JOURNAL OF APPROXIMATION THEORY **85**, 343–347 (1996)
ARTICLE NO. 0047

Another Note on Polynomial vs Rational Approximation

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Communicated by Peter B. Borwein

Received October 17, 1994; accepted June 6, 1995

Let E be a subspace of $C(X)$ and let $R(E)=g/h\colon g,\ h\in E; h>0\}$. We make a simple, yet intriguing observation: if zero is a best approximation to f from E, then zero is a best approximation to f from $R(E)$.

We also prove that if $\{E_n\}$ is dense in $C(X)$ then for almost all f (in the sense of category)

$$\limsup d(f, R(E_n))/d(f, E_n)=1.$$

That extends the results of P. Borwein and S. Zhou who proved it for the case when E_n is the space of algebraic or trigonometric polynomials of degree n. © 1996 Academic Press, Inc.

1. INTRODUCTION

Consider an arbitrary function $f\in C_{[-1,1]}$. Let $\mathscr{P}_n$ stand for the space of polynomials of degree n and let $\mathscr{R}_{n,n}$ stand for rational functions

$$\left\{\frac{g}{h}\colon g,\ h\in\mathscr{P}_n;\ h>0\right\}.$$

Let p_n^* be the best approximation to f from $\mathscr{P}_n$. Then zero is the unique best approximation from $\mathscr{R}_{n,n}$ to $f-p_n^*$.

Here is a short proof:

The function $f-p_n^*$ equioscillates i.e. there are points $\xi_1, ..., \xi_{n+2}\in[-1,1]$ such that $(f-p_n^*)(\xi_j)=\lambda(-1)^j\,\|f-p_n^*\|$ where $\lambda=\pm 1$ (say $\lambda=-1$). Now if $\|f-p_n^*-g/h\|\leqslant\|f-p_n^*\|$ then $g/h(\xi_j)\geqslant 0$ for j even and $g/h(\xi_j)\leqslant 0$ for j odd. Since h is strictly positive, the function $g\in\mathscr{P}_n$ should satisfy the same condition

$$g(\xi_j)\geqslant 0\quad\text{for } j \text{ even}\qquad\text{and}\qquad g(\xi_j)\leqslant 0\quad\text{for } j \text{ odd}.$$

0021-9045/96 $18.00

That forces g to have $n+1$ zeros and hence $g=0$. Examining this proof it is easy to conclude that it has nothing to do with the nature of h, as long as it is strictly positive. The only property of g that we used is that g and p^* belong to the same Chebyshev subspace of $C_{[-1,1]}$.

It turns out that this statement (aside from uniqueness) holds true for rational functions where the numerator and denominator come from arbitrary subspaces of $C(X)$. This is the content of Theorem 2.1.

We then use this theorem to prove that for most of the functions (in the sense of category) in $C(X)$ the rate of best approximation and the rate of best rational approximation is the same. This is known for specific subspaces (cf [1], [3]). We prove it for arbitrary subspaces of $C(X)$.

2. The Best Rational Approximation

Let X be a compact Hausdorff space, let $C(X)$ be the space of real-valued continuous functions on X. If G and H are subspaces of $C(X)$ we use

$$R(G, H) := \{ g/h : g \in G,\ h \in H,\ h(x) > 0 \text{ for all } x \in X \}.$$

To avoid trivialities we will always assume that H contains a strictly positive function.

We will identifying the dual space $(C(X))^*$ with the space of regular Borel measures on X: $\mathscr{M}(X)$, and the same letter may mean a measure or a functional.

Finally if A is a subset of $C(X)$, and $f \in C(X)$

$$d(f, A) := \inf \|f-a\| : a \in A \}.$$

THEOREM 2.1. *Let $f \in C(X)$ and $G \subset C(X)$ be a subspace such that there exists $g \in G$ with $\|f-g^*\| = d(f, G)$. Then for every subspace $H \subset C(X)$*

$$d(f-g^*, R(G, H)) = \|f-g^*\|.$$

Hence zero is a best approximation to $f-g^*$ from $R(G, H)$. Moreover if $X=[a, b]$ and G is Chebyshev then zero is the unique best approximation from $R(G, H)$ to $f-g^*$.

Proof. Since g^* is the best approximation from G to f, hence there exists a functional $\mu \in \mathscr{M}(X)$ such that

$$\mu \perp G \text{ i.e. } \mu(g) = 0 \qquad \text{for all} \quad g \in G. \tag{1}$$

$$\|\mu\| = 1; \quad \mu(f-g^*) = \|f-g^*\| \ \ \|\mu\|. \tag{2}$$

We adopt the logic of [2] for this particular case.

Let $a > 0$ be such that $d(f - g^*, R(G, H)) < a$. Then there exists $g \in G$, $h \in H$, $h > 0$ such that for $\tilde{f} := g/h$ we have $\|(f - p^*) - \tilde{f}\| < a$.

On the other hand,

$$0 \neq \|f - p^*\|(|\mu|\,(h)) = ((f + p^*)\mu)(h) = \int (f - p^*)\, h\, d\mu$$

$$= \int ((f - p^*) - \tilde{f})\, h\, d\mu + \int \tilde{f} h\, d\mu = \int ((f - p^*) - \tilde{f})\, h\, d\mu + \int g d\mu$$

$$(\text{since } \tilde{f}h = g) = \int ((f - p^*) - \tilde{f})\, h\, d\mu$$

$$(\text{since } \mu \perp G) \leqslant \|(f - p^*) - \tilde{f}\| \int hd\,|\mu| \; (\text{since } h \text{ is positive}) < a\, |\mu|(h).$$

Thus $a > \|f - p^*\| = d(f, G)$. The "moreover" part of the Theorem was already proved in the Introduction. ▌

Remark. Theorem 2.1 is a very simple observation. Yet even in the simple case of $C(X) = C_{[-1,1]}$; $R(G, H) = \mathscr{R}_{n,n}$ it is somewhat surprising. First of all it provides a large class of functions for which the best rational approximation is easily computed.

