NORTH-HOLLAND

Extension Constants of Unconditional Two-Dimensional Operators

B. L. Chalmers
Department of Mathematics
University of California, Riverside
Riverside, California 92521

and

B. Shekhtman
Department of Mathematics
University of South Florida
Tampa, Florida 33620

Submitted by Richard A. Brualdi

ABSTRACT

It is shown that the (absolute) extension constant $e(T)$ of an operator T such that $Tv_k = \lambda_k v_k$, $k = 1, 2$, for some unconditional basis (v_1, v_2) of a two-dimensional real normed space is less than or equal to $(|\lambda_1| + |\lambda_2| + 2\sqrt{\lambda_1^2 - |\lambda_1 \lambda_2| + \lambda_2^2})/3$. In fact, it is demonstrated that $e(T)$ is attained by exactly one unconditional two-dimensional space (up to an isometry).

1. INTRODUCTION AND PRELIMINARIES

Let V be an *unconditional* n-dimensional Banach space, and let T be an operator on V such that $Tv_k = \lambda_k v_k$, $k = 1, \dots, n$, for a fixed unconditional basis $(v_1, \dots, v_n)$ of V ($\|\sum_{k=1}^n \alpha_k v_k\| = \|\sum_{k=1}^n |\alpha_k| v_k\|$).

NOTATION. For any Banach space X, with $V \subset X$, set $e(T, X) = \inf\|\tilde{T}\|$, where $\tilde{T}$ runs through all operators from X into V agreeing with T on V. Then $e(T, X)$ is called the *extension constant of T relative to X*. The number

LINEAR ALGEBRA AND ITS APPLICATIONS 240:173–182 (1996)

0024-3795/96/$15.00
SSDI 0024-3795(94)00196-K

$e(T) = \sup_X e(T, X)$ is called the (*absolute*) *extension constant of* T. Any X for which $e(T, X) = e(T)$ is called a *maximal overspace for* T.

It is well known [5] that in the case $T = I$ ($\lambda_1 = \cdots = \lambda_n = 1$), we have $e(T) \leqslant \sqrt{n}$. It had been conjectured (see, e.g., [4, p. 465] and [8, pp. 273–274]) that if $n = 2$ and the field is $\mathbb{R}$, then $e(T) \leqslant \frac{4}{3}$. In [6] it was shown that this conjecture is true and, moreover, that $\frac{4}{3}$ is attained by precisely one two-dimensional space (up to isometry). In the present paper, we extend (the unconditional part of) this result to operators on unconditional two-dimensional subspaces. The procedure yields an operator $\tilde{T} = u_1 \otimes v_1 + u_2 \otimes v_2$ in L^1 (not in general minimal) of norm $\leqslant (|\lambda_1| + |\lambda_2| + 2\sqrt{\lambda_1^2 - |\lambda_1 \lambda_2| + \lambda_2^2})/3$. [For $v \in L^1$ and $u \in L^\infty$, $(u \otimes v)(x) := \langle x, u \rangle v$.] The functionals u_i are 3-piecewise constants in L^∞. The location of the breakpoints is determined by Lemma 2 of Section 2. Numerical solutions of Lemma 2 thus give a very simple way of determining the operator explicitly (see, e.g., the Remark in Section 2). Also it should be noted that, by restricting attention to the case $T = I$, the procedure gives a different (elementary) proof of the "$\frac{4}{3}$" result of [7] in the case of unconditional spaces.

THEOREM A (e.g., [7, 9]). *Any two-dimensional real normed space is* (*isometric to*) *a subspace of* $L^1[-\pi/2, \pi/2]$.

THEOREM B [1, 7]. *If* $\dim V = 2$, *the field is* $\mathbb{R}$, *and* $V \subset L^1$, *then* $e(T, L^1) = e(T)$.

2. MAIN RESULT

(In the following the linear span of w and z will be denoted by $[w, z]$.)

