On Interpolation by and Banach Spaces of Polynomials

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In this paper, we review the relationships between interpolation properties of polynomials and the geometric properties of the unit ball of the Banach spaces of polynomials. We extend some classical results to polynomials with gaps and address some open problems. In particular, problems raised by P. Erdős and J. Szabados.

1. CLASSICAL RESULTS AND OPEN PROBLEMS

Let $\Lambda \subset \mathbb{Z}$ be an ordered set consisting of n integers. We use $H(\Lambda_n)$ to denote polynomials on the unit circle $\mathbb{T}$ with frequencies in Λ_n:

$$(1.1) \qquad H(\Lambda_n) := \operatorname{span}\{z^\lambda \ : \ \lambda \in \Lambda_n\}.$$

In the case when Λ_n is an arithmetic sequence we use $H_n := H(\Lambda_n)$.

Theorem 1.1 (Faber). *There exists a constant $C_1 > 0$ such that for every projection $P_n \ : \ C(\mathbb{T}) \to H_n$*

$$(1.2) \qquad \|P_n\| \geq C_1 \log n.$$

In particular, let $\Delta_n \subset \mathbb{T}$ be a set of n points and let $P(\Delta_n)$ be the Lagrange interpolating projection onto H_n then

Theorem 1.2. *For every* $\Delta_n \subset \mathbb{T}$

$$(1.3) \qquad\qquad\qquad \left\| P(\Delta_n) \right\| \geq C_1 \log n.$$

This estimate is sharp.

Theorem 1.3. *If* Δ_n *consists of the* n *equidistributed points in* $\mathbb{T}$ *then there exists a constant* C_2 *independent of* n *such that*

$$(1.4) \qquad\qquad\qquad \left\| P(\Delta_n) \right\| \leq C_2 \log n.$$

These estimates can be improved if the degree of polynomials is greater than the number of points.

Theorem 1.4 (Bernstein). *Let* $n > am;\ a > 1$. *Then there exist a constant* $C(a) > 0$, *a set* $\Delta_m \subset \mathbb{T}$ *and a mapping* $L_n(\Delta_m) : C(\mathbb{T}) \to H_n$ *such that*

$$(1.5) \qquad\qquad\qquad \left(L_n(\Delta_m)f - f \right)\big|_{\Delta_m} = 0$$

and

$$(1.6) \qquad\qquad\qquad \left\| L_n(\Delta_m) \right\| \leq C(a).$$

The comparison of the theorems of Faber and Bernstein prompted

Problem 1.5. *Let* $m = m(n) < n$ *be such that* $n/m(n) \to 1$ *i.e.,* $n - m(n) = o(n)$. *Let* $\Delta_m \subset \mathbb{T}$ *be arbitrary, and let* $F_n(\Delta_n)$ *be arbitrary mapping* $F_n(\Delta_m) : C(\mathbb{T}) \to H_n$ *satisfying* (1.5). *Is it true that*

a) $\left\| F_n(\Delta_m) \right\| \to \infty$? (P. Erdős)

b) $\left\| F_n(\Delta_m) \right\| \geq C \log\left(n/(n-m) \right)$? (J. Szabados)

While the problem is still open, some partial results have been established:

Theorem 1.6 (Szabados [12]). *Let* Δ_m *be equidistributed points on* $\mathbb{T}$. *Then*

$$(1.7) \qquad\quad \limsup \left(\left\| F_n(\Delta_m) \right\| / \log\left(n/(n-m) \right) \right) > 0.$$

Theorem 1.7 (A. Privalov [6], B. Shekhtman [9]). *Let* $\Delta_m \subset \mathbb{T}$ *be arbitrary and* $L_n(\Delta_m)$ *be arbitrary linear mappings* $L_n(\Delta_m) : C(\mathbb{T}) \to H_n$ *satisfying* (1.5) *then*

$$(1.8) \qquad\quad \liminf \left(\left\| L_n(\Delta_m) \right\| / \log\left(n/(n-m) \right) \right) > 0.$$

