COVERING BY COMPLEMENTS OF SUBSPACES

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June 25, 1994

Abstract. Let V be an n-dimensional vector space over a field F. We attempt to determine the least positive integer $\gamma = \gamma(k, n, F)$ for which there exists a family $U_1, U_2, \dots, U_\gamma$ of k-dimensional subspaces of V such that for every $(n-k)$-dimensional subspace W of V there is an $i \in \{1, 2, \dots, \gamma\}$ satisfying $U_i \oplus W = V$. We find upper and lower bounds for $\gamma(k, n, F)$ in general. In a few special cases we find exact values.

1. Introduction.

Take $V = V(n, F)$ to be an n-dimensional vector space over the field F. As usual a subspace W of V is a *complement* of the subspace U of V if $V = U \oplus W$, i.e., if $U + W = V$ and $U \cap W = \{0\}$. We let $c(U)$ denote the set of all complements of U in V and we write $\begin{bmatrix} V \\ k \end{bmatrix}$ for the set of all k-subspaces ($=$ k-dimensional subspaces) of V. If $U \in \begin{bmatrix} V \\ k \end{bmatrix}$ then $c(U) \subseteq \begin{bmatrix} V \\ n-k \end{bmatrix}$. Define $\gamma(k, n, F)$ to be the least positive integer t such that there exists k-subspaces $U_1, U_2, \dots, U_t$ of V satisfying

$$c(U_1) \cup c(U_2) \cup \dots \cup c(U_t) = \begin{bmatrix} V \\ n-k \end{bmatrix}. \tag{1}$$

. If (1) holds we say that all $(n-k)$-subspaces of V are *covered* by the U_i's.

Let $G = G(k, n, F)$ be the graph whose vetex set is $\begin{bmatrix} V \\ n-k \end{bmatrix} \cup \begin{bmatrix} V \\ k \end{bmatrix}$. Say that vertices U and W are adjacent if $U \oplus W = V$. $\gamma(k, n, F)$ is closely relate to the vertex domination number of the graph G. Recall that a set of vertices is a *vertex dominating set* for a graph if every vertex is either in the set or adjacent to a vertex in the set. The *vertex domination number* $\sigma(G)$ of the graph G is the minimum cardinality of a vertex dominating set of G (see, e.g.,[4]). If $k \neq \frac{n}{2}$ then clearly the vertex domination number of $G(k, n, F)$ is at most $2\gamma(k, n, F)$ and if $k = \frac{n}{2}$ then it is at most $\gamma(k, n, F)$. See [5] for more properties of the graph $G(k, n, F)$.

It is known [1] or [8] that if F has finite cardinality q the number of k-subspaces of $V(n, F)$ is given by the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \dots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \dots (q - 1)}. \tag{2}$$

Key words and phrases. vector space, subspace, complement, domination number, graph.

It is also known that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$ is a monic polynomial in q of degree $k(n-k)$ with non-negative integer coefficients. On the other hand if F has q elements then a k-subspace U has $q^{k(n-k)}$ complements ([3], Lemma 9.3.2, p. 269). It follows that for large values of q the proportion of $(n-k)$-subspaces of V that are complements of U is close to 1. So one would expect to be able to find a small number of spaces $U_1, U_2, \ldots, U_t$ satisfying (1).

2. Upper and Lower Bounds on $\gamma(k, n)$

Throughout $V = V(n, F)$ and without loss of generality we take V to be the vector space F^n of n-tuples over F. We sometimes identify the k-subspaces of V with the $k \times n$ reduced row echelon matrices of rank k over F. This is justified by the fact that the reduced row echelon form of a matrix is uniquely determined by the row space of the matrix ([2], Exercise 6, page 322).

Proposition 1. $\gamma(k, n, F) \leq \binom{n}{k}$ *for every field* F.