THEOREM. *If* V *is an unconditional two-dimensional real Banach space and* T *is an operator on* V *such that* $Tv_k = \lambda_k v_k$, $k = 1, 2$, *then* $e(T) \leqslant (|\lambda_1| + |\lambda_2| + 2\sqrt{\lambda_1^2 - |\lambda_1 \lambda_2| + \lambda_2^2})/3$. *Also, if* $\lambda_1 \lambda_2 \neq 0$, *then* $[(1, u, 0), (0, u, 1)] \subset l_3^1$, *where* $u = d - 1 + \sqrt{d^2 - d + 1}$, $d = |\lambda_1/\lambda_2|$, *gives the unique space* (*up to isometry*) *where equality is attained.*

For the proof of the theorem, we will need the following two lemmas. (Note that in the following Lemma 1, in the "projection" case $T = I$, (b) and (c) are vacuous.)

NOTATION. In the sequel the notation "$\pm\lambda_2$" is used instead of "$|\lambda_2|$" so that the sign configuration of $\tilde{T}$ discussed in the proof of Lemma 1 will correspond.

LEMMA 1. *An arbitrary unconditional two-dimensional subspace of* l_3^1 *is isometric to a space* $V = [v_1, v_2] \subset l_3^1$, *with unconditional basis* $(v_1, v_2) = ((1, 2a, 1), (-1, 0, 1))$, $a \geqslant 0$, *each* v_k *being unique up to a scalar multiple. Let* $\rho = 2(1 + a + a^2)$. *If* $\pm\lambda_2 \geqslant \lambda_1 \geqslant 0$ *and* $\lambda_1 + a(\lambda_1 \mp \lambda_2) \geqslant 0$, *or if* $\lambda_1 \geqslant \pm\lambda_2 \geqslant 0$ *and* $\pm\lambda_2(a+1) - \lambda_1 \geqslant 0$, *set*

$$\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} \lambda_1 + a(\lambda_1 \mp \lambda_2) & -\rho\lambda_2/2 \\ a\lambda_1 \pm \lambda_2 & 0 \\ \lambda_1 + a(\lambda_1 \mp \lambda_2) & \rho\lambda_2/2 \end{pmatrix}. \tag{a}$$

If $\pm\lambda_2 \geqslant \lambda_1 \geqslant 0$ *and* $\lambda_1 + a(\lambda_1 \mp \lambda_2) < 0$, *set*

$$\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\lambda_2 \\ \lambda_1/a & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{b}$$

If $\lambda_1 \geqslant \pm\lambda_2 \geqslant 0$ *and* $\pm\lambda_2(a+1) - \lambda_1 < 0$, *set*

$$\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{pmatrix} = \frac{1}{2(a+1)}\begin{pmatrix} \lambda_1 & -(a+1)\lambda_2 \\ \lambda_1 & 0 \\ \lambda_1 & (a+1)\lambda_2 \end{pmatrix}. \tag{c}$$

For $u_1 = (c_1, c_2, c_3)$ *and* $u_2 = (d_1, d_2, d_3)$, *the operator* $\tilde{T} = u_1 \otimes v_1 + u_2 \otimes v_2$ *is an operator from* l_1^3 *into* V, *with* $\tilde{T}v_k = \lambda_k v_k$, $k = 1, 2$, *and* $\|\tilde{T}\| = (1 + a)(a\lambda_1 + |\lambda_2|)/(1 + a + a^2)$, $|\lambda_2|$, λ_1 *in* (a), (b), (c), *respectively. Further,* $\|\tilde{T}\| \leqslant (\lambda_1 + |\lambda_2| + 2\sqrt{\lambda_1^2 - |\lambda_1\lambda_2| + \lambda_2^2})/3$, *and if* $\lambda_1\lambda_2 \neq 0$, *then equality occurs if and only if* $a = d - 1 + \sqrt{d^2 - d + 1}$, $d = |\lambda_1/\lambda_2|$.