Theorem 1.8 (cf. [7]). *Let* $n - m = o(\log^2 n)$. *Let* $\Delta_m \subset \mathbb{T}$ *be arbitrary and* $F_n(\Delta_m) : C(\mathbb{T}) \to H_n$ *be arbitrary mapping satisfying* (1.5). *Then*

$$(1.9) \qquad\qquad \big\| F_n(\Delta_m) \big\| \to \infty.$$

Let $\Delta \subset \mathbb{T}$ and denote

$$(1.10) \qquad\qquad \|f\|_\Delta = \sup \big\{ \big| f(t) \big|, \ t \in \Delta \big\}.$$

An immediate consequence of the Faber theorem is

Corollary 1.9. *Let* Δ_n *be arbitrary n-point subset of* $\mathbb{T}$. *Then*

$$(1.11) \qquad \sup \big\{ \|p\| / \|p\|_{\Delta_n} : p \in H_n; \ p \neq 0 \big\} \geq C_1 \log n.$$

An analog of the Bernstein theorem in this case is

Theorem 1.10 (H. Ehlich and K. Zeller [2]). *Let* $m > an$, $a > 1$. *Then there exist points* $\Delta_m \subset \mathbb{T}$ *and a constant* $K(a)$ *such that*

$$(1.12) \qquad \sup \big\{ \|p\| / \|p\|_{\Delta_m} : p \in H_n; \ p \neq 0 \big\} \leq K(a).$$

Again it is natural to ask

Problem 1.11 (B. Shekhtman [11]). *Let* $m - n = o(n)$. *Is it true that for every* $\Delta_m \subset \mathbb{T}$

a) $\sup \big\{ \|p\| / \|p\|_{\Delta_m} : p \in H_n, \ p \neq 0 \big\} \to \infty$?

b) $\sup \big\{ \|p\| / \|p\|_{\Delta_m} : p \in H_n, \ p \neq 0 \big\} \geq C \log \big(n/(m - n) \big)$?

In a particular case when $m - n = o(\log^2 n)$ we have an analog of Theorem 1.8.

Theorem 1.12 (cf. [11]). *Let* $m - n = o(\log^2 n)$ *and let* $\Delta_m \subset \mathbb{T}$ *be arbitrary. Then*

$$(1.13) \qquad \sup \big\{ \|p\| / \|p\|_{\Delta_m} : p \in H_n, \ p \neq 0 \big\} \to \infty.$$

Further results on the behavior of the quantity (1.11) for some specific choices of points Δ_m have been obtained with the aid of potential theory.

In this review, we plan to

A) Interpret the above results in terms of the Banach Space Theory.

B) Extend these results to polynomials with gaps $H(\Lambda_n)$.

C) Use the Banach Space Theory to generalize these results and give partial answers to the problems mentioned in this section.

2. TOOLS AND PHILOSOPHY OF BANACH SPACES

Let $H^p(\Lambda_n)$ be a *Banach space* of polynomials $H(\Lambda_n)$ equipped with the $L_p(\mathbb{T})$ norm. By considering it as an n-dimensional Banach space we are restricting ourselves to an isomorphic characteristic of this space. In particular, we can compare these spaces to other (simpler) Banach spaces such as ℓ_p^n. It is in this "isomorphic" framework that we intend to interpret and generalize the result mentioned in the previous section. We start with some notations. Let X and Y be finite-dimensional Banach spaces with $\dim X = \dim Y$. We define the Banach–Mazur distance between X and Y as

$$(2.1) \qquad d(X,Y) := \inf \left\{ \|T\|\,\|T^{-1}\| \, : \, T \text{ isomorphism from } X \text{ to } Y \right\}.$$

This allows us to compare the isomorphic characteristics of different Banach spaces.

If X is a subspace of another Banach space E, we define relative and absolute projection constants as

$$(2.2) \qquad \lambda(X,E) := \inf \left\{ \|P\| \, : \, P \text{ is a projection from } E \text{ onto } X \right\}$$

$$(2.3) \qquad \lambda(X) := \sup \left\{ \lambda(X,E) \, : \, X \subset E \right\}.$$

We now list some well-known properties of these notions

$$(\text{P.1}) \qquad d(X,Y) = d(Y,X) = d(X^*,Y^*) \geq 1$$

$$(\text{P.2}) \qquad d(X,Y) \leq d(X,Z) \cdot d(Z,Y)$$

$$(\text{P.3}) \qquad \lambda(X) \leq d(X,Y)\lambda(Y)$$

$$(\text{P.4}) \qquad \lambda(X) \leq d\big(X,\ell_\infty^{(n)}\big); \quad n = \dim X$$

$$(\text{P.5}) \qquad \lambda(X,E) \leq \min \left\{ \sqrt{\dim X}; \ \sqrt{\operatorname{codim} X} + 1 \right\}.$$

The converse to the inequality (P.4) is probably the most important remaining unsolved problem in Banach space theory.