Proof 1. Fix an ordered basis $e_1, e_2, \ldots, e_n$ for V. For each k-subset $I \subseteq [n] = \{1, 2, \ldots, n\}$ define $U_I = span\{e_i | i \in I\}$. This gives us a collection of $\binom{n}{k}$ k-subspaces of V. We claim that every $(n-k)$-subspace W of V has a complement of the form U_I for some I. Let A be the unique $(n-k) \times n$ reduced row echelon matrix whose row space is W. Let $j_1, j_2, \ldots, j_{n-k}$ be the indices of the columns of A that contain the leading 1's. Let $I = \{i_1, i_2, \ldots, i_k\} = [n] - \{j_1, j_2, \ldots, j_{n-k}\}$.Then the $k \times n$ reduced row echelon matrix B with row space U_I has 1's in columns with indices $i_1, i_2, \ldots, i_k$. To show that $V = U_I \oplus W$ it suffices to show that the rank of the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is n. This is clear since by permuting the rows of $\begin{bmatrix} A \\ B \end{bmatrix}$ it can be put in upper triangular form with 1's on the main diagonal. $\square$

Proof 2. We give another proof that every $(n-k)$-space W of V has one of the spaces U_I as a complement. In this case we use the exterior (or Grassman) algebra $E(V)$ of V. We require the following facts about $E(V)$ (see, e.g., Jacobson [7], chapter 7, section 7.2):

(i) If $v_1, v_2, \ldots, v_r \in V$ then

$$v_1 v_2 \ldots v_r \neq 0 \iff v_1, v_2, \ldots, v_t \quad \text{are linearly independent}$$

(ii) The subspace V^r spanned by all products $v_1 v_2 \ldots v_r$ where $v_1, v_2, \ldots, v_r \in V$, has the basis

$$\{e_I \mid i \subseteq [n]\}$$

where $e_I = e_{i_1} e_{i_2} \ldots e_{i_r}$ when $I = \{i_1, i_2, \ldots, i_r\}$ and $1 \leq i_1 < i_2 < \cdots < i_r \leq n$.

It follows that if U and W are subspaces of V with bases $u_1, \ldots, u_k$ and $w_1, \ldots, w_\ell$, respectively, then $U \cap W = 0 \iff u_1 u_2 \ldots u_k w_1 w_2 \ldots w_\ell \neq 0$

Now let W be any $(n-k)$-subspace of V and let $w_1, w_2, \ldots, w_{n-k}$ be a basis for W. Extend this to a basis

$$w_1, w_2, \ldots, w_{n-k}, v_1, \ldots, v_k$$

for V.

Again we want to show that $W \cap U_I = 0$ for some k-subset $I \subseteq [n]$. Suppose to the contrary that $W \cap U_I \neq 0$ for all k-subsets I. This implies that $w_1 w_2 \ldots w_{n-k} e_I = 0$ for all I. But by (ii) $v_1 v_2 \ldots v_k = \sum a_I e_I$ where the a_I are elements of F and hence

$$\begin{aligned} 0 \neq w_1 \ldots w_{n-k} v_1 \ldots v_k &= w_1 \ldots w_{n-k} (\sum a_I e_I) \\ &= \sum a_I w_1 \ldots w_{n-k} e_I \\ &= 0 \end{aligned}$$

This contradiction proves that $W \cap U_I = 0$ for at least one I as claimed. $\square$

Remarks. A third proof of Propostion 1 may be obtained from the recursive bound given below in Proposition 8.

The proof above shows that every $(n-k)$-space in an n-space V is a complement of some k-space in $\{U_I | I \in \binom{[n]}{k}\}$ Note however that no proper subset of this set of spaces has this property, for if U_I is omitted where $I = \{i_1, i_2, \ldots, i_k\}$ then the $(n-k)$-space U_J where $J = [n] - I$ intersects non-trivially all spaces U_K, $K \in \binom{[n]}{k} - \{I\}$.

