ON THE NORMS OF SOME PROJECTIONS*

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I. *Introduction*

In this note we summarize some results about the norms of the projections onto subspaces that are natural in Approximation Theory. We are mostly concerned with the subspaces in $C(K)$ spaces although most of the results are also valid for $L_1(K)$ spaces.

We use the usual notations: $\lambda(X)$ for the projectional constant of X and $d(X,Y)$ for the Banach-Mazur distance.

II. *Preliminary Results*

In this section we collect some technical results that we list as propositions.

Proposition 1 (Olevskii). Let (K,μ) be a probability space and $(\varphi_j)_{j=1}^{\infty}$ be an orthonormal system in $L_2(\mu)$. Suppose in addition $\|\varphi_j\|_\infty = O(1)$. Then for the projection F_n given by

$$F_n f = \sum_{j=1}^{n} \left(\int \bar{\varphi}_j \cdot f d\mu \right) \varphi_j$$

acting from $L_\infty(\mu) \to L_\infty(\mu)$, we have $\limsup \frac{\|F_n\|}{\log n} > 0$.

Proposition 2. Let $(\varphi_j)_{j=1}^{n}$ be a sequence of functions from $L_\infty(\mu)$ where μ is a probability measure. Let $(\eta_j)_{j=1}^{n}$ be a sequence of positive numbers and let f be a norm one positive functional on $L_\infty(\mu)$ such that $f(|\sum c_j\varphi_j|) \geq \sum \eta_j |c_j|$ for any set of real numbers (c_j). Consider an operator $A : L_\infty(\mu) \to L_\infty(\mu)$ given by $A_n = \sum \mu_j \otimes \varphi_j$ where $\mu_j \in [L_\infty(\mu)]^*$. Then $\|A_n\| \geq \sum \eta_j \|\mu_j\|$.

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* This research was supported by National Science Foundation Grant MCS-8301646.

Proof: Let σ be a probability measure corresponding to f. Let $\nu = \sum |\mu_j|/2^j$. Then there are functions $g_j \in L_1(\nu)$ so that

$$A_n x = \int (\sum g_j(s)\varphi_j)x(s)d\nu$$

and

$$\begin{aligned}\|A_n\| &= \mu - \operatorname{ess\,sup}_t \int |\sum g_j(s)\varphi_j(t)|d\nu(s)\\ &\geq \int\int |\sum g_j(s)\varphi_j(t)|d\nu(s)d\sigma(t)\\ &= \int [\int |\sum g_j(s)\varphi_j(t)|d\sigma(t)]d\nu(s)\\ &\geq \int (\sum \eta_j |g_j(s)|)d\nu(s)\\ &= \sum \eta_j \|\mu_j\|.\end{aligned}$$

Proposition 3. Let G be a compact connected group. Let $(\varphi_j)_{j=1}^n$ be an increasing (with respect to an order on $\hat{G}$) sequence of continuous characters on G. Then there exists a universal constant C such that

$$\int |\sum_{j=1}^n c_j\varphi_j|d\mu \geq C\sum_{j=1}^n \frac{1}{j}|c_j|$$

for any sequence (c_j).

Proof: The proof is the same as in the case $G = \mathbf{T}$. It consists of the construction of the function $F \in C(G)$ so that $\|F\|_\infty \leq 1$, $\int \bar{F}\varphi d\mu = 0$ for $\varphi > \varphi_n$ and $Re[c_j \cdot \int \bar{F}\varphi_j d\mu] \geq C\frac{|c_j|}{j}$. As in [1] we represent $(\varphi_j) = \cup(\varphi_j)_{j=n_k}^{n_{k+1}}$ so that $n_{k+1} - n_k = 4^k$ and construct $F_k = \frac{1}{4^k}\sum_{j=n_k}^{n_{k+1}} \varphi_j \cdot d_j$ where $d_j = \frac{c_j}{|c_j|}$. Let h_k be a harmonic conjugate (with respect to the linear order in $\hat{G}$) to the function $\frac{1}{4}|F_k|$. We define $G_0 = F_0$, $G_{j+1} = G_j e^{-h_{j+1}} + F_{j+1}$ and $F = G_n$. The proof that the function F has the desired properties is identical with the one in [1].

III. *Projections Onto Translation-invariant Subspaces*

In this section we give some immediate corollaries from Propositions 1-3.