Problem 2.1 (P_λ-problem). Does there exist a bounded function φ such that

$$(2.4) \qquad d\big(X, \ell_\infty^{(n)}\big) \leq \varphi\big(\lambda(X)\big)?$$

For an operator $T : X \to Y$ we use

$$(2.5) \qquad \nu(T), \quad \pi_1(T), \quad \gamma_\infty(T)$$

to denote the nuclear, absolutely summing and L_∞-factorization norms of the operator. The definition and properties of these ideal norms can be found in [14].

If X is a subspace of a $C(K)$ space, we define

(2.6)

$$\tilde{\lambda}(X) = \inf\big\{\, \|P\| \,:\, P \text{ is an interpolating projection from } C(K) \text{ onto } X \big\}.$$

There is an easy

Proposition 2.2. *For every $X \subset C(K)$; $\dim X = n$:*

$$(2.7) \qquad d\big(X, \ell_\infty^{(n)}\big) \leq \tilde{\lambda}(X).$$

Proof. Let $t_1, \ldots, t_n \in K$ and let P be an interpolating projection given by

$$(2.8) \qquad Pf = \sum_{j=1}^n f(t_j) x_j$$

where $\{x_j\}$ is a basis for X. Let $\bar{\alpha} = (\alpha_j)_{j=1}^n \in \ell_\infty^{(n)}$. Then there exists a function $f \in C(K)$ such that

$$(2.9) \qquad f(t_j) = \alpha_j; \quad \|f\| = \max\big\{|\alpha_j|;\ j = 1, \ldots, n\big\} = \|\bar{\alpha}\|_\infty.$$

Then

$$(2.10) \qquad \bigg\| \sum_{j=1}^n \alpha_j x_j \bigg\| = \|Pf\| \leq \|P\|\,\|f\| = \|P\|\,\|\bar{\alpha}\|_\infty.$$

On the other hand

$$(2.11) \qquad \bigg\| \sum_{j=1}^n \alpha_j x_j \bigg\| = \max_k \bigg| \Big(\sum_{j=1}^n \alpha_j x_j \Big)(t_k) \bigg| \geq \|\bar{\alpha}\|_\infty.$$

Combining these to inequality, we have

$$(2.12) \qquad \|\bar{\alpha}\|_\infty \leq \left\| \sum_{j=1}^n \alpha_j x_j \right\| \leq \|P\| \|\bar{\alpha}\|_\infty.$$

Let T be an operator from $\ell_\infty^{(n)}$ onto X defined by

$$(2.13) \qquad T\bar{\alpha} = \sum_{j=1}^n \alpha_j x_j \in X.$$

Then (2.12) implies

$$(2.14) \qquad \|T\| \leq \|P\|; \quad \|T^{-1}\| \leq 1$$

and hence

$$(2.15) \qquad d\left(X, \ell_\infty^{(n)}\right) \leq \|T\| \|T^{-1}\| \leq \|P\|. \qquad \blacksquare$$

Combining the Faber theorem and (P.4) we get

$$(2.16) \qquad d\left(H_n^\infty, \ell_\infty^{(n)}\right) \geq C_1 \log n.$$

On the other hand, the combination of Theorem 1.3 and Proposition 2.2 yields

$$(2.17) \qquad d\left(\ell_\infty^{(n)}, H_n^\infty\right) \leq C_2 \log n.$$

Hence we have

Proposition 2.3. $d\left(\ell_\infty^{(n)}, H_n^\infty\right) \sim \log n.$

This proposition indicates how the knowledge of projections and interpolation projections onto the space give us information about the geometry of the space. Geometrically, Proposition 2.3 says that the Hausdorff distance between the unit ball of H_n^∞ and the n-dimensional unit cube is asymptotically behaving as "$\log n$".