Proof. First, by applying a linear transformation, it is immediate that any two-dimensional subspace of l_3^1 is isometric to a subspace with a basis of the form (w_1, w_2), where $w_1 = (1, a', 0)$ and $w_2 = (0, b', \epsilon_2)$, with $|\epsilon_2| = 1$ and

$0 \leqslant a'b'$. Thus, for any real α and β,

$$\|\alpha w_1 + \beta w_2\| = |\alpha| + |\alpha a' + \beta b'| + |\beta| = \|\alpha \tilde{v}_1 + \beta \tilde{v}_2\|,$$

where $\tilde{v}_1 = (1, a, 0)$, $\tilde{v}_2 = (0, b, 1)$, $a = |a'|$, $b = |b'|$. That is, we can take $V = [(1, a, 0), (0, b, 1)]$. Further, V being unconditional implies, either as a consequence of the more general discussion prior to Lemma 2 [Equation (1)] or by a straightforward check, that either $b = a$ and the unconditional basis is uniquely (up to a scalar multiple of each v_i) $(v_1, v_2) = ((1, 2a, 1), (-1, 0, 1))$, or that at least one of a or b is 0, whence V is isometric to $[(1, 0, 1), (-1, 0, 1)]$. This establishes the isometry of Lemma 1.

Secondly, a simple direct check shows that $\tilde{T}$ is an operator from l_3^1 into V such that $\tilde{T}v_k = \lambda_k v_k$, i.e., $\langle v_k, u_l \rangle = \lambda_k \delta_{kl}$. For the sequel, introduce the notation $\vec{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, and let $\vec{\alpha} \cdot \vec{\beta} = \alpha_1 \beta_1 + \alpha_2 \beta_2$ denote the usual dot product. (The fact that $\vec{\alpha} \cdot \vec{\beta} = |\vec{\alpha}|\,|\vec{\beta}| \cos \gamma$, where γ is the angle between the two vectors $\vec{\alpha}$ and $\vec{\beta}$ in $\mathbb{R}^2$, plays a vital role in motivating the geometric proof of the theorem, and for this reason we choose to emphasize the vector notation by an overhead arrow.)

Finally, to calculate $\|\tilde{T}\|$, denote (w_{1k}, w_{2k}) by $\vec{w}(k)$, and form the operator matrix $\tilde{T} = (t_{ij})_{3\times 3} = (\vec{u}(j) \cdot \vec{v}(i))_{3 \times 3}$: If $\pm\lambda_2 \geqslant \lambda_1 \geqslant 0$ and $\lambda_1 + a(\lambda_1 \mp \lambda_2) \geqslant 0$, or if $\lambda_1 \geqslant \pm\lambda_2 \geqslant 0$ and $\pm\lambda_2(a+1) - \lambda_1 \geqslant 0$, then

$$\tilde{T} = \frac{1}{\rho}\begin{pmatrix} \lambda_1 + a(\lambda_1 \mp \lambda_2) + \rho\lambda_2/2 & a\lambda_1 \pm \lambda_2 & \lambda_1 + a(\lambda_1 \mp \lambda_2) - \rho\lambda_2/2 \\ 2a[\lambda_1 + a(\lambda_1 \mp \lambda_2)] & 2a(a\lambda_1 \pm \lambda_2) & 2a[\lambda_1 + a(\lambda_1 \mp \lambda_2)] \\ \lambda_1 + a(\lambda_1 \mp \lambda_2) - \rho\lambda_2/2 & a\lambda_1 \pm \lambda_2 & \lambda_1 + a(\lambda_1 \mp \lambda_2) + \rho\lambda_2/2 \end{pmatrix}.$$

Note that $\lambda_1 + a(\lambda_1 \mp \lambda_2) \mp \rho\lambda_2/2 = -(a+1)[\pm\lambda_2(a+1) - \lambda_1]$ and $\lambda_1 + a(\lambda_1 \mp \lambda_2) \pm \rho\lambda_2/2 = (a+1)\lambda_1 \pm \lambda_2(1 + a^2)$. By noting that the sign configuration of $\tilde{T}$ is

