

# Interpolating Subspaces in $\mathbf{R}_n$

## Abstract

A  $k$ -interpolating subspace of  $C(\mathbf{R}_n)$  is a subspace  $F \subset C(\mathbf{R}_n)$  such that for every choice of distinct points  $t_1, \dots, t_k \in \mathbf{R}_n$  and every choice of scalars  $\alpha_1, \dots, \alpha_k \in \mathbf{R}$  there exists  $f \in F$  with  $f(t_j) = \alpha_j$ ,  $j = 1, \dots, k$ .

We prove that

$$\min\{\dim F : F \subset C(\mathbf{R}_n) \text{ is } k\text{-interpolating}\} = n + k - 1$$

for  $k = 2, 3$ .

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## 1 Introductory Remarks

We use  $C(\mathbf{R}_n)$  to denote the space of real-valued continuous functions on  $\mathbf{R}_n$ .

**Definition 1.** Let  $F$  be a subspace of  $C(\mathbf{R}_n)$ . We say that  $F$  is  $k$ -interpolating if for every choice of distinct points  $t_1, \dots, t_k \in \mathbf{R}_n$  and for every choice of scalars  $\alpha_1, \dots, \alpha_k \in \mathbf{R}$  there exists  $f \in F$  such that  $f(t_j) = \alpha_j$ ,  $j = 1, \dots, k$ .

**Definition 2.** An interpolating index  $i(n, k)$  is defined to be

$$i(n, k) = \min\{\dim F : F \subset C(\mathbf{R}_n); F \text{ is } k\text{-interpolating}\}.$$

A  $k$ -interpolating subspace  $F \subset C(\mathbf{R}_n)$  is called minimal if  $\dim F = i(n, k)$ .

A problem of determining the number  $i(n, k)$  and the minimal subspaces was brought to my attention by E.B. Saff.

In this paper we prove that  $i(n, k) = n + k - 1$  for  $k = 2, 3$ . We also exhibit the corresponding minimal subspaces. The Corrolary 1 came as somewhat of a surprise. The case  $k = 2$  is dealt with in Section 2 and Section 3 resolves the case  $k = 3$ . The rest of this section consists of the remarks of general character.

**Remark 1.** It is easy to see that the constants form a one-dimensional one-interpolating subspace of  $C(\mathbf{R}_n)$  for all  $n$ . Hence  $i(n, 1) = 1$  for all  $n$ .

By strictly dimensional considerations

$$i(n, k) \geq k \quad \text{for all } n.$$

It is also well-known and easy to see from the Vandermond determinant that the functions  $f_j(x) = x^j$ ,  $j = 0, \dots, k - 1$  span a  $k$ -interpolating subspace of  $C(\mathbf{R})$  and hence

$$i(1, k) = k, \quad \text{for all } k.$$

A converse to it was conjectured by S. Mazur and proved by J.C. Mairhuber (cf. [3]).

The Mairhuber theorem states that if  $i(n, k) = k$  for some  $k > 1$  then  $n = 1$ .

Thus the index  $i(n, k)$  characterizes 1-dimensional spaces. In this regard our results show that the indices  $i(n, 2)$  and  $i(n, 3)$  characterize the  $n$ -dimensional spaces.

**Remark 2.** F.R. Cohen and D. Handel (cf [1]) obtained a remarkable inequality:

$$2k - \eta(k) \leq i(2, k) \leq 2k - 1 \quad (1.1)$$

where  $\eta(k)$  is the number of ones in the binary representation of the integer  $k$ .

This result was recently rediscovered by V. A. Vasiliev (cf [4]).

Both papers are written as short communications and use very heavy tools of Algebraic Topology. I am not aware of any straight-forward proof of the left-hand side of (1.1) even in the case when the interpolating subspace consists of polynomials.

I could not even find a direct proof that any *five-dimensional* subspace of the spaces of polynomials of degree 3 is not a 4-interpolating subspace of  $C(\mathbf{R}_2)$ .

Here is a simple proof of the right-hand side:

Consider points  $z_j = (x_j, y_j) \in \mathbf{R}_2$  as points  $z_j = x_j + iy_j \in \mathbf{C}$ ,  $j = 1, \dots, k$ . It is well known that for every choice of real numbers  $\alpha_1, \dots, \alpha_k$  there exist complex numbers  $c_n = a_n + ib_n$ ,  $n = 0, \dots, k-1$  such that the function

$$f(z) = f(x, y) = \sum_{n=0}^{k-1} c_n z^n = \sum (a_n + ib_n)(x + iy)^n$$

interpolates  $\alpha_1, \dots, \alpha_k$  at the points  $z_1, \dots, z_k$ . Since  $\alpha_j$ 's are real,  $\operatorname{Re} f(z)$  also interpolates  $\alpha_j$ .

