DOMINATION NUMBERS OF Q-ANALOGUES OF KNESER GRAPHS

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Abstract. We investigate the domination number and total domination number of the graph $K_q(n,k)$ whose vertices are all the k-subspaces of an n-dimensional vector space over a field with q elements and whose edges are the pairs $\{U,W\}$ of vertices such that $U \cap W = \{0\}$. Bounds are obtained in general and exact results are obtained for $n \geq k^2+k-1$ and in other cases when q is sufficiently large relative to n and k. We also consider analogous bipartite graphs.

1. Introduction.

The *Kneser graph* $K(n,k)$ is the graph whose vertices are all the k-subsets of the set $[n] = \{1,2,\ldots,n\}$ and whose edges are the pairs $\{A,B\}$ of k-subsets such that A and B are disjoint [8]. By the q-analogue of $K(n,k)$ we mean the graph $K_q(n,k)$ whose vertices are all the k-subspaces of the n-dimensional vector space $V(n,q)$ over the finite field with q elements and whose edges are the pairs $\{U,W\}$ of vertices such that $U \cap W = \{0\}$. Here we concern ourselves with the domination numbers of the graphs $K_q(n,k)$. We will consider the domination numbers of $K(n,k)$ elsewhere.

Recall some standard notation: Let $G = (V,E)$ be a (simple) graph with vertex set V and edge set E. As usual we define $N(x) = \{y \in V | \{x,y\} \in E\}$ and $N[x] = N(x) \cup \{x\}$. A set $S \subseteq V$ is a *dominating set* for G if

$$V = \bigcup_{x \in S} N[x],$$

and S is a *total dominating set* for G if

$$V = \bigcup_{x \in S} N(x).$$

The *domination number* (resp., *total domination number*) of G is the cardinality $\gamma(G)$ (resp., $\gamma_t(G)$) of a minimum dominating set (resp., total dominating set) for G. The invariants $\gamma(G)$ and $\gamma_t(G)$ have been investigated for numerous graphs (see, *e.g.*, [1] and [7]). To our knowledge there has been only one previous paper

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on domination in Kneser graphs, namely, [8], in which the authors established $\gamma(K(n,2))$, $\gamma_t(K(n,2))$, and the domatic number of $K(n,2)$.

In this paper we show that for $k \geq 2$: if $n \geq k(k+1)$

$$\gamma(K_q(n,k)) = \gamma_t(K_q(n,k)) = k+1$$

and if either $n = k(k+1)-1$ or $n \geq (k^2+k)/2$ and $q \geq c(n,k)$ then

$$\gamma(K_q(n,k)) = \gamma_t(K_q(n,k)) = k+2$$

We also obtain the bounds

$$k+2 \leq \gamma(K_q(n,k)) \leq \gamma_t(K_q(n,k)) \leq \left\lfloor \frac{k(n-k)}{n-2k+1} \right\rfloor + 1$$

when $2k \leq n \leq k(k+1)-2$ and $q \geq c(n,k)$. The upper bound is derived from an upper bound [4] for $\gamma(G)$ which depends only on the number of vertices of G and the minimum degree of G. To apply this result we require special properties of gaussian polynomials.

In section 3 we study the total domination numbers of the bipartite graph $K_q(n,k,\ell)$, $n > k > \ell$, whose vertex set consists of all k-subspaces and all ℓ-subspaces of $V(n,q)$; $\{U,W\}$ is an edge when U is a k-subspace, W is an ℓ-subspace and $U \cap W = \{0\}$.

The present paper was largely motivated by the paper [5] where the authors considered (in a different language) the problem of finding the total domination numbers of the graphs $K_q(n,k,n-k)$ where $GF(q)$ was allowed to be any field. Exact results were obtained in [6] via algebraic geometry in the case when the field is algebraically closed.

2. Domination Numbers for $K_q(n,k)$.

We use the following upper bound from [4]. We include a proof of for the convenience of the reader.

Lemma 1. *Let $G = (V,E)$ be a graph with p vertices and minimum degree δ.*

(i) If t is a positive integer such that

$$(p-\delta-1)^t < p^{t-1} \tag{1}$$

then $\gamma(G) \leq t$.

