Pergamon

Computers Math. Applic. Vol. 30, No. 3-6, pp. 191–196, 1995

0898-1221/95 \$9.50 + 0.00

0898-1221(95)00096-8

Interpolation of Individual Functions

B. SHEKHTMAN
Department of Mathematics
University of South Florida
Tampa, FL 33620-5700, U.S.A.

Abstract—We investigate convergence at interpolation projections for an arbitrary but fixed set at functions.

Keywords—Interpolation, Projections, Portion of an interval, Best approximation, Polynomial.

1. INTRODUCTION

A well-known Faber theorem asserts that given any sequence P_n of projections from $C([0,1])$ onto the space $\mathfrak{P}_n$ of polynomials of degree $n-1$, there exists a continuous function f such that $P_n f$ does not converge to f in the uniform norm.

In particular, let $\Delta_n : 0 \le t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)} \le 1$ be a sequence of partitions on the interval $[0,1]$. Let

$$L(\Delta_n) : C([0,1]) \longrightarrow \mathfrak{P}_n$$

be the usual Lagrange interpolation operators. Then there exists $f \in C([0,1])$ such that $L(\Delta_n) f \not\to f$.

On the other hand, if the function $f \in C([0,1])$ is given, then one can always find a sequence $\Delta_n \subset [0,1]$ such that $L(\Delta_n) f \to f$. The familiar proof of it goes as follows: Let $b_n(f)$ be a polynomial of best approximation to f. Then $f - b_n(f)$ alternates sign at least $n+1$ times on $[0,1]$; hence (by the intermediate value theorem), there exists at least n points $t_1^{(n)}, \dots, t_n^{(n)} \in [0,1]$ such that $b_n(f)\left(t_j^{(n)}\right) = f\left(t_j^{(n)}\right)$. Pick these points to be Δ_n. Then $L(\Delta_n) f = b_n(f)$ and $L(\Delta_n) f \to f$.

What happens if at least one of the arguments in this proof does not apply? As far as I know, *nothing* is known. Here is a partial list of questions:

1. Given two functions $f, g \in C([0,1])$, does there exist a sequence $\Delta_n \subset [0,1]$ such that $L(\Delta_n) f \to f$ and $L(\Delta_n) g \to g$?
2. Let $X \subset [0,1]$ be a compact subset. Let $f \in C(X)$. Does there exist $\Delta_n \subset X$ such that $L(\Delta_n) f \to f$ in $C(X)$?
3. Let V_n be n-dimensional subspaces in $C[0,1]$ which are dense in $C[0,1]$. Given $f \in C[0,1]$, does there exist a sequence $\Delta_n \subset [0,1]$ such that $P(\Delta_n) f \to f$, where $P(\Delta_n)$ is an interpolation projection from $C([0,1])$ onto V_n?

The set of questions goes on (cf. [1]). The set of answers remains empty (cf. $\emptyset$). In this paper, I describe some partial results relating to questions 1, 2, and 3 above. I also hope to stimulate an interest to these problems.

2. SOME POSITIVE RESULTS

Most of the results in this section are based on the variation of the Fekete points.

PROPOSITION 1. *Let $k \le n$ be arbitrary integers. Let $X \subset [0,1]$ be a compact subset with more than n elements. Let $x_1, \dots, x_{n-k} \in X$ be $(n-k)$ distinct points. Then there exist distinct points $x^*_{n-k+1}, \dots x^*_n$ such that for all m satisfying $n-k+1 \le m \le n$ we have*

$$\left| \frac{\prod_{j=1}^{n-k} (x - x_j) \cdot \prod_{\substack{j=n-k+1 \\ j \neq m}}^{n} (x - x^*_j)}{\prod_{j=1}^{n-k} (x^*_m - x_j) \cdot \prod_{\substack{j=n-k+1 \\ j \neq m}}^{n} (x^*_m - x_j)} \right| \le 1 \tag{2.1}$$

for all $x \in X$.