This and other geometric properties of polynomials had been investigated in [8] and [3]. Despite the effort, even the asymptotics of $d\left(H_n^\infty, \ell_1^{(n)}\right)$ is not known.

We will finish this section with one more powerful

Theorem 2.4 ([Generalized Hardy inequality]). *Let* $\Lambda_n = \lambda_1 < \lambda_2 < \cdots < \lambda_n \in \mathbb{Z}$. *Then there exists a constant* C_3 *such that*

$$(2.18) \qquad \int_{-\pi}^{\pi} \left| \sum_{k=1}^{n} a_k e^{i\lambda_k \theta} \right| d\theta \geq C_3^{-1} \sum_{k=1}^{n} \frac{|a_k|}{k}.$$

Corollary 2.5. *Define an operator* $S_n : H^1(\Lambda_n) \to \ell_1^n$ *by*

$$(2.19) \qquad S_n : e^{i\lambda_j \theta} \to \frac{1}{j} (\delta_{kj})_{k=1}^n \in \ell_1^{(n)}.$$

Then $\|S_n\| \leq C_3$.

Using Theorem 2.4, it is easy to obtain (cf. [7]).

Proposition 2.6. *Let* Λ_n *be as above. Then*

$$(2.20) \qquad C_3 \log n = \lambda\big(H^\infty(\Lambda_n) \big) \leq d\big(H^\infty(\Lambda_n), \ell_\infty^{(n)} \big).$$

In the next section, we will provide further generalizations of the Faber theorem.

3. GENERALIZED FABER THEOREM

In the Banach space terminology, the Faber theorem states that

$$(3.1) \qquad \lambda\big(H_n^\infty, C(\mathbb{T}) \big) \geq C_1 \log n.$$

Since for a subspace X of a $C(K)$ space we have

$$(3.2) \qquad \lambda\big(X, C(K) \big) = \lambda(X)$$

we conclude that

$$(3.3) \qquad \lambda(H_n^\infty) \geq C_1 \log n.$$

And since (cf. (P.3)) the quantity $\lambda(H_n^\infty)$ is an isomorphic invariant, we are seeking intrinsic isomorphic characteristics of the space $H^\infty(\Lambda_n)$ that validate (3.3). In other words, neither the analytic characters of the polynomials, nor its position with regard to other functions in $C(\mathbb{T})$ is responsible

for the theorem. Just the geometric shape of the unit ball of H_n^∞ should matter with regard to the quantity $\lambda\big(X, C(K)\big)$. In fact, the P_λ-problem aims at precisely that. If true, it would confirm that the spaces which admit a well-bounded projection are spaces with cube-like unit ball.

As luck would have it, the unit ball of H_n^∞ is hard to get ahold of. In fact, for years now I have been asking

Question 3.1. Provide a fair isomorphic description of the dual space $(H_n^\infty)^*$. Equivalently, it asks for n functionals on the space H_n^∞ that provide adequate asymptotic information on the norms of the polynomial. Aside from some vague references to the B.M.O. spaces, I got nowhere. The main theorem of this section explains the difficulty of this question.

Proposition 3.2. *Let X be a finite dimensional subspace of a Banach space E. Let P be a projection from E onto X. Then P^* is a projection from E^* onto its finite-dimensional subspace $U = P^*E^*$ and*

$$(3.4) \qquad d(X^*, U) = d(X^*, P^*E^*) \leq \|P\| \leq \lambda(X, E).$$

Proof. The proof is trivial. It is obvious that $U = P^*E^*$ is an n-dimensional subspace of E^*. Let $f \in U$. Then $f = P^*f$. Let us consider the functional $\tilde{f}$ on X defined as $f \mid X$. Algebraically, $\tilde{f}$ is exactly the same as f. But the norm of $\tilde{f}$ is

$$(3.5) \qquad \|\tilde{f}\| = \sup\left\{\, |\tilde{f}(x)| \; : \; \|x\| \leq 1; \; x \in X \right\},$$

while

$$(3.6) \qquad \|f\| = \sup\left\{\, |f(e)| \; : \; \|e\| \leq 1; \; e \in E \right\}.$$

The latter norm "sups" over a larger set and hence it is concievable that $\|f\| \leq \|\tilde{f}\|$.