$\operatorname{Re} f(z)$

$$= a_0 + \sum_{n=1}^{k-1} \left[ a_n \left( \sum_{2m \leq n} (-1)^m x^{2m} y^{n-2m} \right) + b_n \left( \sum_{2m+1 \leq n} (-1)^n x^{2m+1} y^{n-2m-1} \right) \right]$$

is a function in the  $2k - 1$  dimensional space spanned by

$$\left\{ 1, \sum_{2m \leq n} (-1)^m x^{2m} y^{n-2m}, \sum_{2m+1 \leq n} (-1)^m x^{2m+1} y^{n-2m-1} \right\}_{n=1}^{k-1}.$$

The inequality (1.1) is sharp if  $k = 2^m$ . Surprisingly for  $k = 3$  it is the lower bound that gives the accurate answer i.e.  $i(2, 3) = 4$ .

Indeed the functions  $f_1(x, y) \equiv 1$ ,  $f_2(x, y) = x$ ,  $f_3(x, y) = y$ ,  $f_4(x, y) = x^2 + y^2$  span 3-interpolating subspace of  $C(\mathbf{R}_2)$ . To prove this we need to show that for every three points  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$

$$\text{rank} \begin{bmatrix} 1, & x_1, & y_1, & x_1^2 + y_1^2 \\ 1, & x_2, & y_2, & x_2^2 + y_2^2 \\ 1, & x_3, & y_3, & x_3^2 + y_3^2 \end{bmatrix} = 3.$$

We postpone the formal verification until Section 3 and give a geometric proof instead. If the points are in general position (they lie on a circle and  $f_4$  does not contribute) then

$$\det \begin{bmatrix} 1, & x_1, & y_1 \\ 1, & x_2, & y_2 \\ 1, & x_3, & y_3 \end{bmatrix} \neq 0$$

and we are done. If not then we can assume (by linear change of variables) that all three points lie on the  $x$ -axis, and the problem is equivalent to interpolating three points in  $\mathbf{R}_1$  by  $[1, x, x^2]$ .

**Remark 3.** The condition that the functions  $f_j$  be continuous is essential. There exists a  $k$ -dimensional subspace of functions from  $\mathbf{R}_n$  into  $\mathbf{R}$  that interpolate arbitrary data at arbitrary  $k$  points.

Indeed since  $\mathbf{R}$  and  $\mathbf{R}_n$  has the same cardinality there exists a one-to-one, onto map  $\sigma : \mathbf{R}_n \rightarrow \mathbf{R}$ . Let  $F$  be a  $k$ -dimensional,  $k$ -interpolating subspace

of  $C(\mathbf{R})$ . Then  $\{f \circ \sigma; f \in F\}$  is a  $k$ -dimensional  $k$ -interpolating subspace on  $\mathbf{R}_n$ .

In view of the last remark it is not surprising that we employ topological methods in the proofs.

Finally I would like to express my gratitude to my colleagues E.B. Saff and J. Pedersen for many useful discussions. I am especially thankful to E. Clark who took an active interest in this research and enlightened me on a number of fine points in this paper.

## 2 $i(n, 2) = n + 1$

Let  $x = (x_1, \dots, x_n) \in \mathbf{R}_n$  we use  $\pi_k$  to denote the function in  $C(\mathbf{R}_n)$  defined by  $\pi_j(x) := x_j$ .

**Theorem 1.**  $i(n, 2) = n + 1$ . If  $F = \text{span}[1, \pi_1, \dots, \pi_n] \subset C(\mathbf{R}_n)$  then  $F$  is a minimal 2-interpolating space.

**Proof.** We first show that  $F$  is 2-interpolating. Indeed if

$$\text{rank} \begin{bmatrix} 1, & \pi_1(x), & \pi_2(x), & \dots, & \pi_n(x) \\ 1, & \pi_1(y), & \pi_2(y), & \dots, & \pi_n(y) \end{bmatrix} < 2$$

for some  $x, y \in \mathbf{R}_n$ , then each determinant

$$\det \begin{bmatrix} 1 & \pi_j(x) \\ 1 & \pi_j(y) \end{bmatrix} = 0, \quad j = 1, \dots, n.$$

Hence  $\pi_j(x) = \pi_j(y)$  for all  $j$  and  $x = y$ .