PROOF. Consider a function

$$\varphi(x_{n-k+1}, \dots, x_n) = \left| \det \left[x_j^{m-1} \right]_{j.m=1}^{n} \right| \tag{2.2}$$

as a function from $X^k \to \mathbb{R}_+$. This is a continuous function from a compact set into $\mathbb{R}$, and hence, it attains its maximum. Let $x^*_{n-k+1}, \dots, x^*_n$ be the points where the maximum is attained; i.e.,

$$\varphi\left(x^*_{n-k+1}, \dots, x^*_n\right) \ge \varphi(x_{n-k+1}, \dots, x_n) \tag{2.3}$$

for all $x_{n-k+1}, \dots x_n \in X$. In particular, it implies that the points $x_1, \dots, x_{n-k}$, $x^*_{n-k+1}, \dots x^*_n$ are all distinct. Observe that the function in (2.1) is the unique polynomial which is equal to zero for $x = x_j$, $1 \le j \le n-k$ and for $x = x^*_j$; $n-k+1 \le j \le n$; $j \ne m$ and is equal to one for $x = x^*_m$. The function

$$\ell_m(x) := \frac{\varphi\left(x^*_{n-k+1}, \dots, x^*_{m-1}, x, x^*_{m+1}, \dots, x^*_n\right)}{\varphi\left(x^*_{n-k+1}, \dots, x^*_{m-1}, x^*_m, x^*_{m+1}, \dots, x^*_n\right)}$$

is also a polynomial of x that has the same properties, since the determinate is zero if two columns are equal. By (2.2), we conclude that $|\ell_m(x)| \le 1$. ∎

As a simple corollary, we obtain the following proposition.

PROPOSITION 2. *Let X be a compact subset of $[0,1]$. Let $x_1, \dots, x_{n-k}$ be distinct points in X. Then there exist $\Delta_n \subset X$ with $x_1, \dots, x_{n-k} \in \Delta_n$ and with the following property:*

Let $f \in C(X)$ and let $p \in \mathfrak{P}_n$ interpolate f at the points $x_1, \dots, x_{n-k}$. Then

$$\|f - L(\Delta_n) f\|_X \le (k+1) \|f - p\|_X .$$

PROOF. Pick $x^*_{n-k+1}, \dots, x^*_n$ as in Proposition 1. To simplify the notation, we rename x^*_j as x_j for $n-k+1 \le j \le n$. Let Δ_n consist of points $x_1, \dots, x_n$. By Proposition 1, the Lagrange fundamental polynomials

$$\ell_m(x) = \prod_{\substack{j=1 \\ j \neq m}}^{n} (x - x_j) \Bigg/ \prod_{\substack{j=1 \\ j \neq m}}^{n} (x_m - x_j)$$

are bounded by 1 on X.

Since $(f-p)(x_j) = 0$ for $j = 1, \dots, n-k$, we have

$$L(\Delta_n)(f-p) = \sum_{m=n-k+1}^{n} (f-p)(x_m)\, \ell_m(x).$$

Using the fact that $L(\Delta_n)p = p$, we obtain

$$f - L(\Delta_n)\, f = (f-p) - \sum_{m=n-k+1}^{n} (f-p)(x_m)\, \ell_m(x)$$

and by triangle inequality

$$\|f - L(\Delta_n)\, f\| \le \|f-p\| + \sum_{m=n-k+1}^{n} \|f-p\|\, \|\ell_m(x)\|.$$

Since $\|\ell_m(x)\| \le 1$ we get the desired result. ∎

COROLLARY 3. *Let g, $f \in C([0,1])$ be such that*

$$\frac{f}{g} = \frac{p}{q} \in C([0,1]), \qquad p, q \in \mathfrak{P}_N.$$

Then there exists a sequence $\Delta_n \subset [0,1]$ such that

$$L(\Delta_n) f \longrightarrow f \quad \text{and} \quad L(\Delta_n) g \longrightarrow g.$$

