On a Conjecture of Tomas Sauer Regarding Nested Ideal Interpolation

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Abstract

Tomas Sauer conjectured in [4] that if an ideal complements polynomials of degree less than n then it is contained in a larger ideal that complements polynomials of degree less than $n-1$. We construct a counterexample to this conjecture for polynomials in three variables with $n = 3$.

1 Introduction and Preliminaries

Let $\mathbb{C}[x_1, ..., x_d]$ denotes the space of polynomials in d variables with complex coefficients, let $\mathbb{C}_{<n}[x_1, ..., x_d]$ denotes its subspace of polynomials of degree less than n and let $(\mathbb{C}[x_1, ..., x_d])' :=\mathrm{Hom}_{\mathbb{C}}\ (\mathbb{C}[x_1, ..., x_d], \mathbb{C})$ denotes the algebraic dual of $\mathbb{C}[x_1, ..., x_d]$, i.e., the space of all linear functionals on $\mathbb{C}[x_1, ..., x_d]$.

Definition 1 *Let Λ be a subspace of $(\mathbb{C}[x_1, ..., x_d])'$ and E be a subspace of $\mathbb{C}[x_1, ..., x_d]$. We say that Λ* ***is correct fo****r E if for every $f \in \mathbb{C}[x_1, ..., x_d]$ there exists unique $g \in E$ such that $\lambda(g) = \lambda(f)$ for every $\lambda \in \Lambda$.*

With every subspace $\Lambda \subset (\mathbb{C}[x_1, ..., x_d])'$ we associate a subspace $\ker \Lambda \subset \mathbb{C}[x_1, ..., x_d]$ defined by

$$\ker \Lambda := \{f \in \mathbb{C}[x_1, ..., x_d] : \lambda(f) = 0 \text{ for all } \lambda \in \Lambda\}.$$

The purpose of this note is to construct a counterexample to the following conjecture of Tomas Sauer:

Conjecture 2 *([4], Conjecture 4.1): Let Λ be a subspace of $(\mathbb{C}[x_1, ..., x_d])'$ such that Λ is correct for $\mathbb{C}_{<n}[x_1, ..., x_d]$ and $\ker \Lambda$ is an ideal in $\mathbb{C}[x_1, ..., x_d]$. Then there exists a subspace $\Lambda_0 \subset \Lambda$ such that Λ_0 is correct for $\mathbb{C}_{<n-1}[x_1, ..., x_d]$ and $\ker \Lambda$ is an ideal in $\mathbb{C}[x_1, ..., x_d]$.*

Some specific spaces Λ for which the conjecture is valid can be found in [5] and [6]. In particular if $\ker \Lambda$ is the radical ideal then the conjecture is verified. If $\ker \Lambda$ is radical, then the associated variety consists of the maximal possible number of points.

The counterexample, presented in the next section, is constructed in the space of polynomials of three variables with $n = 3$. The corresponding ideal $\ker \Lambda$ is primary, that is the other extreme, the variety of $\ker \Lambda$ consists of unique point.

We will use the rest of this section to recall some well-known facts regarding duality (apolarity, inverse systems) for $\mathbb{C}[x_1, ..., x_d]$.

We will identify the space $(\mathbb{C}[x_1, ..., x_d])'$ with the space $\mathbb{C}[[x_1, ..., x_d]]$ of formal power series as follows:

With every element $F(x_1, ..., x_d) \in \mathbb{C}[[x_1, ..., x_d]]$ we associate a differential operator $F(D) \in \mathbb{C}[\partial_{x_1}, ..., \partial_{x_1}]$ by formally replacing variables in F with the appropriate powers of differential operators. For instance, if $F(x_1, x_2) = x_1^4 + 3x_1^2 x_2 + 1$ then

$$F(D) = \frac{\partial^4}{\partial x_1^4} + 3\frac{\partial^3}{\partial x_1^2 \partial x_2} + I.$$