Next we need to prove that  $i(n, 2) > n$ . Assume that  $g_0, g_1, \dots, g_{n-1} \in C(\mathbf{R}_n)$  are arbitrary. We want to show that there exist two points  $x, y \in \mathbf{R}_n$

such that

$$\text{rank} \begin{bmatrix} g_0(x), & g_1(x), & \dots, & g_{n-1}(x) \\ g_0(y), & g_1(y), & \dots, & g_{n-1}(y) \end{bmatrix} < 2. \quad (2.1)$$

Without loss of generality we can assume that  $g_0$  is not identically zero. Let  $\mathcal{U} = \{x \in \mathbf{R}_n : f_0(x) \neq 0\}$ . Since  $\mathcal{U}$  is a non-empty open set in  $\mathbf{R}_n$  it contains a closed ball. Let  $S^{n-1}$  be the boundary of this ball.

Let  $f_j := g_j/g_0 \in C(\mathcal{U})$ . Then (2.1) is equivalent to

$$\text{rank} \begin{bmatrix} 1, & f_1(x), & \dots, & f_{n-1}(x) \\ 1, & f_1(y), & \dots, & f_{n-1}(y) \end{bmatrix} < 2. \quad (2.2)$$

Consider a map  $\phi : S^{n-1} \rightarrow \mathbf{R}_{n-1}$  defined by

$$\phi(x) = (f_1(x), \dots, f_{n-1}(x)).$$

Then by the Borsuk Theorem (cf. [2]) there exists a pair at distinct points  $x, y \in S^{n-1}$  such that  $\phi(x) = \phi(y)$ . Hence

$$\det \begin{bmatrix} 1 & f_j(x) \\ 1 & f_j(y) \end{bmatrix} = 0 \quad \text{for all } j = 1, \dots, n-1,$$

as well as

$$\det \begin{bmatrix} f_j(x), & f_m(x) \\ f_j(y), & f_m(y) \end{bmatrix} = 0 \quad \text{for all } j, m = 1, \dots, n-1. \quad \blacksquare$$

**Corollary 1.** *Let  $X$  be a linear topological space. Let  $F$  be a finite-dimensional  $k$ -interpolating subspace of  $C(X)$ , for some  $k \geq 2$ . Then  $X$  is finite-dimensional.*

### 3 $i(n, 3) = n + 2$

In this section we prove

**Theorem 2.**  $i(n, 3) = n + 2$ . If  $F = \text{span} \{1, \pi_1, \dots, \pi_n, \sum_{j=1}^n \pi_j^2\}$  then  $F$  is a minimal 3-interpolating subspace in  $C(\mathbf{R}_n)$ .

**Proof.** A) We first show that  $i(n, 3) > n + 1$ . As before we assume that  $f_0, \dots, f_n$  are arbitrary functions in  $C(\mathbf{R}_n)$  with  $f_0$  not identically zero. Let  $\mathcal{U} = \{x \in \mathbf{R}_n : f_0(x) \neq 0\}$  and let  $g_j = f_j/f_0 \in C(\mathcal{U})$ . We want to prove that there exist three points  $x, y, z \in \mathcal{U}$  so that

$$\text{rank} \begin{bmatrix} g_0(x), & g_1(x), & \dots, & g_n(x) \\ g_0(y), & g_1(y), & \dots, & g_n(y) \\ g_0(z), & g_1(z), & \dots, & g_n(z) \end{bmatrix} = \text{rank} \begin{bmatrix} 1, & f_1(x), & \dots, & f_n(x) \\ 1, & f_1(y), & \dots, & f_n(y) \\ 1, & f_1(z), & \dots, & f_n(z) \end{bmatrix} < 3. \quad (3.1)$$

If all the vectors  $(f_j(x), f_j(y), f_j(z))$  are constant vectors then the inequality (3.1) is trivial. Hence we may assume that there exist  $x_0, z \in \mathcal{U}$  such that  $f_1(x_0) \neq f_1(z)$ . Since  $f_1$  is a continuous function on the non-empty open set  $\mathcal{U}$  there exists a closed ball  $B$  centered at  $x_0$  such that  $f_1(x) \neq f_1(z)$  for all  $x \in B$ . Let  $S^{n-1}$  be the boundary of this ball, thus a closed sphere in  $\mathbf{R}_n$ . It is convenient now to introduce functions

$$D_j(x, y) = f_j(x) - f_j(y), \quad j = 1, \dots, n.$$

It is easy to verify by direct calculation that

$$\det \begin{bmatrix} 1, & f_1(x), & f_j(x) \\ 1, & f_1(y), & f_j(y) \\ 1, & f_1(z), & f_j(z) \end{bmatrix} = D_1(x, z)D_j(y, z) - D_1(y, z)D_j(x, z). \quad (3.2)$$