Observe, however, that

$$(3.7) \qquad f(e) = P^*f(e) = f(Pe) = \tilde{f}(Pe) \leq \|\tilde{f}\|\,\|P\|.$$

Hence

$$(3.8) \qquad \|f\| = \sup\left\{ f(e) \; : \; e \in E; \; \|e\| \leq 1 \right\} \leq \|\tilde{f}\|\,\|P\|$$

and

$$(3.9) \qquad \|\tilde{f}\| \leq \|f\| \leq \|\tilde{f}\|\,\|P\|.$$

Thus, the isomorphism $T : U \to X^*$ defined by $Tf = \tilde{f}$ satisfies

$$(3.10) \qquad \|T\| \, \|T^{-1}\| \leq \|P\|. \qquad \blacksquare$$

So, if there is a projection $P : C(\mathbb{T}) \to H^\infty(\Lambda_n)$ with a small norm, then there exists a subspace $U \subset \left(C(\mathbb{T})\right)^* := M(\mathbb{T})$ that is close to $\left(H^\infty(\Lambda_n)\right)^*$ in the sense of the Banach–Mazur distance.

Theorem 3.3 ([Generalized Faber theorem]). *Let* $U \subset \left(C(\mathbb{T})\right)^*$ *be an arbitrary n-dimensional space. Then*

$$(3.11) \qquad d\!\left(\left(H^\infty(\Lambda_n)\right)^*, U\right) \geq C_3^{-1} \log n.$$

Proof. Suppose that J is an embedding operator

$$(3.12) \qquad J : \left(H^\infty(\Lambda_n)\right)^* \to U \hookrightarrow \left(C(\mathbb{T})\right)^*; \quad \|J^{-1}\| = 1$$

considered as a mapping from $\left(H^\infty(\Lambda_n)^* \hookrightarrow \left(C(\mathbb{T})\right)^*\right)$. Then $J^* : L_\infty \to H^\infty(\Lambda_n)$ is a quotient mapping and (cf. [14]), there exist functions

$$(3.13) \qquad h_j \in L_\infty : J^* h_j = e^{ij\theta}; \quad \|h_j\| \leq 1.$$

Let $I_\infty^1 : H_\infty(\Lambda_n) \to H_1(\Lambda_n)$ be a formal identity, and let e_j be the canonical vector basis in $\ell_1^{(n)}$. We define an operator $B : \ell_1^{(n)} \to \ell_1^{(n)}$ as follows:

$$(3.14) \qquad B : \ell_1^{(n)} \xrightarrow{A} L_\infty \xrightarrow{J^*} H^\infty(\Lambda_n) \xrightarrow{I_\infty^1} H'(\Lambda_n) \xrightarrow{S_n} \ell_1^{(n)}$$

where S_n is defined in Corollary 2.5 and $A : \ell_1^{(n)} \to L_\infty$ is defined by $Ae_j = h_j$. Obviously, we have

$$(3.15) \qquad \|A\| \leq 1.$$

Now by the definition of S_n

$$(3.16) \qquad \sum_{j=1}^{n} \frac{1}{j} = \operatorname{tr} B = \operatorname{tr}\left(S_n \cdot I_\infty^1 \cdot J^* \cdot A\right) = \operatorname{tr}\left(AS_n I_\infty^1 \cdot J^*\right).$$

By "trace-duality" (cf. [5])

$$(3.17) \qquad \sum_{j=1}^{n} \frac{1}{j} \leq \nu(AS_n I_\infty^1)\|J^*\|.$$

Since the range of A is an L_∞ space and by the fact that $\pi_1(I_\infty^1) = \|I_\infty^1\|$, we have (cf. [5])

$$(3.18) \qquad \nu(AS_n I_\infty') = \pi_1(AS_n I_\infty') \le \|A\|\,\|S_n\|\pi_1(I_\infty') \le C_3\|I_\infty'\| = C_3.$$