(ii) If t is a positive integer such that

$$(p-\delta)^t < p^{t-1} \tag{2}$$

then $\gamma_t(G) \leq t$.

Proof. Write X^t for the cartesian product of X with itself t times and define

$$B = \bigcup_{y \in V} (V - N[y])^t. \tag{3}$$

Note that $B \subseteq V^t$. It is easy to see that a t-set $\{x_1, x_2, \ldots, x_t\} \subseteq V$ is a dominating set for G if and only if the sequence $(x_1, x_2, \ldots, x_t)$ is not in B. Hence if $|B| < |V^t| = p^t$ then there exists a dominating set of cardinality t. Now $\delta + 1 \leq |N[y]|$ for all vertices y and hence

$$|B| \leq |V|(|V - N[y]|)^t| \leq p(p - \delta - 1)^t.$$

So if (1) holds then $|B| < |V^t|$ and we have $\gamma(G) \leq t$. This proves (i).

If we replace $N[y]$ by $N(y)$ in the above proof of (i) we obtain a proof of (ii). $\square$

Remark. It is obvious that $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$. In general γ and γ_t are not equal, but as we shall see when n is large relative to k for the graphs $K_q(n,k)$ they are equal. To show this it suffices to show that an upperbound for γ_t is equal to a lower bound for γ. If $k = 1$ then $K_q(n,k)$ is a complete graph and so $\gamma(K_q(n,1)) = 1$ and $\gamma_t(K_q(n,1)) = 2$. We know of no case when $k > 1$ when $\gamma(K_q(n,1)) \neq \gamma_t(K_q(n,1))$. If $n < 2k$ then $K_q(n,k)$ is trivial and so we are only interested in the cases $k \geq 2$ and $n \geq 2k$.

Lemma 2. *If $k \geq 2$ and $n \geq 2k$, then $\gamma(K_q(n,k)) \geq k+1$ for all prime powers q.*

Proof. Suppose $S = \{U_1, \ldots, U_k\}$ is a dominating set for $K_q(n,k)$. Select $0 \neq x_i \in U_i$ for $i = 1, \ldots, k$ and let U be the span of $x_1, \ldots, x_k$. Assume first that $dim\ U < k$. If we extend U to a k-subspace W then W still meets all U_i's. We have a contradiction if we can show there is at least one such W that is not in S. It is easy to show that if $dim\ U = t < k$ there are precisely $\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$ k-subspaces W of $V(n,q)$ containing U. Now

$$(4) \qquad \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q \geq \begin{bmatrix} n-t \\ 1 \end{bmatrix}_q = \frac{q^{n-t}-1}{q-1} \geq \frac{q^{k+1}-1}{q-1} = q^k + \cdots + q + 1 > k+1.$$

So there are more than $k+1$ such W's and therefore at least one is not in S and we have our contradiction.

So we can assume that U has dimension exactly k. Now if $U_i \cap U_j = 0$ for all $i \neq j$ then we clearly cannot have $U = U_i$. On the otherhand, if for some pair $i \neq j$ we have $U_i \cap U_j \neq 0$, then we may choose $x_i = x_j$ and it follows that U has dimension less than k which is the case already handled. $\square$

Lemma 3. *Let $k \geq 2$ and $2k \leq n \leq k(k+1) - 1$ then $\gamma(K_q(n,k)) \geq k+2$.*

Proof. Suppose $S = \{U_1, \ldots, U_{k+1}\}$ is a dominating set of $K_q(n,k)$. Since each U_i is a k-subspace of $V(n,q)$ and $n \leq (k+1)k - 1$ we know that $U_1 + \cdots + U_{k+1}$ is not a direct sum. Hence there exists an index i and a non-zero vector $x_i \in U_i$ that is a linear combination of some non-zero vectors $x_j \in U_j$ for $j \in J \subseteq [k+1] - \{i\}$. Thus we can select for every $j \in [k+1]$ a non-zero vector $x_j \in U_j$ such that the space $U = span(x_1, \ldots, x_{k+1})$ has dimension at most k. Now we consider two cases:

Case I: For some choice of non-zero $x_j \in U_j$ we have $dim\ U = t < k$.