Proposition 2. *For $0 \leq k \leq n$ we have $\gamma(k, n, F) = \gamma(k, n-k, F)$ for all fields F.*

Proof. We let $U^\perp$ denote the "orthogonal complement" of U relative to the standard inner product on $V = F^n$. Note that in general $U^\perp$ need not be a complement to U since $U \cap U^\perp$ need not be $\{0\}$. Nevertheless the mapping $U \mapsto U^\perp$ is a duality of the lattice of all subspace of V. That is, $(U+W)^\perp = U^\perp \cap W^\perp$, $(U \cap W)^\perp = U^\perp + W^\perp$, and $U^{\perp\perp} = U$. Also $dim\ U^\perp = n - dim\ U$ (see, e.g., [7], chapter 6). Hence if $U \oplus W = V$ then $U^\perp \oplus W^\perp = V$. To prove the lemma it suffices to prove that the following two equations are equivalent for k-subspaces U_i:

$$c(U_1) \cup c(U_2) \cup \cdots \cup c(U_t) = \begin{bmatrix} V \\ n-k \end{bmatrix}. \tag{3}$$

and

$$c(U_1^\perp) \cup c(U_2^\perp) \cup \cdots \cup c(U_t^\perp) = \begin{bmatrix} V \\ k \end{bmatrix}. \tag{4}$$

Assume (3) holds. Let W be any k-subspace of V so $W^\perp$ is an $(n-k)$ subspace of V. Hence from (3) $W^\perp \oplus U_i = V$ for some i and it follows that $W \oplus U_i^\perp = V$. Hence (4) holds. Similarly (4) implies (3). $\square$

Lemma 1. *For $t \geq 1$ and $1 \leq k \leq n$ the following are equivalent:*

(a) If $U_1, U_2, \ldots, U_t$ is any length t sequence of k-subspaces of $V = V(n, F)$ then

$$U_1 \cap U_2 \cap \cdots \cap U_t \neq 0.$$

(b) $t(n-k) < n$.

Proof. (a) $\Longrightarrow$ (b): Suppose (a) holds but $n \leq t(n-k)$. To get a contradiction it suffices to find a set of $t' \leq t$ k-subspaces whose intersection is 0. Write $n =$

$q(n-k)+r$, where $0 \le r \le n-k$. Then $q(n-k)+r \le t(n-k)$ so $q \le t$. Clearly if $r \neq 0$ then $q+1 \le t$. Let $t' = q$ if $r = 0$ and $t' = q+1$ if $r > 0$. In any case $t' \le t$. Now since $n = q(n-k)+r$ we can partition any basis B of V as

$$B = X_1 \cup \cdots \cup X_q \cup Y$$

where $|X_i| = n-k$ and $|Y| = r$. Let $W_i = span(X_i)$ and if $|Y| \neq 0$ Let W_{q+1} be any $(n-k)$-subspace contaning $span(Y)$. This clearly gives

$$V = W_1 + W_2 + \cdots + W_{t'}.$$

It follows that

$$0 = W_1^\perp \cap W_2^\perp \cap \cdots \cap W'_{t^\perp}$$

and since each $W_i^\perp$ has dimension k we have a contradiction to (a).

(b) $\Longrightarrow$ (a): Assume (b) and let $U_1, \ldots, U_t$ be any k-spaces in V. Let $W_i = U_i^\perp$, $1 \le i \le t$. Since each W_i has dimension $n-k$ from (b) we obtain $W_1 + \cdots + W_t \nsubseteq V$. It follows that $(W_1 + \cdots + W_t)^\perp \neq 0$. Then $U_1 \cap \cdots \cap U_t = W_1^\perp \cap \cdots \cap W_t^\perp = (W_1 + \cdots + W_t)^\perp \neq 0$ □