Combining with (3.17), we obtain

$$(3.19) \qquad\qquad\qquad \log n \le C_3\|J^*\|$$

and hence $\|J^*\| = \|J\| = \|J\|\,\|J^{-1}\| \ge C_3^{-1}\log n.$ ∎

By Proposition 3.2, this theorem implies the Faber theorem, yet it is not a consequence of it. It is very easy to construct subspaces $X_n \subset C(K)$ such that

$$(3.20) \qquad \lambda(X_n) \to \infty; \quad \exists U_n \subset C^*(K) : d(U_n, X_n^*) = 1.$$

Regarding Question 3.1, the theorem shows that if one is to find "The Riesz Representation" of $(H_n^\infty)^*$ it is not going to be among the measures on $\mathbb{T}$. Finally, it shows that one cannot find the basis $\varphi_1, \ldots, \varphi_n$ in the space H_n^∞ and the weights (w_j) such that the norm of a polynomial

$$(3.21) \qquad\qquad \left\| \sum_{j=1}^n c_j \varphi_j \right\| \sim \sum_{j=1}^n w_j |c_j|.$$

We conclude this section by mentioning an observation of J. Bourgain and A. Pelczynski (cf. [15]) which is in sharp contrast with Theorem 3.3. Let $A(\mathbb{T})$ be the disk algebra.

Theorem 3.4. *For every $n \ge 1$ there exists a subspace $X_n \subset A(\mathbb{T})$ such that*

$$(3.22) \qquad d(X_n, H_n^\infty) \le 2 \quad and \quad \lambda\big(X_n, A(\mathbb{T})\big) \le 2.$$

Together with Proposition 3.2, this theorem implies that there exist n-dimensional subspaces $U_n \subset \big(A(\mathbb{T})\big)^*$ such that

$$(3.23) \qquad\qquad d\big(U_n, (H_n^\infty)^*\big) \le 4.$$

Thus, while the Faber theorem holds in $A(\mathbb{T})$ just as well as in $C(\mathbb{T})$, the Generalized Faber theorem fails in $A(\mathbb{T})$.

4. The Sharpness of the Bernstein Theorem

In the spirit of the last section, we are looking for the isomorphic interpretation of the Erdős–Szabados Problem 1.5. In this section, we review some results in this direction.

Let $\delta_j := \delta_{t_j}$ be the functional on $C(\mathbb{T})$ defined as $\delta_j(f) = f(t_j)$, where $j = 1, \ldots, m$. Let $m < n$. It is clear that

$$(4.1) \qquad \|\Sigma a_j \delta_j\| = \Sigma |a_j|$$

and hence

$$(4.2) \qquad d\big(\operatorname{span}[\delta_j], \ell_1^{(m)}\big) = 1.$$

Suppose that γ is a constant such that for every function $f \in C(\mathbb{T})$ there exists a polynomial $p \in H^\infty(\Lambda_n)$ such that $\|p\| \leq \gamma$ and $p(t_j) = f(t_j)$. Equivalently,

$$(4.3) \qquad \forall (\alpha_j) \in \ell_\infty^m \exists p \in H^\infty(\Lambda_n) : p(t_j) = \alpha_j \quad \text{and} \quad \|p\| \leq \gamma.$$

Let $\tilde{\delta}_j = \delta_j \mid H^\infty(\Lambda_n)$. Then $\tilde{\delta}_j$ can be viewed as elements of $\big(H^\infty(\Lambda_n)\big)^*$ and the condition (4.3) can be interpreted as

$$(4.4) \qquad \sum_{j=1}^m |\alpha_j| \geq \left\| \sum_{j=1}^m \alpha_j \tilde{\delta}_j \right\| \geq \frac{1}{\gamma} \sum_{j=1}^m |\alpha_j|$$

or

$$(4.5) \qquad d\big(\operatorname{span}[\tilde{\delta}_j], \ell_1^{(m)}\big) \leq \gamma.$$