$$\begin{pmatrix} \pm & + & \mp \\ + & + & + \\ \mp & + & \pm \end{pmatrix},$$

it is straightforward to verify that all three absolute column sums $(\Sigma_{i=1}^3 |t_{ij}|$, $j = 1, 2, 3)$ are equal to $(1+a)(a\lambda_1 + |\lambda_2|)/(1 + a + a^2)$. But this latter quantity achieves its maximum $[= (\lambda_1 + |\lambda_2| + 2\sqrt{\lambda_1^2 - |\lambda_1\lambda_2| + \lambda_2^2})/3]$ at $a = d - 1 + \sqrt{d^2 - d + 1}$, $d = |\lambda_1/\lambda_2|$, as is easily seen by solving for the zero of the derivative with respect to a. [For this value of a yielding the maximum, $\lambda_1 + a(\lambda_1 \mp \lambda_2) = \tau(\lambda_1 \mp \lambda_2 + \tau)/(\pm\lambda_2) \geqslant 0$ and $\pm\lambda_2(a+1) - \lambda_1 = \tau \geqslant 0$, where $\tau = \sqrt{\lambda_1^2 \mp \lambda_1\lambda_2 + \lambda_2^2}$.]

If $\pm\lambda_2 \geqslant \lambda_1 \geqslant 0$ and $\lambda_1 + a(\lambda_1 \mp \lambda_2) < 0$,

$$\tilde{T} = \frac{1}{2}\begin{pmatrix} \lambda_2 & \lambda_1/a & -\lambda_2 \\ 0 & 2\lambda_1 & 0 \\ -\lambda_2 & \lambda_1/a & \lambda_2 \end{pmatrix}.$$

By noting that the sign configuration of $\tilde{T}$ is

$$\begin{pmatrix} \pm & + & \mp \\ 0 & + & 0 \\ \mp & + & \pm \end{pmatrix},$$

it is immediate that $\|\tilde{T}\| = |\lambda_2|$.

If $\lambda_1 \geqslant \pm\lambda_2 \geqslant 0$ and $\pm\lambda_2(a+1) - \lambda_1 < 0$, then

$$\tilde{T} = \frac{1}{2(a+1)}\begin{pmatrix} \lambda_1 + (a+1)\lambda_2 & \lambda_1 & \lambda_1 - (a+1)\lambda_2 \\ 2a\lambda_1 & 2a\lambda_1 & 2a\lambda_1 \\ \lambda_1 - (a+1)\lambda_2 & \lambda_1 & \lambda_1 + (a+1)\lambda_2 \end{pmatrix}.$$

By noting that the sign configuration of $\tilde{T}$ is

$$\begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix},$$

it follows that all three absolute column sums ($\sum_{i=1}^{3}|t_{ij}|$, $j = 1, 2, 3$) are equal to λ_1.

Finally note that, if $\lambda_1\lambda_2 \neq 0$, then $\max\{\lambda_1, |\lambda_2|\} < (\lambda_1 + |\lambda_2| + 2\sqrt{\lambda_1^2 - |\lambda_1\lambda_2| + \lambda_2^2})/3$. ■

Now, denote the pair (v_1, v_2) by $\vec{v}$, and let $V = [\vec{v}] := [v_1, v_2] \subset L^1[-\pi/2, \pi/2]$. First note that, via an isometry (replace v_1 by $|v_1|$ and v_2 by $v_2 \operatorname{sgn} v_1$ where $v_1 \neq 0$), we can assume that $0 \leqslant v_1$. Secondly, by rearranging the values of $\vec{v}$ according to the angle of inclination of the radius vectors, we can conclude that V is isometric to and can therefore be replaced by $V = [\vec{v}]$, where $\vec{v}(\theta) = (\cos\theta, \sin\theta)R(\theta)$ on $[-\pi/2, \pi/2]$, and R is a finite measure on $[-\pi/2, \pi/2]$. But, before the rearrangement, we can arbitrarily closely approximate $\vec{v}$ in the norm to insure that, for the approximating space, $R(\theta)$ is absolutely continuous with respect to Lebesgue measure and positive, and therefore conclude that it suffices to consider $V = [v_1, v_2] \subset L^1[-\pi/2, \pi/2]$, where $\vec{v}(\theta) = (\cos\theta, \sin\theta)r(\theta)$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, and $0 < r \in L^1[-\pi/2, \pi/2]$.