Proposition 3. *If $tk < n$ then for all fields F*

$$tk + 1 \le \gamma(k, n, F).$$

Proof. Since $\gamma(k,n) = \gamma(n-k,n)$ it suffices to prove that if $t(n-k) < n$ then $t(n-k)+1 \le \gamma(k,n)$. The result is obvious if $t = 0$ or $k = n$ so we can assume that $t(n-k) \neq 0$. Let

$$U_1, U_2, \ldots, U_{t(n-k)}$$

be k-subspaces of V. We partition these spaces into $n-k$ groups with t k-spaces in each group. By Lemma 1 the spaces in the i-th group contain in their intersection some non-zero vector x_i. Let W be any $(n-k)$-space that contains the vectors $x_1, x_2, \ldots x_{n-k}$. Clearly W will not be a complement of any of the U_i's and the lemma follows. □

Corollary 1. *If $k > 0$ then $n-k+1 \le \gamma(k,n,F)$ for all F.*

Proof. If $k > 0$ then $1(n-k) < n$ and Lemma 4 applies. □

Proposition 4. *If $k = n/2$ and $n \ge 2$ then $n-k+2 \le \gamma(k,n,F)$ for all F.*

Proof. Let $U_1, \ldots, U_{n-k+1}$ be any collection of k-subspaces in V. It suffices to show there is some $(n-k)$-space which non-trivially intersect all of the U_i's. There are two cases: I. $U_1 \cap U_2 \neq 0$ and II. $U_1 \cap U_2 = 0$.

In case I we can choose $0 \neq x_1 \in U_1 \cap U_2$, and non-zero $x_i \in U_i$ for $3 \le i \le n-k+1$. Then any $(n-k)$-space containing $x_1, \ldots, x_{n-k}$ meets each U_i non-trivially.

In case II $U_1 \oplus U_2 = V$ since $k = n/2$ and hence $U_3 \subseteq U_1 \oplus U_2$. Let $0 \neq x \in U_3$ then $x = x_1 + x_2$, $x_i \in U_i$. If $x_1 = 0$ then $0 \neq x \in U_2 \cap U_3$ and we can use case I. Similarly if $x_2 = 0$. So we can assume x_1 and x_2 are both non-zero. Now choose $x_i \in U_{i+1}$ for $3 \le i \le n-k$. Then any $(n-k)$-space containing $x_1, x_2, \ldots, x_{n-k}$ must intersect some U_i non-trivially. □

Proposition 5. *For all fields F*

$$\gamma(1, n, F) = \gamma(n-1, n, F) = n$$

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Proof. By Corollary 1, $n \leq \gamma(1, n, F)$. By Proposition 1 $\gamma(1, n, F) \leq \binom{n}{1} = n$. By Proposition 2 $\gamma(n-1, n, F) = \gamma(1, n, F)$. $\square$

3. The special case $\gamma(2, 4, F)$

By Propositions 1 and 3 we have only $4 \leq \gamma(2, 4, F) \leq 6$. The exact value of $\gamma(2, 4, F)$ turns out to depend on the field F.

A field F is *quadratically closed* if every quadratic polynomial over F has a root in F. In this section we establish the following result.

Proposition 6. *If F is quadratically closed then $\gamma(2, 4, F) = 5$; otherwise, $\gamma(2, 4, F) = 4$.*

Proof. We first show that $\gamma(2, 4, F) \leq 5$ for all F. Consider the five subspaces $U_1, \ldots, U_5$ defined by :

$$U_1 = \text{rowspace of } \; A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$U_2 = \text{rowspace of } \; A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U_3 = \text{rowspace of } \; A_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$U_4 = \text{rowspace of } \; A_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$U_5 = \text{rowspace of } \; A_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We want to show that every 2-subspace W of V is a complement of at least one of these 5 spaces. To do this we only need to show that if R is the 2×4 rank 2 reduced row echelon matrix whose row space is W then the determinant of the matrix $\begin{bmatrix} R \\ A_i \end{bmatrix}$ is non-zero for some $i \in \{1, \ldots, 5\}$.