Now, for every $F \in \mathbb{C}[[x_1, ..., x_d]]$ we define the functional $\hat{F} \in (\mathbb{C}[x_1, ..., x_d])'$ by

$$\hat{F}(f) := (F(D)f)(0) \text{ for every } f \in \mathbb{C}[x_1, ..., x_d] \tag{1}$$

It is well-known (cf. [1], [2] and [3]) that a pairing $F \to \hat{F}$ defined by (1) is an isomorphism between $\mathbb{C}[[x_1, ..., x_d]]$ and $(\mathbb{C}[x_1, ..., x_d])'$. If $M \subset \mathbb{C}[[x_1, ..., x_d]]$ we use $\ker M$ to denote the space $\ker \hat{M}$, i.e.,

$$\ker M := \{f \in \mathbb{C}[x_1, ..., x_d] : \hat{F}(f) = 0 \text{ for all } F \in M\}.$$

Definition 3 *A subspace $M \subset \mathbb{C}[[x_1, ..., x_d]]$ is called D-invariant if for every $F \in M$*

$$\partial_{x_j} F \in M \textit{ for all } j = 1, ..., d.$$

Theorem 4 *(cf. [1], [2] and [3] in its original form): Let M be a finite-dimensional subspace of $\mathbb{C}[[x_1, ..., x_d]]$. Then $\ker M$ is an ideal in $\mathbb{C}[x_1, ..., x_d]$ if and only if M is D-invariant.*

In this terminology, to construct a counterexample to the Conjecture 2, we need to construct a D-invariant subspace $M \subset \mathbb{C}[[x_1, ..., x_d]]$ such that M is correct for $\mathbb{C}_{<n}[x_1, ..., x_d]$ while no D-invariant subspace $N \subset M$ is correct for $\mathbb{C}_{<n-1}[x_1, ..., x_d]$.

As a worm-up, consider the following simple example that already gives a counterexample to the Conjecture 5. 14 of [4] in two variables:

Example 5 *Let M be a subspace of $\mathbb{C}[[x, y]]$ spanned by four polynomials:*

$$F_1 = 1, F_2 = x, F_3 = x^3 - y, F_4 = x^2.$$

It is easy to check that M is D-invariant and that M is correct for the space

$$E := span\ \{f_1 = 1, f_2 = x, f_2 = y, f_4 = x^2\} \subset \mathbb{C}[x, y]$$

since $F_j(f_k) = \delta_{j,k}$ for all $j, k = 1, ..., 4$. On the other hand no three-dimensional D-invariant subspace $N \subset M$ is correct for the space $E := span\ \{1, x, y\}$. Indeed, by virtue of being D-invariant and three dimensional, N could not contain a polynomial of degree 3 for, if it does, the cosequitive partial derivatives of such polynomial would span a four-dimensional subspace. Hence $N = span\{1, x, x^2\}$ is the only three-dimensional D-invariant subspace of M. This space is not correct for E since every functional associated with a polynomial in N vanishes on $y \in E$.

2 Counterexample in three variables:

Expanding on the last example, we will now describe a D-invariant 10-dimensional subspace $M \subset \mathbb{C}[[x, y, z]]$ that is correct for $\mathbb{C}_{<3}[x, y, z]$, such that no 4-dimensional, D-invariant subspace $N \subset M$ is correct for $\mathbb{C}_{<2}[x, y, z]$.

Consider the space $M \subset \mathbb{C}[[x, y, z]]$ spanned by 10 polynomials $F_1, ..., F_{10}$ given by

$$\begin{aligned} F_1 &= 1, F_2 = x, F_3 = \frac{1}{2}x^2, F_4 = x^3 + y, F_5 = \frac{1}{4}x^4 + xy, F_6 = z, F_7 = \frac{1}{2}z^2, F_8 = \frac{1}{20}x^5 + \frac{1}{2}x^2y + xz, \\ F_9 &= \frac{1}{2}y^2 + x^3y + \frac{1}{20}x^6 + 3x^2z, F_{10} = \frac{1}{140}x^7 + \frac{1}{4}x^4y + zx^3 + \frac{1}{2}xy^2 + zy. \end{aligned}$$