Furthermore, V being an unconditional space implies that we can take for our fixed unconditional basis $(v_1, v_2) = (\cos\theta, \sin\theta)r(\theta)$, where v_1 is even and v_2 is odd on $[-\pi/2, \pi/2]$, which is seen as follows:

$$\int_{-\pi/2}^{\pi/2} |\cos\theta + \alpha\sin\theta| r(\theta)\, d\theta = \int_{-\pi/2}^{\pi/2} |\cos\theta - \alpha\sin\theta| r(\theta)\, d\theta \qquad \forall\alpha > 0$$

if and only if [define $r_*(\theta) := \cos\theta\, r(\theta)$]

$$\int_{-\pi/2}^{\pi/2} (|1 + \alpha\tan\theta| - |1 - \alpha\tan\theta|) r_*(\theta)\, d\theta = 0 \qquad \forall\alpha > 0$$

if and only if

$$\int_{-\pi/2}^{\pi/2} \operatorname{sgn}\theta \min\{1, \alpha|\tan\theta|\} r_*(\theta)\, d\theta = 0 \qquad \forall\alpha > 0$$

if and only if

$$\int_{-\tan^{-1}(\alpha-\epsilon)^{-1}}^{\tan^{-1}(\alpha-\epsilon)^{-1}} \operatorname{sgn}\theta \frac{\min\{1, \alpha|\tan\theta|\} - \min\{1, (\alpha-\epsilon)|\tan\theta|\}}{\epsilon} r_*(\theta)\, d\theta = 0 \qquad \forall 0 < \alpha > \epsilon > 0,$$

which implies (on letting $\epsilon \to 0$)

$$\int_{-\psi}^{\psi} \tan\theta\, r_*(\theta)\, d\theta = 0 \qquad \forall\psi = \tan^{-1}\frac{1}{\alpha} \in \left(0, \frac{\pi}{2}\right),$$

which implies (on differentiating w.r.t. ψ)

$$\tan\psi\,[r_*(\psi) - r_*(-\psi)] = 0 \qquad \text{a.e.}(\psi).$$

Thus, we can take

$$\vec{v}(\theta) = (\cos\theta, \sin\theta) r(\theta), \quad r(-\theta) = r(\theta), \qquad -\pi/2 \leqslant \theta \leqslant \pi/2. \tag{1}$$

Hence v_1 is even and v_2 is odd on $[-\pi/2, \pi/2]$, which is what we wanted to show.

Now for arbitrary $\theta' \in [0, \pi/2)$, let $(\theta_0, \theta_1, \theta_2) = (-\pi/2, -\theta', \theta')$, and let

$$(a_i, b_i) = \int_{\theta_{i-1}}^{\theta_i} \vec{v}(\theta)\, d\theta, \qquad i = 1,2,3,$$

where $\theta_3 = \theta_0 + \pi = \pi/2$. Note that by symmetry $a_3 = a_1$, $b_3 = -b_1$, and $b_2 = 0$. Let U be the linear transformation with matrix

$$B = \frac{1}{a_1 b_1}\begin{pmatrix} b_1 & 0 \\ 0 & -a_1 \end{pmatrix}$$

taking (a_1, b_1) into $(1, -1)$ and (a_3, b_3) into $(1, 1)$, and let a be given by $(2a, 0) = U(a_2, b_2)$, i.e., $a = a_2/2a_1$.

Next, let

$$\vec{u}(\theta)(c_i^U, d_i^U) = U^*(c_i, d_i) = \left(\frac{c_i}{a_1}, -\frac{d_i}{b_1}\right), \quad \theta_{i-1} \leqslant \theta < \theta_i, \qquad i = 1,2,3, \tag{2}$$

where U^* (with matrix $B^t = B$) is the adjoint of U, and $c_i = c_i(\theta')$ and $d_i = d_i(\theta')$, $i = 1, 2, 3$, are given by Lemma 1.