Now every 2×4 rank 2 reduced row echelon matrix has one of the following 6 types where x, y, z, w represent arbitrary elements of F:

$$R_1 = \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 1 & x & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & x & y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_4 = \begin{bmatrix} 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

$$R_5 = \begin{bmatrix} 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_6 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now since $det \begin{bmatrix} R_1 \\ A_2 \end{bmatrix} = 1$ all spaces represented by reduced row echelon matrices of type R_1 are complements of U_2. For spaces of type R_2 note we have:

$$det \begin{bmatrix} R_2 \\ A_3 \end{bmatrix} = -z + x - y, \tag{3.1}$$

$$det \begin{bmatrix} R_2 \\ A_4 \end{bmatrix} = 1 - z + x - y. \tag{3.2}$$

Now if $-z + x - y \neq 0$ by (3.1) U_3 is a complement of a space of type R_2. If $-z + x - y = 0$ then by (3.2) U_4 is a complement of a space of type R_2.

For spaces of type R_3 note we have

$$det \begin{bmatrix} R_3 \\ A_1 \end{bmatrix} = y \tag{3.3}$$

$$det \begin{bmatrix} R_3 \\ A_3 \end{bmatrix} = -1 + y \tag{3.4}$$

Now if $y \neq 0$ by (3.3) U_1 is a complement of a space of type R_2. If $y = 0$ by (3.4) U_3 is a complement.

For spaces of type R_4 note we have

$$det \begin{bmatrix} R_4 \\ A_1 \end{bmatrix} = -y \tag{3.5}$$

$$det \begin{bmatrix} R_4 \\ A_3 \end{bmatrix} = 1 - y \tag{3.6}$$

Now if $y \neq 0$ by (3.5) U_1 is a complement of a space of type R_4. If $y = 0$ by (3.6) U_3 is a complement.

For spaces of type R_5 note we have $det \begin{bmatrix} R_5 \\ A_5 \end{bmatrix} = -1$ Hence U_5 is a commplement of any space of type R_5.

The single space of type R_6 is clearly a complement of the space U_1.

This completes the proof that every 2-subspace of V is a complement of at least one of the U_i's and hence $\gamma(2, 4, F) \leq 5$ for all fields. From Lemma 5 we have $4 \leq \gamma(3, 4, F)$. Assume now that there exist four 2-subspaces U_1, U_2, U_3, U_4 of V such that every 2-subspace in V is a complement of at least one of the four spaces.

Claim. *$U_i \cap U_j = 0$ for $1 \le i < j \le 4$.*

Proof of Claim: First suppose x is a non-zero vector in the intersection of three of the spaces. Take y to be a non-zero vector in the fourth space. Then the space $\langle x, y >\rangle$ generated by x and y intersects non-trivially all four spaces. So any three must intersect in 0. If say, U_1 and U_2 contain non-zero x and U_3 and U_4 contain non-zero y then again $\langle x, y\rangle$ meets all four spaces. Suppose U_1 and U_2 contain non-zero x and $U_3 \cap U_4 = 0$. Then $V = U_3 \oplus U_4$ and we can write $x = x_3 + x_4$ where $x_i \in U_i$. If $x_3 = 0$ then x lies in $U_1 \cap U_2 \cap U_4$ which we already shown to be impossible. Hence $x_3 \neq 0$. Similarly $x_4 \neq 0$. Then the space $\langle x_3, x_4\rangle$ meets all four spaces. We conclude that the spaces have pairwise intersections equal to 0.