PROOF. Let $h := f/p = g/q \in C([0,1])$. Then

$$f = h \cdot p, \qquad g = h \cdot q.$$

For $n > N$, pick Δ_{n-N} such that $L(\Delta_{n-N})h \to h$. Define

$$P_n := p \cdot (L(\Delta_{n-N})h); \qquad Q_n := q \cdot (L(\Delta_{n-N})h).$$

Then P_n and Q_n are polynomials of degree n that interpolate f and g at $n-N$ points. By Proposition 2, there exist $\Delta_n \supset \Delta_{n-N}$ so that

$$\|f - L(\Delta_n) f\| \le (N+1)\, \|p\|\, \|h - L(\Delta_{n-N})h\| \longrightarrow 0 \qquad \text{as } n \to \infty;$$

$$\|g - L(\Delta_n) g\| \le (N+1)\, \|q\|\, \|h - L(\Delta_{n-N})h\| \longrightarrow 0 \qquad \text{as } n \to \infty.$$ ∎

COROLLARY 4. *Let $X = \bigcup_{m=1}^{k} I_m$ be a finite union of closed disjoint intervals $I_1 < I_2 < \cdots < I_k$. Let $f \in C(X)$. Then there exists $\Delta_n \subset X$ such that*

$$\|f - L(\Delta_n) f\|_X \longrightarrow 0 \qquad \text{as } n \to \infty.$$

PROOF. Let $b_n(f) \in \mathfrak{P}_n$ be the best approximation polynomial to f on X. The standard proof shows that there are points $\xi_1 < \xi_2 < \cdots < \xi_{n+1}$ in X such that

$$\|(f - b_n(f))(\xi_j)\| = \lambda(-1)^j\, \|f - b_n(f)\| : j = 1, \dots, n+1$$

where $\lambda = \pm 1$.

Since all the points ξ_j are confined to $\bigcup_{m=1}^{k} I_m$, there have to be at least $n-k+1$ alternation of the sign of $f - b_n(f)$ that occur inside the intervals that compose X. Hence, there are points $x_1, \dots, x_{n-k+1} \in X$ such that

$$(f - b_n(f))(x_j) = 0, \qquad \text{for } j = 1, \dots, n-k+1.$$

Hence, $b_n(f)$ is a polynomial of degree n that interpolates f at $n-(k+1)$ points in X. By Proposition 2, there exist $\Delta_n \subset X$ such that

$$\|f - L(\Delta_n) f\|_X \le (k)\, \|f - b_n(f)\|_X \longrightarrow 0, \qquad \text{as } n \to \infty.$$ ∎

REMARK 5. Proposition 2 allows us to find "good" interpolation points $\Delta_n \subset [0,1]$ without knowing the best approximation $b_n(f)$ to f.

Indeed, let p_n be an arbitrary approximation to f (for instance Bernstein polynomials) such that $\|f - p_n\| \to 0$. Consider a function $\widetilde{f} := (f - p_n)/\|f - p_n\|$. Let T_n be the Chebyshev polynomial of degree $n-1$. Then the function $\widetilde{f} - 2T_n$ has at least $(n-1)$ zeroes on $[0,1]$. Hence, $2T_n$ is a polynomial in $\mathfrak{P}_n$ that interpolates $\widetilde{f}$ at $(n-1)$ points. We apply Proposition 2 to obtain Δ_n such that $\left\|\widetilde{f} - L(\Delta_n)\,\widetilde{f}\right\| \le 2\left\|\widetilde{f} - 2T_n\right\| \le 6$. Thus

$$\frac{\|f - L(\Delta_n) f\|}{\|f - p_n\|} \le 6$$

and $\|f - L(\Delta_n)\,f\| \le 6\,\|f - p_n\|$. This remark also shows that it is possible to obtain $\Delta_n \subset [0,1]$ such that $L(\Delta_n) f \to f$ and all the interpolation points, except possibly one, lie between the consecutive extremas of the n^{th} degree Chebyshev polynomials.