LEMMA 2. *For convenience of notation let* $(c_{-1}, d_{-1}) = (c_3, d_3)$, *assume* $\lambda_2 \neq 0$, *and set* $\sigma = \operatorname{sgn}\lambda_2$. *The system of equations*

$$F(\theta_0, \theta_1, \theta_2) = \begin{pmatrix} c_\sigma^U \cos\theta_2 + d_\sigma^U \sin\theta_2 \\ c_{-\sigma}^U \cos\theta_1 + d_{-\sigma}^U \sin\theta_1 \\ c_2^U \cos\theta_0 + d_2^U \sin\theta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

has a solution $(\theta_0, \theta_1, \theta_2) = (-\pi/2, -\theta', \theta')$ *for some* $\theta' \in [0, \pi/2)$.

Proof. Because of the symmetry, the system of equations (3) is equivalent to the following fixed-point problem:

$$\theta' = \tan^{-1}\left(-c_\sigma^U/d_\sigma^U\right),$$

where $(c_\sigma^U, d_\sigma^U) = (c_\sigma^U(\theta'), d_\sigma^U(\theta'))$. Since $a_1 > 0$ and $b_1 < 0$ and since $c_\sigma \geqslant 0$ and $d_\sigma < 0$,

$$G(\theta') = \tan^{-1}\left(-c_\sigma^U/d_\sigma^U\right) = \tan^{-1}(b_1 c_\sigma / a_1 d_\sigma)$$

maps $[0, \pi/2)$ continuously into $[0, \pi/2]$. Moreover, it is easily checked that G extends continuously to $[0, \pi/2]$ so that $G(\pi/2) = 0$. Thus G has a fixed point (not $\pi/2$) in $[0, \pi/2]$, and the equations (3) are solved. ∎

Proof of Theorem. Without loss, assume that $\lambda_1 \geqslant 0$ and $\lambda_2 \neq 0$ and that $\vec{v}$ is given by (1), in accordance with the above discussion describing any unconditional basis. Then, for any choice of θ' in $[0, \pi/2)$, let $\vec{u}$ be given by (2). It then follows immediately from Lemma 1 that if $\tilde{T} = \sum_{k=1}^{2} u_k \otimes v_k$, then $\tilde{T} v_k = \lambda_k v_k$, $k = 1, 2$. It also follows directly, by use of Lemmas 1 and 2, that, for $(\theta_0, \theta_1, \theta_2)$ given by a choice of θ' determined in Lemma 2, cases (b) and (c) of Lemma 1 do not apply and the Lebesgue function of $\tilde{T}$,

$$\begin{aligned} L(\psi) &= \int_{\theta_0}^{\theta_0+\pi} |\vec{u}(\psi)\cdot\vec{v}(\theta)|\,d\theta \qquad \left(-\frac{\pi}{2} \leqslant \psi < \frac{\pi}{2}\right) \\ &= \sum_{i=1}^{3} \left| \int_{\theta_{i-1}}^{\theta_i} \left(c_j^U, d_j^U\right)\cdot(\cos\theta, \sin\theta) r(\theta)\,d\theta \right| \\ &= \sum_{i=1}^{3} \left| \left(c_j^U, d_j^U\right)\cdot(a_i, b_i)\right| \\ &= \left|(c_j, d_j)\cdot(1, -1)\right| + \left|(c_j, d_j)\cdot(2a, 0)\right| + \left|(c_j, d_j)\cdot(1, 1)\right|, \\ &\qquad\qquad j = 1, 2, \text{ or } 3 \end{aligned}$$

$$\text{for } \psi \in [-\pi/2, -\theta'),\ [-\theta', \theta'),\ \text{or } [\theta', \pi/2), \quad \text{respectively},$$

is constant and equals $(1 + a)(a\lambda_1 + |\lambda_2|)/(1 + a + a^2)$ from case (a) of Lemma 1. [Case (b) of Lemma 1 cannot apply, since $0 = (c_\sigma^U, d_\sigma^U)\cdot(a_2, b_2)$

implies (since $c_\sigma^U = 0$) that $\sin\theta' = 0$, i.e., $\theta' = 0$, and hence $a = 0$, which contradicts $\lambda_1 + a(\lambda_1 \mp \lambda_2) < 0$. Case (c) of Lemma 1 cannot apply either, since $0 < (c_\sigma^U, d_\sigma^U)\cdot(a_3, b_3)$ implies that $\theta' > \tan^{-1}(b_3/a_3)$, which is impossible, since $(a_3, b_3) = \int_{\theta'}^{\pi/2}(\cos\theta, \sin\theta)r(\theta)\,d\theta$.]