We require the following result which is known ([6], Prop. 5.1) and easy to prove:

Lemma 2. *If U is a k-subspace of the direct sum $A \oplus B$ of the k-spaces A and B such that $U \cap A = 0$ and $U \cap B = 0$ then there is a non-singular linear mapping $T : A \to B$ such that $U = \{x + T(x) | x \in A\}$.* □

Now due to the above claim $U_1 \cap U_2 = 0$ so $V = U_1 \oplus U_2$. Hence we can choose a basis x_1, x_2, x_3, x_4 for V so that x_1 and x_2 span U_1 and x_3 and x_4 span U_2. By Lemma 2 there is a nonsingular linear map $T : U_1 \to U_2$ such that $\{x_1 + T(x_1), x_2 + T(x_2)\}$ is a basis for U_3 and $\{x_1, x_2, T(x_1), T(x_2)\}$ is a basis for V. Let φ be the automorphism of V that sends

$$\begin{aligned} x_1 &\mapsto e_1 = (1, 0, 0, 0) \\ x_2 &\mapsto e_2 = (0, 1, 0, 0) \\ T(x_1) &\mapsto e_3 = (0, 0, 1, 0) \\ T(x_2) &\mapsto e_4 = (0, 0, 0, 1). \end{aligned}$$

Now also by Lemma 2 there is a nonsingular linear map $S : U_1 \to U_2$ such that $x_1 + S(x_1)$ and $x_2 + S(x_2)$ for a basis for U_4. It follows that

$$\varphi(U_1) = \text{rowspace of } \; B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\varphi(U_2) = \text{rowspace of } \; B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\varphi(U_3) = \text{rowspace of } \; B_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\varphi(U_4) = \text{rowspace of } \; B_4 = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$$

Since the relevant properties are invariant under φ we can assume that $U_i = \varphi(U_i)$. Now we claim the variables a, b, c, d can be chosen so that every 2-subspace is a complement of one of these four spaces if and only if there is a quadratic polynomial that has no roots in F. To prove this we represent the 2-subspaces of V in reduced row echelon form as before.

Define the polynomials in variables x, y, z, w by

$$p_{i,j}(x, y, z, w) = det \begin{bmatrix} R_i \\ B_j \end{bmatrix}.$$

In order for all 2-spaces of type R_i to have a complement among the four spaces $U_1, \ldots, U_4$ it is necessary and sufficient that the system of four equations

$$\begin{aligned} p_{i,1}(x, y, z, w) &= 0 \\ p_{i,2}(x, y, z, w) &= 0 \\ p_{i,3}(x, y, z, w) &= 0 \\ p_{i,4}(x, y, z, w) &= 0 \end{aligned}$$

have no solution in F.

For $i = 1, 2, 3, 4, 5, 6$ computing the necessary determinants we obtain the following 6 systems of equations: (In each case we exhibit only those equations needed to guarantee there is no solution.)

$$p_{1,1} = 1 = 0$$

In this case there is clearly no solution.

(i=2) $p_{2,1} = -y = 0 \quad p_{2,3} = -z + x - y = 0 \quad p_{2,4} = b - za + xd - xzc - y = 0$

In this case the solutions are given by :

$$y = 0, \quad z = x, \quad z = root\ of(cZ^2 + (a - d)Z - b)$$

Hence there is no solution if and only if $cZ^2 + (a - d)Z - b$ has no roots in F.

(i=3) $$p_{3,1} = y = 0 \quad p_{3,3} = -1 + y = 0$$

Clearly these equations have no solution.

(i=4) $$p_{4,1} = -y = 0 \quad p_{4,3} = 1 - y = 0$$

Clearly these equations have no solution.

(i=5) $$p_{5,1} = x = 0 \quad p_{5,5} = -c + x = 0$$

In this case no solution exists if and only if $c \neq 0$.

(i=6) $$p_{6,1} = 1 = 0$$

Clearly this equation has no solution.

Now suppose the polynomial $Z^2+\alpha Z+\beta$ has no root in F. Then if we take$c=1$, $a=\alpha$, $d=0$ and $b=-\beta$ it is clear from the above that none of the six systems of equations have solutions so every 2-space has one of the U_is as a complement.

Conversely if every 2-space has one of the U_i as a complement, each set of equations must have no solution. In patricular for $i=5$ we have no solution and hence $c\neq 0$. Then in order for $i=2$ to have no solutions we must have that $cZ^2+(a-d)Z-b$ has no root in F.