Theorem 1. Let G be a compact connected group and $(\varphi_j)_{j=1}^n$ be a set of continuous characters on G. Then there exists a constant C so that

$$\lambda([\varphi_j]_{j=1}^n) \geq C \cdot \log n.$$

Proof: The combination of Propositions 2 and 3 is the proof.

Remark 1. It follows immediately from this theorem that

$$\lambda([\varphi_j]_{j=1}^n) \leq d([\varphi_j], \ell_\infty^{(n)}) \leq e^{\lambda([\varphi_j])}.$$

Now consider G to be the unit circle and $\varphi_j = z^{\lambda_j}$, where (λ_j) is a sequence of integers. It is well known that if $\lambda_j = j$, then $\lambda([\varphi_j]_{j=1}^n) \sim \log n$. If (λ_j) is a lacunary sequence then $\lambda([\varphi_j]_{j=1}^n) \sim \sqrt{n}$. In the latter case, $d([\varphi_j], \ell_\infty^{(n)}) \sim \sqrt{n}$ also. A similar estimate can be done for $\lambda_j = j$.

Theorem 2. $d([z^j]_{j=1}^n, \ell_\infty^{(n)}) \sim \lambda([z^j]_{j=1}^n) \sim \log n$.

Proof: We only have to prove that $d([z^j], \ell_\infty^{(n)}) \leq C \cdot \log n$. Let $P_n = \sum \delta_{z_j} \otimes \varphi_j$ be an interpolating projection onto $[z^j]$ that interpolates on the roots of unity. Then for any sequence of numbers a_j, we have

$$\max |a_j| = \max_j |\delta_{z_j}(\sum a_j \varphi_j)| \leq \| \sum a_j \varphi_j \|.$$

On the other hand let f be a function of norm $\max |a_j|$ so that $f(z_j) = a_j$, then

$$\| \sum a_j \varphi_j \| = \|P_n f\| \leq \|P_n\| \max |a_j|,$$

but the norm of $\|P_n\|$ is known to grow as $\log n$.

We now turn our attention to the more general groups.

Theorem 3. Let G be a compact group. Let $(\varphi_j)_1^n \subset \hat{G}$. Let $F_n = \sum_1^n \varphi_j \otimes \varphi_j$. Then $\|F_n\| = \int |\sum \varphi_j| d\mu = \lambda([\varphi_j]_{j=1}^n)$.

Proof: Following the standard argument, let P_n be an arbitrary projection from $C(G)$ onto $[\varphi_j]$. For every $s \in G$ define $T_s : C(G) \to C(G)$ by $(T_s x)(t) = x(t \cdot s)$, $\forall\, t \in G$. Then it is easy to check that

$$(F_n f)(t) = \int (T_{s^{-1}} P_n T_s f) d\mu_s$$

and hence $\|P_n\| \geq \|F_n\|$.

Theorem 4. Let G be a compact group such that the identity component of G has finite index m. Then there exists a constant C_m depending on m only, such that

$$\lambda([\varphi_j]_{j=1}^n) \geq C_m \cdot \log n.$$

Proof: It follows that the dual group has exactly m elements of finite order. Now factoring it out and using Proposition 3, we get

$$\int |\sum_{j=1}^n \varphi_j| d\mu \geq C_m \cdot \log n \qquad (n > m).$$

If $\hat{G}$ has an infinite torsion subgroup, then Theorem 4 does not hold.

Theorem 5. Let $\Gamma \subset \hat{G}$ be a finite subgroup of the dual of an arbitrary compact group G. Then

$$\lambda([\varphi]_{\varphi \in \Gamma}) = d([\varphi], \ell_\infty^{(\#\Gamma)}) = 1.$$

Proof: For every $\varphi_0 \in \Gamma$ we have $\sum_{\varphi \in \Gamma} \varphi = \varphi_0 \cdot \sum_{\varphi \in \Gamma} \varphi$. Hence for every point $t \in G$ we have either $\sum_{\varphi \in \Gamma} \varphi(t) = 0$ or $\varphi(t) = 1$ for all $\varphi \in \Gamma$. One way or the other, $\sum_{\varphi \in \Gamma} \varphi \geq 0$ and using Theorem 3, we get $\lambda([\varphi]) = \int (\sum_{\varphi \in \Gamma} \varphi) d\mu = 1$.