Case II: For every choice of non-zero $x_j \in U_j$ such that $dim\ U \leq k$ we have $dim\ U = k$.

We show that both cases lead to contradictions. First assume Case I holds. Then as in the first part of Lemma 2 we see that there are more than $k+1$ k-subspaces W containing U. So at least one of the W's is not in S. This is a contradiction since W meets every U_i in a non-zero vector.

Thus we can assume Case II holds. Then since U is a k-subspace meeting each U_i non-trivially we must have $U \in S$. So we can assume that $U = U_1$. Thus we have $U_1 \cap U_2 \neq 0$ and if we take $0 \neq x_1 = x_2 \in U_1 \cap U_2$ and $0 \neq x_j \in U_j$ for $j \geq 3$ we have $dim\ span(x_1, \ldots, x_{k+1}) = k$ and hence $x_2, x_3, \ldots, x_{k+1}$ are linearly independent. This implies that

$$(U_1 \cap U_2) + U_3 + \cdots + U_{k+1}$$

is a direct sum. Suppose that $k \geq 3$, so $k+1 \geq 4$. Then we let $\{x_3^i | i \in [k]\}$ and $\{x_4^i | i \in [k]\}$ be bases for U_3 and U_4, respectively. Then the k^2 spaces

$$U_{ij} = span(x_2, x_3^i, x_4^j, x_5, \ldots, x_{k+1})$$

are distinct. Since $k^2 \geq k+2$ all of the U_{ij}'s cannot be in S and we have a contradiction.

It remains to consider the case $k = 2$. Then $S = \{U_1, U_2, U_3\}$ and $(U_1 \cap U_2) + U_3$ is a direct sum. So if we take $\{y, z\}$ to be a basis for U_3 and $0 \neq x \in U_1 \cap U_2$ then the two subspaces $W_1 = span(x, y)$ and $W_2 = span(x, z)$ both meet each U_i non-trivially and so must be in S. Since $x \notin U_3$ we can assume that $W_1 = U_1$ and $W_2 = U_2$. This shows that U_1, U_2 and U_3 are all subspaces of the 3-subspace $W = span(x, y, z)$. Now W contains $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = q^2 + q + 1$ 2-subspaces. Any two 2-subspaces of W intersect non-trivially and so there is at least one 2-subspace of W that is not is S and meets non-trivially every element of S. So again we have a contradiction and the proof is complete. $\square$

Theorem 1. *Let $k \geq 2$.*
(a) If $n \geq k(k+1)$ then $\gamma(K_q(n,k)) = \gamma_t(K_q(n,k)) = k+1$
(b) If $n = k(k+1) - 1$ then $\gamma(K_q(n,k)) = \gamma_t(K_q(n,k)) = k+2$

Proof of (a). By Lemma 2 and the fact that $\gamma \leq \gamma_t$ it suffices to find a total dominating set of cardinality $k+1$ for $K_q(n,k)$. Since $dim\ V(n,q) \geq k(k+1)$ we can find k-subspaces $W_1, \ldots, W_{k+1}$ and a subspace Z such that

$$V(n,q) = W_1 \oplus W_2 \oplus \cdots \oplus W_{k+1} \oplus Z.$$

If W is a subspace meeting W_i in a non-zero vector x_i for each i, since the x_i's are linearly independent, W cannot be k-dimensional. Hence the W_i's form a total dominating set. This proves (a).