Thus we have shown that $\gamma(2,4,F)=4$ if and only if F is not quadratically closed. This completes the proof. $\square$

4. The Case $5 \leq \gamma(2,5,F) \leq 7$

Let U_i be the row space of the matrix A_i where A_i is given below for $1 \leq i \leq 7$:

$$A_1=\begin{bmatrix}1&0&1&0&0\\0&1&0&1&0\end{bmatrix}$$

$$A_2=\begin{bmatrix}1&0&0&1&0\\0&1&0&0&1\end{bmatrix}$$

$$A_3=\begin{bmatrix}1&0&0&0&1\\0&1&1&0&1\end{bmatrix}$$

$$A_4=\begin{bmatrix}1&0&1&0&1\\0&1&1&1&1\end{bmatrix}$$

$$A_5=\begin{bmatrix}1&0&1&1&1\\0&1&1&1&0\end{bmatrix}$$

$$A_6=\begin{bmatrix}1&0&1&1&0\\0&1&0&1&1\end{bmatrix}$$

$$A_7=\begin{bmatrix}0&0&1&0&0\\0&0&0&0&1\end{bmatrix}$$

This gives us a set of 7 2-subspaces of $V=F^5$. We claim that every 3-subspace of V is a complement of U_i for some $i\in\{1,2,\ldots,7\}$. To prove this we consider the following 10 types of 3×5 reduced row echelon matrices of rank 3 over F. Every 3-subspace of V is the row space of one of these 10 matrices for some unique choice of the variables x,y,z,w,u,v.

$$R_1=\begin{bmatrix}1&0&0&x&y\\0&1&0&z&w\\0&0&1&u&v\end{bmatrix}$$

$$R_2=\begin{bmatrix}1&0&u&0&y\\0&1&x&0&z\\0&0&0&1&w\end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 0 & y & x & 0 \\ 0 & 1 & w & z & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_4 = \begin{bmatrix} 1 & x & 0 & 0 & y \\ 0 & 0 & 1 & 0 & w \\ 0 & 0 & 0 & 1 & v \end{bmatrix}$$

$$R_5 = \begin{bmatrix} 1 & x & 0 & y & 0 \\ 0 & 0 & 1 & w & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_6 = \begin{bmatrix} 1 & x & y & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_7 = \begin{bmatrix} 0 & 1 & 0 & 0 & y \\ 0 & 0 & 1 & 0 & w \\ 0 & 0 & 0 & 1 & v \end{bmatrix}$$

$$R_8 = \begin{bmatrix} 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_9 = \begin{bmatrix} 0 & 1 & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{10} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now as in section 3 we define for $1 \leq i \leq 10$ the polynomials

$$p_{i,j}(x,y,z,w,u,v) = det \begin{bmatrix} R_i \\ A_j \end{bmatrix} \qquad 1 \leq j \leq 7$$

It remains to prove that for each $i \in \{1, 2, \ldots, 10\}$ the system of 7 equations

$$p_{i,j}(x,y,z,w,u,v) = 0 \qquad 1 \leq j \leq 7$$

has no solution in F. We list only enough equations in each case to guarantee no solution exists. In all cases except for $i = 1, 3, 4$ the systems clearly have no solutions. We give the details in these three cases.

$(i = 1)$:

$$p_{1,1} = v - zv + wu + y + xw - yz = 0$$

$$p_{1,2} = 1 - w - x + xw - yz = 0$$

$$p_{1,3} = u + z - x + xv - yu + xw - yz = 0$$

$$p_{1,4} = -1 + v + z - zv + wu - x + y + xv - yu + xw - yz = 0$$

$$p_{1,5} = -1 + u + z - w - zv + wu + y + xv - yu + xw - yz = 0$$

$$p_{1,6} = 1 - u + v - w - zv + wu - x + y + xw - yz = 0$$

$$p_{1,7} = -u = 0$$

In this case we observe that $p_{1,6} + p_{1,3} - p_{1,2} - p_{1,4} = 1$ which clearly shows that the polynomials have no common set of zeros.