Finally, the theory of [2] (Theorem 2 and Lemma 4) shows that $\tilde{T}$ is minimal if and only if, for some positive function ϕ, $\vec{v}/\phi$ is constant (a.e.) on $[\theta_{i-1}, \theta_i)$ for each $i = 1, 2, 3$, and from this we get the uniqueness conclusion of the theorem by use of Lemma 1. ∎

NOTE. If $T \neq I$, then the "extreme" $\tilde{T}$ $[\|\tilde{T}\| = (|\lambda_1| + |\lambda_2| + 2\sqrt{\lambda_1^2 - |\lambda_1\lambda_2| + \lambda_2^2})/3]$ is not an orthogonal (symmetric) matrix. Thus the results of this paper do not appear to be obtainable by extending the methods of [6].

REMARK. There are in general several essentially different solutions given by Lemma 2. For example, in the case $T = I$ and $r(\theta) = \sin^2\theta$ there are two essentially different solutions $\tilde{T}_1$ and $\tilde{T}_2$ with $\|\tilde{T}_1\| = 1.329642\ldots$ and $\|\tilde{T}_2\| = 1.327513\ldots$. In the (circle-projection) case $T = I$ and $r(\theta) \equiv 1$, the unique solution $\tilde{T}$ has norm $\frac{4}{3}$.

CONJECTURE 1. We conjecture that the value $\lambda'_A = (|\lambda_1| + |\lambda_2| + 2\sqrt{\lambda_1^2 - |\lambda_1\lambda_2| + \lambda_2^2})/3$ of the theorem is the "action constant" (see [3]) for two-dimensional real spaces corresponding to the action matrix $A = (a_{kl})_{2\times 2}$ with real eigenvalues λ_1, λ_2. That is, for V arbitrary (not necessarily unconditional) there exists $\tilde{T} = \sum_{k=1}^{2} u_k \otimes v_k$ into $V \subset X$ with $\langle v_k, u_l\rangle = a_{kl}$ such that

$$\|\tilde{T}\| \leqslant \lambda'_A,$$

and λ'_A is best possible.

CONJECTURE 2. The results of this paper extend to unconditional two-dimensional *complex* spaces.

REFERENCES

1 B. L. Chalmers and F. T. Metcalf, A simple formula showing L^1 is a maximal overspace for two-dimensional real spaces, *Ann. Polon. Math.* 56:303–309 (1992).

2 B. L. Chalmers and F. T. Metcalf, The determination of minimal projections and extensions in L^1, *Trans. Amer. Math. Soc.* 329:289–305 (1992).

3 B. L. Chalmers and K. Pan, Finite dimensional action constants, submitted for publication.
4 B. Grünbaum, Projection constants, *Trans. Amer. Math. Soc.* 95:451–465 (1960).
5 M. I. Kadec and M. G. Snobar, Some functionals over a compact Minkowski space, *Math. Notes* 10:694–696 (1971).
6 H. König and N. Tomczak-Jaegermann, Norms of minimal projections, *J. Funct. Anal.*, 119:253–280 (1994).
7 J. Lindenstrauss, On the extension of operators with a finite-dimensional range, *Illinois J. Math.* 8:488–499 (1964).
8 N. Tomczak-Jaegermann, *Banach-Mazur Distances and Finite-Dimensional Operator Ideals*, Wiley, New York, 1989.
9 D. Yost, L_1 contains every two-dimensional normed space, *Ann. Polon. Math.* 49:17–19 (1988).

Received 7 February 1994; final manuscript accepted 29 August 1994