This situation comes up when considering the Walsh functions or more generally, any periodic multiplicative system. Let $(\varphi_j)_{j=1}^\infty$ be such a system. Then (φ_j) can be modeled as the set of characters on some compact group. The group generated by (φ_j) has infinite torsion subgroups since $\varphi_j^2 = \varphi_j$. By Theorem 3 and Proposition 1 we conclude that $\limsup \lambda([\varphi_j]_{j=1}^n) \to \infty$; yet, Theorem 5 shows that the "lim sup" in this statement cannot be replaced by "lim."

IV. *Projections Onto Algebraic Polynomials*

In this section we turn our attention to the subspaces of algebraic polynomials on the real interval $[a, b]$.

Theorem 6. Let $(\lambda_j)_{j=1}^n$ be an increasing sequence of integers. Then there exists a constant C such that

$$\lambda([\cos \lambda_j \theta]_{j=1}^n) \geq C \cdot \log n$$

$$\lambda([\sin \lambda_j \theta]_{j=1}^n) \geq C \cdot \log n.$$

Proof: We would like to estimate the sum $\int_{-\pi}^{\pi} |\sum a_j \cos \lambda_j \theta| d\theta$. Using $\cos \lambda_j \theta = (e^{i\lambda_j \theta} + e^{-i\lambda_j \theta})/2$ and Proposition 3, we easily get a constant C such that

$$\int_{-\pi}^{\pi} |\sum_{j=1}^{n} a_j \cos \lambda_j \theta| d\theta \geq C \cdot \sum_{j=1}^{n} \frac{|a_j|}{n-j}.$$

For an arbitrary projection $P_n = \sum \mu_j \otimes \cos \lambda_j \theta$, we have $\mu_j(\cos \lambda_j \theta) = 1$ and hence $\|\mu_j\| \geq 1$. Now by Proposition 2 we obtain the first inequality. The second one can be obtained in the same way.

Corollary 1. Let T_j and U_j be Chebyshev polynomials of the first and second kind on an arbitrary interval $[a, b]$. Let $(\lambda_j)_{j=1}^n$ be a sequence of integers. Then

$$\lambda([T_{\lambda_j}]_{j=1}^n) \geq C \cdot \log n$$

$$\lambda([U_{\lambda_j}]_{j=1}^n) \geq C \cdot \log n.$$

Proof: It follows from the fact that the distances $d([T_{\lambda_j}]^n, [\cos \lambda_j \theta]^n)$ and $d([U_{\lambda_j}]^n, [\sin \lambda_j \theta]^n)$ are bounded uniformly in n and (λ_j).

Corollary 2. Consider $[t^j]_{j=0}^n \subset C_{[a,b]}$. Then

$$\lambda([t^j]_{j=0}^n) \sim d([t^j]_{j=0}^n, \ell_\infty^{(n+1)}) \sim \log n.$$

Proof: Since $[t^j]_{j=0}^n = [T_j]_{j=0}^n$ we have $\lambda([t^j]) = \lambda([T_j]) \geq C \cdot \log n$ and hence $d([t^j], \ell_\infty^{(n+1)}) \geq C \log n$.

To prove the reverse inequality, we again consider the interpolating projection P_n from $C_{[a,b]} \to [t^j]_{j=0}^n$ that interpolates at the zeros of the T_{n+1}. The norm of P_n is known to grow as $\log n$. The rest of the argument is the same as in Theorem 2.

In view of Corollaries 1 and 2, one is tempted to ask:

Problem 1. Does Corollary 1 hold for other sequences of orthogonal polynomials?

Problem 2. Does Corollary 2 hold if we replace $[t^j]_{j=0}^n$ by $[t^{\lambda_j}]_{j=1}^n$ for some (λ_j)?

In the rest of the section we will describe some results about Problem 2. Proposition 4 is a quick consequence of Corollary 2.

Proposition 4. Let $\lambda_j \leq j + o(\log^2 j)$. Then $\lambda([t^{\lambda_j}]_{j=1}^n) \to \infty$ as $n \to \infty$.