Proof of (b). By Lemma 3 and the fact that $\gamma \leq \gamma_t$ it suffices to find a total dominating set of cardinality $k+2$ for $K_q(n,k)$. Since $dim\ V(n,q) \geq k(k+1) - 1$ we can find k-subspaces $U_1, \ldots, U_k$ and a $(k-1)$-subspace C such that

$$V(n,q) = U_1 \oplus U_2 \oplus \cdots \oplus U_k \oplus C.$$

Let x, y be linearly independent vectors in U_1 and set

$$U_{k+1} = C \oplus span(x), \qquad U_{k+2} = C \oplus span(y).$$

We claim that $S = \{U_1, \ldots, U_{k+2}\}$ is a total dominating set for $K_q(n,k)$. Suppose U is a k-space meeting U_i, $i = 1, \ldots, k+2$, in a non-zero vector x_i. Since $U_1 + \cdots + U_k$ is a direct sum $x_1, \ldots, x_k$ are linearly independent and so $U = span(x_1, \ldots, x_k)$ and $U \cap U_i = span(x_i)$, $i = 1, \ldots, k$. Now $x_{k+1} = c_1 + \alpha x$ where $c_1 \in C$ and α is a non-zero scalar. Since x_{k+1} is in $U \subseteq U_1 + \cdots + U_k$ it follows that $c_1 \in C \cap (U_1 + \cdots + U_k) = 0$. It follows that $c_1 = 0$ and so $x_{k+1} = \alpha x$. Similarly $x_{k+2} = \beta y$ for some non-zero scalar β. This shows that $x, y \in U \cap U_1 = span(x_1)$. This contradicts the fact that x and y are linearly independent and completes the proof. $\square$

Now to apply Lemma 1 we note that the number of vertices of $K_q(n,k)$, that is, the number of k-subspace of $V(n,q)$ is given by the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^n - 1)(q^{n-1} - 1) \ldots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \ldots (q - 1)}. \tag{5}$$

and by [3], p. 306,, the degree of each vertex is $\begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2}$. Thus if we let $U_q(n,k)$ be the least positive integer t such that

$$(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2} - 1)^t < \begin{bmatrix} n \\ k \end{bmatrix}^{t-1} \tag{6}$$

from Lemma 1 we have

Theorem 2. *If $n \geq 2k$ then*

$$\gamma(K_q(n,k) \leq U_q(n,k). \quad \square \tag{7}$$

Theorem 3. *For $k \geq 2$ and $n \geq 2k$ there exists an integer $c(k,n)$ such that for $q \geq c(k,n)$:*

$$\gamma(K_q(n,k)) \leq \gamma_t(K_q(n,k)) \leq \left\lfloor \frac{k(n-k)}{n-2k+1} \right\rfloor + 1 \tag{8}$$

Proof. It is well-known (see, e.g., [2]) that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\ell=0}^{k(n-k)} p(k, n-k, \ell) q^\ell \tag{9}$$

where $p(k, n-k, \ell)$ is the number of partitions of ℓ with at most $n-k$ parts and each part is at most k. In particular $\begin{bmatrix} n \\ k \end{bmatrix}$ is a monic polynomial in q of degree $k(n-k)$. Now consider the polynomial

$$f(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2}. \tag{10}$$

Assume for a moment that we have shown that $f(n,k)$ is a monic polynomial of degree $k(n-k)-(n-2k)-1$. Now to apply Lemma 1 we seek the smallest positive integer t such that the polynomial

$$h_t(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}^{t-1} - f(n,k)^t \tag{11}$$

is monic. It will follow that for sufficiently large q $h_t(n,k)$ will be positive. To see how large t should be we note that $\begin{bmatrix} n \\ k \end{bmatrix}^{t-1}$ has degree $k(n-k)(t-1)$ and by our momentary assumption $f(n,k)^t$ has degree $(k(n-k)-(n-2k)-1)t$. So we need

$$k(n-k)(t-1) > (k(n-k)-(n-2k)-1)t$$

which is equivalent to

$$t > \frac{k(n-k)}{n-2k+1}.$$

So if we take the least positive such t we get

$$t = \left\lfloor \frac{k(n-k)}{n-2k+1} \right\rfloor + 1$$

and the theorem is proved.

