $(Pf)(0,0) = f(0,0)$, $\frac{\partial}{\partial x}(Pf)(0,0)) = \frac{\partial}{\partial x}(f)(0,0))$, $(Pf)(0,1) = f(0,1)$. It is easy to check that $Px^2 = 0, Pxy = 0$ and $Py^2 = y$, Hence the two multiplication operators are*

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Let $S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, *hence* $S^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. *Then*

$$SM_1S^{-1} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & [0] \end{bmatrix}$$

and

$$SM_2S^{-1} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & [1] \end{bmatrix}$$

is a simultaneous block-diagonalization of M_1 and M_2 consisting of two blocks corresponding to an $\mathbf{M}_P$*-invariant decomposition*

$$V = span\{1, x\} \oplus span\{y\}.$$

This is the maximal $\mathbf{M}_P$*-invariant decomposition and thus a maximal block-diagonalization.*

We conclude this paper by discussing the relationship between **ideal decomposition** of an ideal projector P onto the space V and the $\mathbf{M}_P$-invariant decomposition of V.

Definition 3.7 *Let P be an operator from $\mathbb{C}[\mathbf{x}]$ onto its subspace V. We say that*

$$P = \sum_{k=1}^{t} P_k \tag{3.2}$$

is an ideal decomposition of P if each P_k, $(k = 1, ..., t)$ is an ideal projector and

$$P_k P_s = 0 \text{ if } k \neq s. \tag{3.3}$$

Theorem 3.8 *If (3.2) is an ideal decomposition of P then P is an ideal projector and*

$$V = \oplus_{k=1}^{t} (ran\ P_k) \tag{3.4}$$

is an $\mathbf{M}_P$-invariant decomposition of V.

Conversely, if P is an ideal projector onto V and

$$V = V_1 \oplus V_2 \oplus ... \oplus V_t \tag{3.5}$$

is an $\mathbf{M}_P$-invariant decomposition of V then it generates an ideal decomposition (3.2) of P with ran $P_k = V_k$.

Proof. We have

$$\begin{aligned} P(fg) - P(fPg) &= \sum_{k=1}^{t} P_k(fg) - \sum_{k=1}^{t} P_k(f \sum_{s=1}^{t} P_s g) \\ &\overset{\text{by (3.1)}}{=} \sum_{k=1}^{t} P_k(fg) - \sum_{k=1}^{t} P_k(f \sum_{s=1}^{t} P_k P_s g) \\ &\overset{\text{by (3.3)}}{=} \sum_{k=1}^{t} P_k(fg) - \sum_{k=1}^{t} P_k(fP_k g) \overset{\text{by (3.1)}}{=} \sum_{k=1}^{t} P_k(fg) - \sum_{k=1}^{t} P_k(fg) = 0 \end{aligned}$$

and by Theorem 3.2, P is an ideal projector. Decomposition (3.4) easily follows from (3.2) and (3.3). It remains to show that decomposition (3.4) is $\mathbf{M}_P$-invariant. Let $f \in$ (ran P_k). Then

$$\begin{aligned} M_j f := P(x_j f) &\overset{\text{by (3.1)}}{=} P(x_j P f) \overset{\text{by (3.3)}}{=} P(x_j P_k f) = \sum_{s=1}^{t} P_s(x_j P_k f) \\ &\overset{\text{by (3.1)}}{=} \sum_{s=1}^{t} P_s(x_j P_s P_k f) \overset{\text{by (3.3)}}{=} P_k(x_j P_k f) \end{aligned}$$

and $M_j f \in$ (ran P_k).

Conversely, let P be an ideal projector onto V and suppose that (3.5) is an $\mathbf{M}_P$-invariant decomposition of V . Then

$$M_j(g) = P(x_j g) \in V_k \text{ for every } g \in V_k$$

and thus

$$P(fg) \in V_k \text{ for every } g \in V_k.$$

Let Q_k be the projector from V onto V_k parallel to $\oplus_{s\neq k} V_s$ and define $P_k := Q_k P$. We have

$$I_V = \sum Q_k \text{ and } Q_k Q_s = 0 \text{ for } k \neq s \tag{3.6}$$

from which (3.2) and (3.3) follows. Clearly P_k is a projector onto V_k and we only have left to check that $\ker P_k$ is an ideal. This follows from the following sequence of implications:

$$\begin{aligned}
&f \in \ker P_k \Rightarrow Pf \in \ker Q_k \Rightarrow Pf \in \oplus_{s\neq k} V_s \overset{\text{by (3.1) and (3.6)}}{\Rightarrow} \\
&P(gf) = P(gPf) \in \oplus_{s\neq k} V_s \text{ for every } g \in \mathbb{C}[\mathbf{x}] \Rightarrow \\
&Q_k P(gf) = 0 \Rightarrow gf \in \ker P_k.
\end{aligned}$$

This proves the theorem. ■

Combinig theorems 2.6 and 3.8 we immediately obtain

Theorem 3.9 *Let P be an ideal projector onto the N-dimensional subspace V. Let*

$$\ker P = J_1 \cap J_2 \cap ... \cap J_m \tag{3.7}$$

be the minimal primary decomposition of $\ker P$*. Then the projector P has a unique ideal decomposition*

$$P = \sum_{k=1}^{m} P_k$$

and this decomposition is maximal in the sense that if

$$P = \sum_{k=1}^{t} \tilde{P}_k$$

is an ideal decomposition of P then $t \leq m$.

Example 3.10 *Let P be the ideal projector defined in the example 3.6. Define ideal projectors P_1 onto span$\{1, x\}$ and P_2 onto span$\{y\}$ by requiring $(P_1 f)(0,0) = f(0,0)$, $\frac{\partial}{\partial x}(P_1 f)(0,0)) = \frac{\partial}{\partial x}(f)(0,0))$, $(P_2 f)(0,1) = f(0,1)$. Then $P_1 P_2 = 0$ and $P = P_1 + P_2$ is the maximal ideal decomposition of P.*

Remark 3.11 *The existence (but not uniqueness or maximality) of ideal decomposition (3.2) also follows from the description of ideal projectors in [8], cf. also [3]. Thus the size of the blocks in the maximal block-diagonalization of $\mathbf{M}_P$ is the multiplicity of zeroes of the corresponding primary ideals in (3.7).*

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