Proof: Let $M = [t^j]_{j=1}^{\lambda_n}$ and P_n be a projection from $C_{[0,1]}$ onto $[t^{\lambda_j}]_{j=1}^n$. Then $M = [t^{\lambda_j}]_{j=1}^n \oplus N$, where $N = \ker(P_n|M); \dim N \leq o(\log^2 n)$. Then there exists a projection Q_n from $C_{[0,1]}$ onto N with the norm $\|Q_n\| \leq o(\log n)$. It is easy to see that $P_n + Q_n - Q_n P_n$ is a projection onto M, and by Corollary 2 we have

$$\log \lambda_n \leq \|P_n + Q_n - Q_n P_n\| \leq (1 + \|P_n\|)\|Q_n\|.$$

Hence $1 + \|P_n\| \geq \frac{\log \lambda_n}{\|Q_n\|} \to \infty$ as $n \to \infty$.

Next we consider one very special case.

Theorem: Let $\lambda_j = j^2$. Then $\lambda([t^{\lambda_j}]_{j=1}^n) \geq C \cdot \log n$.

Proof: Without loss of generality we can assume $t = \cos\theta$. Then any projection P_n from $C_{[-\pi,\pi]}$ onto $[\cos^{\lambda_j}\theta]_{j=1}^n$ can be written as

$$P_n = \sum \mu_j \otimes \cos^{\lambda_j}\theta.$$

On the other hand, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$. Hence

$$\cos^k\theta = \frac{1}{2^k}\sum\binom{k}{j}e^{i(k-2_j)\theta}. \tag{1}$$

Therefore the projection P_n can be written as

$$P_n = \sum_{j=1}^{\lambda_n} \nu_j \otimes e^{i(\lambda n-2j)\theta},$$

where ν_j are some linear combinations of μ_j. By Propositions 2 and 3 we get $\|P_n\| \geq \sum^{\lambda_n} \frac{1}{j}\|\nu_j\|$, and the proof boils down to estimating the norms $\|\nu_j\|$.

For sufficiently large n and m, let

$$\frac{m^2}{2} - \frac{m}{2} \leq j \leq \frac{m^2}{2} + \frac{m}{2}. \tag{2}$$

At this point we have to remark that for sufficiently large k, the coefficients in the polynomial (1) are distributed almost normally and thus there are $\approx \sqrt{k}$ coefficients that are of size $\frac{1}{\sqrt{k}}$ (we will call them essential) and the rest are negligible.

Now we estimate

$$\mu_j(\cos^{m^2}\theta - \cos^{(m-1)^2}\theta).$$

Since P_n is a projection, this is the j-th coefficient in the polynomial $\cos^{m^2}\theta - \cos^{(m-1)^2}\theta$. By (2) this coefficient is $\sim \frac{1}{m}$ since the corresponding coefficient in $\cos^{(m-1)^2}\theta$ is negligible. Thus $\mu_j(\cos^{m^2}\theta - \cos^{(m-1)^2}\theta) = \frac{1}{m}$. On the other hand,

$$\|\cos^{m^2}\theta - \cos^{(m-1)^2}\theta\| \sim \frac{1}{m}$$

and $\|\mu_j\| \geq 1$. Hence

$$\sum_{(m^2-m)/2}^{(m^2+m)/2} \frac{1}{j}\|\mu_j\| \sim \ell n\frac{m+1}{m-1} = \ell n\Big[(1+\frac{2}{m-1})^{\frac{m-1}{2}\cdot\frac{2}{m-1}}\Big]$$
$$= \frac{2}{m-1}\ell n\Big([1+\frac{2}{m-1}]^{\frac{m-1}{2}}\Big) \sim \frac{2}{m-1}.$$

Thus $\|P_n\| \geq \sum_{m>N}^{n} \frac{2}{m-1} \sim \log n.$

Similarly to Proposition 4, we can obtain Proposition 5.

Proposition 5. Let $|\lambda_j - j^2| \leq o(\log^2 j)$. Then $\lambda([t^{\lambda_j}]_{j=1}^n) \to \infty$ as $n \to \infty$.

In Section III we observed that the projectional constant of the complex polynomials is the largest when (λ_j) is lacunary. The algebraic polynomials exhibit the opposite behavior.

Proposition 6. Let (λ_j) be lacunary; then

$$\lambda([t^{\lambda_j}]_j^n = 1) \leq d([t^{\lambda_j}]_{j=1}^n, \ell_\infty^{(n)}) \leq O(1).$$

Proof: Our proof follows from the results of [2], where it is proved that (t^{λ_j}) spans a copy of c_0 and forms a basis in its span.

I am grateful to Professors W. Johnson and L. Harper for many useful suggestions.

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