So it remains to show that

$$f(n,k) \quad \text{is monic and} \quad deg(f(n,k)) = k(n-k)-(n-2k)-1. \tag{12}$$

The case $n = 2k$ is trivial so we assume $n > 2k$. Let $N = k(n-k)$ and $M = k(n-2k)$ then

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\ell=0}^{N} a_\ell q^\ell \tag{13}$$

and

$$\begin{bmatrix} n-k \\ k \end{bmatrix} = \sum_{\ell=0}^{M} b_\ell q^\ell. \tag{14}$$

These polynomials are symmetric, that is $a_\ell = a_{N-\ell}$ and $b_\ell = b_{M-\ell}$ for all ℓ (see [2]). Note that $M + k^2 = N$ so $\begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2}$ have the same degree. So to prove (12) it suffices to show that

$$a_\ell = b_\ell, \qquad \ell = 0, 1, \ldots, n-2k \tag{15}$$

and

$$a_\ell - b_\ell = 1, \qquad \ell = n-2k+1. \tag{16}$$

To establish this we let $S(k,m,\ell)$ be the set of all sequences $(\lambda_1,\dots,\lambda_t)$ satisfying

$$\ell = \lambda_1 + \lambda_2 + \cdots + \lambda_t \tag{17}$$

$$k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 1 \tag{18}$$

$$m \geq t \geq 1 \tag{19}$$

It is easy to see that for $n > 2k$:

$$S(k, n-2k, \ell) = S(k, n-k, \ell), \qquad \ell \in \{1, 2, \dots, n-2k\}$$

and

$$S(k, n-2k, \ell) + 1 = S(k, n-k, \ell), \qquad \ell = n - 2k + 1.$$

From this we immediately obtain (15) and (16) and the proof is complete. $\square$

Lemma 4. *Let* $U(n,k) = \left\lfloor \frac{k(n-k)}{n-2k+1} \right\rfloor + 1$. *Then*

(a) $U(n,k)$ *is non-increasing in* n *for fixed* k.
(b) $k^2 + 1 \geq U(n,k) \geq k + 1$ *for* $n \geq 2k$.
(c) $U(n,k) = k + 1 \Longleftrightarrow n \geq k^2 + k$.
(d) $U(n,k) = k + 2 \Longleftrightarrow \frac{1}{2}k^2 + \frac{3}{2}k \leq n \leq k^2 + k - 1$. $\square$

Using Lemma 4 we may summarize Lemma 3, Theorem 1 and Theorem 3 as follows:

Theorem 4. *If* $k \geq 2$ *and* $n \geq k^2 + k - 1$ *then*

$$\gamma(K_q(n,k)) = \gamma_t(K_q(n,k)) = U(n,k) \tag{20}$$

Moreover if $n \geq \frac{1}{2}k^2 + \frac{3}{2}k$ *then (20) holds whenever* $q \geq c(n,k)$. *In any case for* $q \geq c(n,k)$ *and* $n \geq k(k+1) - 1$

$$k + 2 \leq \gamma(K_q(n,k)) \leq \gamma_t(K_q(n,k)) \leq U(n,k) \leq k^2 + 1. \quad \square$$

Remark. In [5] we showed that

$$\gamma(K_q(2,4)) = \gamma_t(K_q(2,4)) = 4$$

for all q. So (20) fails in at least one instance.

3 The Bipartite Graphs $K_q(n,k,\ell)$.

For integers $n > k > \ell \geq 1$ and $n \geq k+\ell$ define the bipartite graph $K_q(n,k,\ell)$ as follows: The vertices of $K_q(n,k,\ell)$ are the k-subspaces and the ℓ-subspaces of $V(n,q)$ and (U,W) is an edge if and only if U is a k-subspace, W is an ℓ-subspace and $U \cap W = 0$. Using the methods of section 2 it is easy to prove:

Theorem 5. *If $n > k > \ell > 1$ and $n \geq k\ell + k$ then the total domination number of $K_q(n,k,\ell)$ is $k+\ell+2$.* $\square$

The following is a bipartite version of Lemma 1.

Lemma 5. *Let $G = (V,E)$ be a bipartite graph with parts V_1 and V_2. Let δ_i be the minimum degree of the vertices in V_i and let $p_i = |V_i|$ for $i = 1,2$. Let s and t be positive integers such that both of the following hold:*

$$p_1(p_2 - \delta_1)^t < p_2^t \tag{21}$$

and

$$p_2(p_1 - \delta_2)^s < p_1^s. \tag{22}$$

Then $\gamma_t(G) \leq s+t$.