Let $\tilde{U} = \operatorname{span}[\delta_j]$. Then $\tilde{U}$ is a subspace of $H^\infty(\Lambda_n)$ and $d\big(\tilde{U}, \ell_1^{(m)}\big) \leq \gamma$. In order to show that for large m

$$(4.6) \qquad \gamma > \log \frac{n}{n - m + 1}.$$

we need to show that the n-dimensional space $\big(H^\infty(\Lambda_n)\big)^*$ does not contain a subspace of a small codimension $\tilde{U}$ which is close to ℓ_1^m. It seems apparent, since from (P.1) and (3.11) we have

$$(4.7) \qquad d\big(H^\infty(\Lambda_n)^*, \ell_1^{(n)}\big) \geq C \log n.$$

So if the space $\left(H^\infty(\Lambda_n)\right)^*$ is far removed from $\ell_1^{(n)}$, then any "large" subspace $\tilde{U} \subset \left(H^\infty(\Lambda_n)\right)^*$ should be far from $\ell_1^{(m)}$. At present, we do not know how to make this argument since the space $\left(H^\infty(\Lambda_n)\right)^*$ is not easy to handle. The rub is that $\tilde{U}$, being a subspace of $\left(H^\infty(\Lambda_n)\right)^*$, does not imply that $(\tilde{U})^*$ is similar to an n-dimensional subspace of $H^\infty(\Lambda_n)$.

Conjecture 4.1. *There exists a constant C_4 such that if U is an m-dimensional subspace of $\left(H^\infty(\Lambda_n)\right)^*$ $(m \leq n)$ then*

$$(4.8) \qquad d\left(U, \ell_1^{(m)}\right) \geq C_4 \left(\log \frac{n}{n-m+1} \right).$$

If this conjecture is true, it would imply the Erdős–Szabados conjecture (Problem 1.5). For now, the best we can do is to prove the following

Theorem 4.2. *There exists a constant $C_4 > 0$ such that if V_m is a subspace of $H^\infty(\Lambda_n)$ with $\dim V_m = m \leq n$. Then*

$$(4.9) \qquad d(V_m, \ell_\infty^m) \geq C_4 \cdot \left(\log \frac{n}{n-m+1} \right)^{\frac{1}{2}}.$$

The proof of this theorem is based on the combinatorial

Lemma 4.3. *Let A and B be $m \times n$ and $m \times n$ matrices such that*

    a) $A = (a_{ij})$; $i = 1, \ldots, m$, $j = 1, \ldots, n$; $|a_{ij}| \leq 1$

    b) $q(n) = n - m$

    c) $A \cdot B = I$ *an identity matrix on $\mathbb{C}_m$.*

Then there are $m - q(n)$ rows of the matrix $B = (b_{ij})$, $k_1, \ldots, k_{m-q(m)}$ such that

$$(4.10) \qquad \sum_{j=1}^{m} |b_{k_i,j}| \geq \frac{1}{2} \quad \text{for} \quad i = 1, \ldots, m - q(m).$$

The meaning of this lemma becomes clear by considering the case $m = n$. Then A and B are square matrices with $AB = I$. Hence, $BA = I$ and for every row in B we have

$$(4.11) \qquad 1 = \Sigma a_{ij} b_{ji} \leq \left(\Sigma |b_{ji}| \right) \max |a_{ij}| \leq \Sigma |b_{ji}|.$$

Hence, there are $m - q(n) = m$ rows satisfying (4.10).

The same lemma is the key to proving

Theorem 4.4. *Let P_m be an interpolating projection from $C(\mathbb{T})$ onto an m-dimensional subspace of $H^\infty(\Lambda_n)$. Then,*

$$(4.12) \qquad \|P_m\| \geq C_4 \log \frac{n}{n - m + 1}.$$

The proof of Theorems 4.2 and 4.4 are similar to the proof in [9]. We will forego the details of these proofs.

Comparing the theorems, we arrive at

Conjecture 4.5. *The estimate in Theorem 4.2 can be improved from $\sqrt{\log \frac{n}{n-m+1}}$ to $\log \frac{n}{n-m+1}$.*

If this conjecture is correct, then Theorem 4.4 is a simple corollary from Theorem 4.2 and Proposition 2.2.