The verification that M is D-invariant is by straight-forward computations presented in the table below:

$\partial_x F_1 = 0$	$\partial_y F_1 = 0$	$\partial_z F_1 = 0$
$\partial_x F_2 = 1 = F_1$	$\partial_y F_2 = 0$	$\partial_z F_2 = 0$
$\partial_x F_3 = x = F_2$	$\partial_y F_3 = 0$	$\partial_z F_3 = 0$
$\partial_x F_4 = 3x^2 = 3F_3$	$\partial_y F_4 = 1 = F_1$	$\partial_z F_4 = 0$
$\partial_x F_{15} = x^3 + y = F_4$	$\partial_y F_5 = x = F_2$	$\partial_z F_5 = 0$
$\partial_x F_6 = 0$	$\partial_y F_6 = 0$	$\partial_z F_6 = 1 = F_1$
$\partial_x F_7 = 0$	$\partial_y F_7 = 0$	$\partial_z F_7 = z = F_6$
$\partial_x F_8 = \frac{1}{4}x^4 + yx + z = F_5 + F_6$	$\partial_y F_8 = \frac{1}{2}x^2 = F_3$	$\partial_z F_8 = x = F_2$
$\partial_x F_9 = \frac{3}{10}x^5 + 3yx^2 + 6zx = 6F_8$	$\partial_y F_9 = x^3 + y = F_4$	$\partial_z F_9 = 3x^2 = 6F_3$
$\partial_x F_{10} = \frac{1}{20}x^6 + x^3y + 3zx^2 + \frac{1}{2}y^2 = F_9$	$\partial_y F_{10} = \frac{1}{4}x^4 + yx + z = F_5 + F_6$	$\partial_z F_{10} = x^3 + y = F_4$

The polynomials $f_1, ..., f_{10}$ defined by

$$f_1 = 1, f_2 = x, f_3 = x^2, f_4 = y, f_5 = xy, f_6 = z, f_7 = z^2, f_8 = xz, f_9 = y^2, f_{10} = zy,$$

in this order, form a basis for $\mathbb{C}_{<3}[x, y, z]$ and $\hat{M}$ is correct for $\mathbb{C}_{<3}[x, y, z]$ since $\hat{F}_j(f_k) = \delta_{j,k}$.

It remains to prove that no D-invariant subspace $N \subset M$ is correct for

$$\mathbb{C}_{<2}[x, y, z] = \text{span}\ \{1, x, y, z\}.$$

Proof. Observe that if N is correct for $\mathbb{C}_{<2}[x,y,z]$ and D-invariant then N is 4-dimensional and thus cannot contain polynomials of degree 4 or larger, since the consecutive partial derivatives of a polynomial of degree k span a subspace of dimension $k+1$. Thus N is in the linear span of

$$F_1 = 1, F_2 = x, F_3 = \frac{1}{2}x^2, F_4 = x^3 + y, F_6 = z, F_7 = \frac{1}{2}z^2. \tag{2}$$

Now assume that N is correct for $\mathbb{C}_{<2}[x,y,z]$. Then N contains a polynomial F such that the functional $\hat{F}$ satisfies

$$\hat{F}(y) = 1, \hat{F}(0) = \hat{F}(x) = \hat{F}(z) = 0. \tag{3}$$

The polynomial $F \in N$ that corresponds to such functional must be of the form

$$F = F_4 + aF_3 + bF_7$$

for some constants a and b, for, if it contains a non-zero summand of any other polynomial in (2), then at least one of the last three equalities in (3) would fail. By D-invariance,

$$\partial_x(F) = 3x^2 + ax \in N$$

and by repeated differentiation, 1 and x belong to N. Since N is four-dimensional, it follows that

$$N = \text{span } \{1, x, x^2, F = x^3 + y + \frac{a}{2}x^2 + \frac{b}{2}z^2\}$$

and hence every functional that corresponds to a polynomial in this space annihilate $z \in \mathbb{C}_{<2}[x,y,z]$. This contradict the assumption that N is correct for $\mathbb{C}_{<2}[x,y,z]$. ■

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