Proof. Let $B = \bigcup_{y \in V_1} (V_2 - N(y))^t$. Then $B \subseteq V_2^t$ and the t-set $\{x_1,\ldots,x_t\} \subseteq V_2$ is a dominating set for the vertices in V_1 if and only if the sequence $(x_1,\ldots,s_t)$ lies in $V_2^t - B$. Now, $\delta_1 \leq N(y)$ for all $y \in V_1$ so $|B| \leq p_1(p_2-\delta_1)^t$. So if (20) holds $|B| < |V_2|^t = p_2^t$ and there is a dominating set for V_1 of size t. Similarly if (21) holds there is a dominating set for V_2 of size s. So if both hold then $\gamma_t(G) \leq s+t$. $\square$

To apply Lemma 5 to $K_q(n,k,\ell)$ where we assume $n > k > \ell \geq 1$ and $n \geq k+\ell$ we take

$V_1 =$ the set of all k-subspace of $V(n,q)$ and $V_2 =$ the set of all ℓ-subspaces of $V(n,q)$. Then from [3], page 306, we have

$$p_1 = \begin{bmatrix} n \\ k \end{bmatrix} \qquad p_2 = \begin{bmatrix} n \\ \ell \end{bmatrix} \qquad \delta_1 = \begin{bmatrix} n-k \\ \ell \end{bmatrix} q^{k\ell} \qquad \delta_2 = \begin{bmatrix} n-\ell \\ k \end{bmatrix} q^{k\ell}.$$

Thus (20) and (21) become

$$\begin{bmatrix} n \\ k \end{bmatrix} \left(\begin{bmatrix} n \\ \ell \end{bmatrix} - \begin{bmatrix} n-k \\ \ell \end{bmatrix} q^{k\ell} \right)^t < \begin{bmatrix} n \\ \ell \end{bmatrix}^t \tag{23}$$

$$\begin{bmatrix} n \\ \ell \end{bmatrix} \left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-\ell \\ k \end{bmatrix} q^{k\ell} \right)^s < \begin{bmatrix} n \\ k \end{bmatrix}^s \tag{24}$$

then via the same approach used in the proof of Theorem 1 one obtains:

Theorem 6. *If $n > k > \ell$ and $n \geq k + \ell$ then there is a constant $c(n,k)$ such that for $q \geq c(n,k)$ then $\gamma_t(K_q(n,k,\ell))$ is at most*

$$\left\lfloor \frac{k(n-k)}{n-k-\ell+1} \right\rfloor + \left\lfloor \frac{\ell(n-\ell)}{n-k-\ell+1} \right\rfloor + 2. \quad \square \tag{25}$$

Remark. A case of special interest is the case $\ell = n - k$. In this case $\{A, B\}$ is an edge if and only if $V(n,q) = A \oplus B$; and the above upper bound becomes

$$\gamma_t(K_q(n,k,n-k) \leq 2k(n-k) + 2. \tag{26}$$

Let $K_F(n,k,n-k)$ denote the graph whose vertices are the k-subspaces and the $(n-k)$-subspaces of the n-dimensional vector space $V(n,F)$ over an arbitrary field F and whose edges are the pairs $\{U, W\}$ such that $U \cap W = \{0\}$. In [5] and [6] the authors investigated the total domination number of $K_F(n,k,n-k)$. It follows from [6] that if F is algebraically closed then

$$\gamma_t(K_F(n,k,n-k)) = 2k(n-k) + 2, \quad \text{if} \quad k < n/2$$

and

$$\gamma_t(K_F(2k,k,k)) = k^2 + 1.$$

The problem is open for arbitrary F; however, we proved in [5] that $\gamma_t(K_F(4,2,2))$ is 5 if F is quadratically closed and, 4 otherwise. In particular, as mentioned previously, for any prime power q

$$\gamma_t(K_q(4,2)) = \gamma(K_q(4,2)) = 4$$

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