Privaloff [6] proved that every projection (and not just an interpolating projection) from $C(\mathbb{T})$ onto an m-dimensional subspace of H_n^∞ satisfies (4.12). Hence

Conjecture 4.6. *Every projection P from $C(\mathbb{T})$ onto an m-dimensional subspace of $H^\infty(\Lambda_n)$ satisfies (4.12).*

The positive answer to this conjecture would affirm Conjecture 4.5 and would follow from the validity of

Conjecture 4.7. *Let V_m be an arbitrary m-dimensional subspace of $H^\infty(\Lambda_n)$ and let U_m be an arbitrary subspace of $\left(C(\mathbb{T})\right)^*$. Then*

$$(4.13) \qquad d(V_m^*, U_m) \geq C_4 \log \frac{n}{n - m + 1}.$$

Geometrically, the questions and results of this section address the following: Proposition 2.3 says that the unit ball of $H_\infty(\Lambda_n)$ is at least "$\log n$" removed from an n-dimensional unit cube. We want to conclude that every m-dimensional cross-section of the unit ball of $H_\infty(\Lambda_n)$ is "$\log \frac{n}{n-m+1}$" removed from the m-dimensional unit cube.

5. Embeddings of $H^\infty(\Lambda_n)$ into ℓ_∞^m

This short section is dedicated to the relationship between isomorphic embeddings of $H^\infty(\Lambda_n)$ into ℓ_∞^m and Theorem 1.10 and Problem 1.11.

Let $\Delta_m \subset \mathbb{T}$ be a collection of m distinct points, $m \geq n$. Consider a mapping

$$(5.1) \qquad J(\Delta_m) : H_\infty(\Lambda_n) \to \ell_\infty^m$$

defined by

$$(5.2) \qquad J(\Delta_m)f = (f \mid \Delta_m).$$

The range of this map is an n-dimensional subspace

$$(5.3) \qquad V_n \subset \ell_\infty^m.$$

Clearly, $J(\Delta_m)$ is an isomorphism from $H^\infty(\Lambda_n)$ onto V_n, and

$$(5.4) \qquad \big\| J(\Delta_m) \big\| = 1.$$

Also

$$(5.5) \quad \big\| J^{-1}(\Delta_m) \big\| = \sup\left\{ \frac{\big\| J^{-1}(\Delta_m)x \big\|}{\|x\|};\ x \in V_m \subset \ell_\infty^m;\ x \neq 0 \right\}$$

$$= \sup\left\{ \frac{\|f\|}{\big\| J(\Delta_m)f \big\|};\ f \in H^\infty(\Lambda_n) \right\}$$

$$= \sup\left\{ \frac{\|f\|}{\|f\|_\Delta};\ f \in H^\infty(\Lambda_n) \right\}.$$

Thus, the quantity discussed in Problem 1.11 relates to the Banach–Mazur distance $d\big(V_n, H^\infty(\Lambda_n)\big)$:

$$(5.6)\ \sup\left\{ \frac{\|f\|}{\|f\|_\Delta} : f \in H^\infty(\Lambda_n) \right\} = \big\| J^{-1}(\Delta) \big\| = \big\| J^{-1}(\Delta) \big\|\, \big\| J(\Delta) \big\|$$

$$\geq d\big(V_n, H^\infty(\Lambda_n)\big).$$

In particular, Theorem 1.10 implies

Corollary 5.1. *Let $a > 1$ and $m > an$. Then there exists a constant $K(a)$ and a subspace $V_n \subset \ell_\infty^{(m)}$ such that*

$$(5.7) \qquad\qquad d(H_n^\infty, V_n) \leq K(a).$$

Conjecture 5.2. *Let V_n be an arbitrary subspace of $\ell_\infty^{(m)}$. Then*

$$(5.8) \qquad\qquad d\big(H^\infty(\Lambda_n), V_n\big) \geq \log \frac{m}{m - n + 1}.$$

The affirmative resolution of this conjecture would immediately answer Problem 1.11.

The main reason for including this section in the paper is to demonstrate the "duality" of these questions with the problems posed in the previous two sections.

6. Conclusions

The result of Section 3 (Faber's theorem) demonstrate the relationship between the n-dimensional cube and the unit ball of $H^\infty(\Lambda_n)$. The results of Section 4 (Bernstein's theorem) hint on the relationships between the m-dimensional sections of the unit ball of $H^\infty(\Lambda_n)$ and the m-dimensional cube. The understanding of the relationship between the uniform and discrete norms of a polynomial shed light on the connection between the unit ball of $H^\infty(\Lambda_n)$ and n-dimensional sections of the unit ball of ℓ_∞^m.

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