Second, it shows how easy it is to spoil a function for rational approximation.

For instance $d(|x|, \mathscr{R}_{n,n}) \sim e^{\alpha\sqrt{n}}$. Add to $|x|$ a polynomial of degree n (namely $-p_n^*$) and the rate of approximation drops, and drops significantly to $1/n$, since $d((|x| - p_n^*), \mathscr{R}_{n,n}) = \|\,|x| - p_n^*\| \sim 1/n$.

3. Rates of Approximation

We now use the Theorem 2.1 to extend a result of P. Borwein and S. Zhou (cf. [1], Theorem 1) from $\mathscr{R}_{n,n}$ to $R(G_n, H_n)$ for arbitrary G_n, $H_n \subset C(X)$.

THEOREM 3.1. *Let X be an infinite compact Hausdorff space. Let $G_n \subset C(X)$ be a sequence of finite-dimensional subspaces such that*

$$d(f, G_n) \to 0 \qquad \textit{for all} \quad f \in C(X).$$

Then for all finite-dimesnional subspaces $H_n \subset C(X)$, the set

$$A := \{f \in C(X)\colon \limsup [d(f, R(G_n, H_n))/d(f, G_n)] \geqslant 1\}$$

is the set of second category in $C(X)$.

Proof. We proceed as in [1]. Let $\tilde{G}_n := C(X)\backslash G_n$. Then $\tilde{G}_n$ is open and dense in $C(X)$. We consider sets

$A_n = \{f \in C(X)$: there exists $m(n) > n$ with

$$d(f, R(G_m, H_m))/d(f, G_m) > 1 - \frac{1}{n}; \; d(f, G_m) \neq 0\}.$$

Then $A = (\bigcap_{n=1}^{\infty} A_n) \cap (\bigcap_{n=1}^{\infty} \tilde{G}_n)$. It remains to prove that each set A_n is open and dense in $C(X)$. The proof that A_n is open is exactly the same as in ([1], Theorem 1) and we refer to it for technical details. The idea, however is very simple. For a fixed $f \in A_n$ choose ε and δ so small that for all $\tilde{f}$ with $\|f - \tilde{f}\| < \delta$ we have

$$\tilde{f} \notin G_n; \quad |d(\tilde{f}, R(G_n, H_n)) - d(f, R(G_n, H_n))| < \varepsilon;$$
$$|d(f, G_n) - d(\tilde{f}, G_n)| < \varepsilon.$$

Since ε is "very small" the ratio $d(\tilde{f}, R(G_n H_n))/d(\tilde{f}, G_n)$ is still greater than $1 - 1/n$.

We now turn to the density of A_n. Let $f \in C(X)$. For arbitrary $\varepsilon > 0$ pick $\eta = \varepsilon/2$ and let $g_m \in G_m$ be such that $\|f - g_m\| < \eta$; $m > n$. Let

$$E_m := \operatorname{span}\{g_m \cdot h_m + g'_m \colon g_m, g'_m \in G_m, h_m \in H_m\}.$$

Since G_m and H_m are of finite dimension, so is E_m. Let F be an arbitrary function in $C(X)\backslash E_m$. Since E_m is finite-dimensional, there exists e^*_m which is a best approximation to F from E_m. Denote

$$F := (F - e^*_m)/\|F - e^*_m\|.$$

We now consider the function

$$\varphi(x) = g_m(x) + \eta F^*(x).$$

Observe that $\|f - \varphi\| < \eta + \eta = \varepsilon$. It remains to show that $\varphi \in A_n$. Indeed, since $G_m \subset E_m$ we have

$$\eta \geqslant d(\varphi, G_m) = d(\eta F^*, G_m) \geqslant d(\eta F^*, E_m) = \eta.$$

Therefore

$$d(\varphi, G_m) = \eta. \tag{3}$$

Now let e_m/h_m be an arbitrary element in $R(E_m, H_m)$. Then

$$\varphi - \frac{e_m}{h_m} = \eta F^* + g_m - \frac{e_m}{h_m} = \eta F^* + \frac{g_m h_m - e_m}{h_m} = \eta F - \frac{e'_m}{h_m}$$

where $e'_m = e_m - g_m h_m$ is an arbitrary element in E_m. Thus

$$d(\varphi, R(E_m, H_m)) = d(\eta F^*, R(E_m, H_m)) = \eta.$$

The last equality follows from the Theorem 2.1.

Since $G_m \subset E_m$ we have

$$d(\varphi, R(G_m, H_m)) \geqslant d(\varphi, R(E_m, H_m)) = \eta;$$

which together with (3) implies

$$d(\varphi, R(G_m, H_m))/d(\varphi, G_m) \geqslant \eta/\eta = 1 > 1 - \frac{1}{n}$$

and hence $\varphi \in A_n$. ▮

ACKNOWLEDGMENT

I am grateful to Professor Zhou for pointing out the reference [1] and [3] to me. A conversation with him sparked off my interest in this subject.

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