3. SOME NEGATIVE RESULTS

It is easy to see that for every pair of functions $f, g \in C([0,1])$ there exist projections P_n from $C([0,1])$ onto $\mathfrak{P}_n$ such that

$$P_n f \longrightarrow f \quad \text{and} \quad P_n g \longrightarrow g.$$

In fact, one can construct such projections to be supported on $n+2$ points; i.e., the value $P_n f$ depends only on the value of f at $n+2$ points. Here is an outline of the proof (cf. [1]): Consider a Banach space $H_{n+2} := [f,g] \oplus \mathfrak{P}_n$. Then $\mathfrak{P}_n$ is an n-dimensional subspace of an $(n+2)$-dimensional Banach space H_{n+2}. Hence, the codim $\mathfrak{P}_n = 2$ and by the general estimates of projectional constants (cf. [2]) there exist projections $P_n : H_{n+2} \to \mathfrak{P}_n$ with $\|P_n\| \le 1 + \sqrt{2}$. These projections P_n can be represented as

$$P_n f = \sum_{j=0}^{n-1} \mu_j(f)\, x^j,$$

where μ_j are some functionals on H_{n+2}. Since H_{n+2} is an $(n+2)$-dimensional subspace of $C([0,1])$ there exist points $t_1, \dots, t_{n+2} \in [0,1]$ such that each μ_j can be written as $\sum_{k=1}^{n+2} a_{jk}\delta_{t_k}$ in the sense that for every j there are numbers $a_{j1}, \dots, a_{jn}$ such that

$$\mu_j(h) = \sum_{k=1}^{n+2} a_{jk} h(t_k)$$

for every $h \in H_{n+2}$. Define projections $P_n : C([0,1])$ onto $\mathfrak{P}_n$ by

$$P_n h = \sum_{j=0}^{n-1} \left(\sum_{k=1}^{n+2} a_{jk} h(t_k) \right) t^j \in \mathfrak{P}_n.$$

These are the desired projections.

My hope was (and still is) to use the existence of such projections and to be able to remove two points out of the support to obtain an interpolation projection. The examples of this section will show that it is not possible in general (without using the connectedness of $[0,1]$ or some special properties of polynomials $\mathfrak{P}_n$). In particular it is, in general, impossible to remove *one point* from the support of the projection without drastically altering its norm.

PROPOSITION 6. *For every $n \geq 2$ there exists a compact Hausdorff space X_n and an $(n-1)$-dimensional subspace $E_n \subset C(X_n)$ such that:*

(a) *There exists a projection $P_n : C(X_n)$ onto E_n with*

$$\|P_n\| \leq 2;$$

(b) *There exists a function $f \in C(X)$ such that for every interpolating projection Q_n from $C(X_n) \to E_n$*

$$\|Q_n f\| \geq n-3.$$

PROOF. Let X_n be a set consisting of n points; i.e., $C(X) = \ell_\infty^{(n)}$. Let $e_1, \ldots, e_n$ be the canonical vector basis in $\ell_\infty^{(n)}$. Consider a subspace $E_n \subset \ell_\infty^{(n)}$ spanned by the vectors $v_j := e_1 + e_j$, $j = 2, \ldots, n$. Then $\dim E_n = n-1$ and codim $E_n = 1$. Consequently, (cf. [2]) there exists a projection $P_n : C(X_n)$ onto E_n with $\|P_n\| \leq 2$. Now let $\widetilde{e}_j$ be the canonical vector basis in $\ell_1^{(n)}$. Then the interpolating projection at the points $2, \ldots, n$ is

$$P_1 v := \sum_{j=2}^{n} \widetilde{e}_j(v)\, v_j$$

and $P_1\left(\sum_{j=1}^n e_j\right) = \sum v_j = (n-1)e_1 + \sum e_j$. Hence, $\left\|P_1\left(\sum_{j=1}^n e_j\right)\right\| = n-1$.