Since  $D_1(x, z) \neq 0$  for all  $x \in S^{n-1}$  we consider a continuous function  $\phi : S^{n-1} \rightarrow \mathbf{R}_{n-1}$  given by

$$\phi(x) = (D_2(x, z)/D_1(x, z), \dots, D_n(x, z)/D_1(x, z)).$$

By the Borsuk Theorem there exist  $x, y \in S^{n-1}$  such that  $x \neq y$  and  $\phi(x) = \phi(y)$ . By (3.2) it follows that

$$\det \begin{bmatrix} 1, & f_1(x), & f_j(x) \\ 1, & f_1(y), & f_j(y) \\ 1, & f_1(z), & f_j(z) \end{bmatrix} = 0 \quad \text{for all } j = 2, \dots, n.$$

By our assumption the vectors  $(1, 1, 1)$  and  $(f_1(x), f_1(y), f_1(z))$  are linearly independent, whence the vector  $(f_j(x), f_j(y), f_j(z))$  lie on the plane spanned by  $(1, 1, 1)$  and  $(f_1(x), f_1(y), f_1(z))$ .

Using the ‘‘Piogen Hall Principle’’ we conclude that all the column vectors in the matrix (3.1) lie on the same plane and hence the rank of this matrix is at most 2.

B) We now need to prove that the space  $F$  defined in the statement of the Theorem is 3-interpolating. To prove that we need to show that

$$\text{rank} \begin{bmatrix} 1, & \pi_1(x), & \dots, & \pi_n(x), & \sum_{j=1}^n \pi_j^2(x) \\ 1, & \pi_1(y), & \dots, & \pi_n(y), & \sum_{j=1}^n \pi_j^2(y) \\ 1, & \pi_1(z), & \dots, & \pi_n(z), & \sum_{j=1}^n \pi_j^2(z) \end{bmatrix} = 3.$$

for any choice of distinct  $x, y, z \in \mathbf{R}_n$ . Since  $x, y, z$  are distinct, not each of the vectors  $(\pi_j(x), \pi_j(y), \pi_j(z))$  is a constant vector. Assume without loss of generality that  $(\pi_1(x), \pi_1(y), \pi_1(z))$  is not a constant. Consider the determinants

$$\det \begin{bmatrix} 1, & \pi_1(x), & \pi_j(x) \\ 1, & \pi_1(y), & \pi_j(y) \\ 1, & \pi_1(z), & \pi_j(z) \end{bmatrix}, \quad j = 2, \dots, n.$$

If at least one of these determinants is non-zero, we are done.

If each one of them is zero, it means that the points  $(\pi_1(x), \pi_j(x))$ ,  $(\pi_1(y), \pi_j(y))$ ,  $(\pi_1(z), \pi_j(z))$  lie in the same line. Hence there are constants  $\alpha_j, b_j$  such that

$$\pi_j(x) = \alpha_j \pi_1(x) + b_j; \quad \pi_j(y) = \alpha_j \pi_1(y) + b_j; \quad \pi_j(z) = \alpha_j \pi_1(z) + b_j. \quad (3.3)$$

Let  $\alpha^2 := 1 + \sum_{j=2}^n \alpha_j^2$ ;  $b^2 := \sum_{j=2}^n b_j^2$ ;  $\beta = 2 \sum_{j=2}^n \alpha_j b_j$ . In this case

$$\begin{aligned} & \det \begin{bmatrix} 1, & \pi_1(x), & \sum_{j=1}^n \pi_j^2(x) \\ 1, & \pi_1(y), & \sum_{j=1}^n \pi_j^2(y) \\ 1, & \pi_1(z), & \sum_{j=1}^n \pi_j^2(z) \end{bmatrix} \\ &= \det \begin{bmatrix} 1, & \pi_1(x), & \alpha^2 \pi_1^2(x) + \beta \pi_1(x) + b^2 \\ 1, & \pi_1(y), & \alpha^2 \pi_1^2(y) + \beta \pi_1(y) + b^2 \\ 1, & \pi_1(z), & \alpha^2 \pi_1^2(z) + \beta \pi_1(z) + b^2 \end{bmatrix} \\ &= \alpha^2 \det \begin{bmatrix} 1 & \pi_1(x) & \pi_1^2(x) \\ 1 & \pi_1(y) & \pi_1^2(y) \\ 1 & \pi_1(z) & \pi_1^2(z) \end{bmatrix}. \end{aligned}$$

To show that this determinant is non-zero we need to show that it does not contain two identical rows. Assume that  $\pi_1(x) = \pi_1(y) \neq \pi_1(z)$ . By (3.2) it implies

$$\pi_j(x) = \pi_j(y) \quad \text{for all } j = 1, \dots, n$$

and  $x = y$ . ■

## References

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