Let P_j be an interpolating projection at the points $\{1, \ldots, j-1, j+1, \ldots, n\}$. We consider functions (vectors)

$$u_1^{(j)} = v_j = e_1 + e_j \in E;$$
$$u_k^{(j)} = v_k - v_j = e_k - e_j \in E, \qquad k = 2, \ldots, n, \ k \neq j.$$

Clearly, P_j is defined by

$$P_j v = \sum_{k \neq j} \widetilde{e}_k(v)\, u_k^{(j)} = \sum_{k \neq j} \widetilde{e}_k(v)\, e_k + \widetilde{e}_1(v)\, e_j - \sum_{\substack{m \geq 2 \\ m \neq j}} \widetilde{e}_m(v)\, e_j.$$

Then

$$P_j\left(\sum_{k=1}^n e_k\right) = \sum_{k \neq j} e_k + e_j - \sum_{\substack{m \geq 2 \\ m \neq j}} e_j = \sum_{k \neq j} e_k - (n-3)\, e_j.$$

Hence, $\|P_j\left(\sum_{k=1}^n e_k\right)\| = n-3$. ∎

REMARKS.

1. The space E_n constructed in the proposition is Chebyshev in the sense that the dimension of the space E_n restricted to any $(n-1)$ points is equal to the $\dim E_n = n-1$.
2. The estimate $\|P\| \leq 2$ cannot be improved to $\|P\| \leq 1$ since it is known that if E has a contractive projection in $C(X)$ then E has an interpolation projection of norm 1.
3. The actual norms of P_j constructed in the proof of the proposition are

$$\|P_j\| = \left\|P_j\left(e_1 - \sum_{k=2}^n e_k\right)\right\| = n-1.$$

This estimate is the best possible. Using the "Fekete points" for any $(n-1)$-dimensional subspace $E \subset C(X)$, one can construct an interpolating projection $Q : C(X)$ onto E with $\|Q\| \leq (n-1)$.

COROLLARY 7. *There exists an infinite space X and a sequence of n-dimensional subspaces $E_n \subset C(X)$ such that:*

(a) $E_n \subset E_{n+1}$;

(b) *There exist projections $P_n : C(X) \to E_n$ with*

$$\|P_n\| \leq 2;$$

(c) *There exists an element $f \in C(X)$ such that*

$$\|Q_n f\| \geq n - 4$$

for any interpolating projections $Q_n : C(X) \to E_n$.

PROOF. Let X be the one-point compactification of $\mathbb{N}$, i.e., $C(X) = c$, the space of all convergent sequences. Let E_{n-1} be again spanned by $e_1 + e_j$, $j = 2, \dots, n$. Then clearly for the vector $f = (1, 1, \dots, 1, \dots) \in C$ we have the desired result. ∎

Of course, in this setting, the spaces E_n are no longer Chebyshev; hence, the choice of interpolating points is limited to the n out of the first $(n+1)$ points.

Another weakness of this example is that $\cup E_n$ is not dense in $C(X)$.

COROLLARY 8. *Let X be as in Corollary 7. Then there exists a sequence of n-dimensional subspaces $E_n \subset C(X)$ such that:*

(a) $\operatorname{dist}(f, E_n) \to 0$ *for every* $f \in C(X)$;

(b) *There exist projections $P_n : C(X) \to E_n$ with $\|P_n\| \leq 2$;*

(c) *There exists an element $f \in C(X)$ such that for every sequence of interpolating projections $Q_n : C(X) \to E_n$*

$$\|Q_n f\| \geq n - 4.$$

PROOF. Choose $E_n = \operatorname{span}[e_{n+1} + e_j]_{j=1}^n$ and choose $f = (1, 1, \dots, 1, \dots)$. ∎

REFERENCES

1. B. Shekhtman, Some simple open problems on interpolation of individual functions, In *Constructive Theory of Functions*, pp. 259–268, Sofia, (1992).
2. N. Tomczak-Jaegermann, *Banach-Mazur Distances and Finite-Dimensional Operator Ideals*, Pitman Monographs.