$(i = 2)$:

$$p_{2,7} = -1 = 0$$

$(i = 3)$

$$p_{3,1} = 1 - z - y - xw + yz = 0$$

$$p_{3,2} = w + yz - xw = 0$$

$$p_{3,3} = x - xw + yz = 0$$

$$p_{3,4} = 1 - z - y + x - xw + yz = 0$$

$$p_{3,5} = w - z - y + x - xw + yz = 0$$

$$p_{3,6} = 1 + w - z - y - xw + yz = 0$$

$$p_{3,7} = 0 = 0$$

In this case we note that $p_{3,6} - p_{3,5} + p_{3,4} - p_{3,1} = 1$ which shows that the polynomials have no common set of zeros.

$(i = 4)$:

$$p_{4,1} = w + xv + y = 0$$

$$p_{4,2} = v + y - x = 0$$

$$p_{4,3} = -1 - x + xw + y = 0$$

$$p_{4,4} = -1 + w - x + xv + xw + y = 0$$

$$p_{4,5} = -1 + v + w + xv + xw + y = 0$$

$$p_{4,6} = v + w - x + xv + y = 0$$

$$p_{4,7} = 0 = 0$$

Assume this system has a solution. Then $0 = p_{4,1} - p_{4,6} = -v + x$ and hence $x = v$. Substituting this into $p_{4,2} = 0$ we obtain $y = 0$ Substituting $x = v$ and $y = 0$ in $p_{4,4} - p_{4,5} = 0$ we obtain $-2v = 0$ and if we assume $char(F) \neq 2$ we have $v = 0$. Then substituting $x = v = 0$ in $p_{4,3} = 0$ yields $-1 = 0$. This proves that the system has no solution if $char(F) \neq 2$.

$(i = 5)$:

$$p_{5,4} = -w + x - xw - y = 0$$

$$p_{5,5} = 1 - w + x - xw - y = 0$$

$(i = 6)$:

$$p_{6,1} = -1 + y = 0$$

$$p_{6,2} = y = 0$$

$(i = 7)$:

$$p_{7,1} = v + y = 0$$

$$p_{7,6} = -1 + v + y = 0$$

$(i = 8)$:

$$p_{8,1} = 1 - y = 0$$

$$p_{8,2} = -y = 0$$

$(i = 9)$:

$$p_{9,1} = x = 0$$

$$p_{9,3} = -1 + x = 0$$

$(i = 10)$:

$$p_{10,1} = 1 = 0$$

The above together with Lemma 3 proves the following result:

Proposition 7. $5 \leq \gamma(2,5,F) \leq 7$ *for fields F of characteristic different from 2.* $\square$

Example.

From the proof of Proposition 6 one might conjecture that a large enough collection of pairwise disjoint ($\Leftrightarrow$ intersection equal 0) k-subspaces of V would "cover" all $(n-k)$-subspaces. The following example shows that this need not hold even for infinitely many such spaces (if F is infinite). Consider the following set of k-subspaces of $V = F^n$ where $k \geq 2$ and $n \geq 2k$:

$$U_\alpha = \{(x_1, \alpha x_1, x_2, \alpha x_2, \ldots, x_k, \alpha x_k, 0, 0, \ldots, 0) | x_i \in F\} \qquad \alpha \in F$$

Clearly $U_\alpha \cap U_\beta = 0$ if $\alpha \neq \beta$. On the other hand the 2-space

$$W = \{(x_1, x_2, \ldots, x_{n-k}, 0, 0, \ldots, 0) | x_i \in F\}$$

meets each U_α non-trivially.

5. A recursive inequality for $\gamma(k,n,F)$

In this section we prove the following:

Proposition 8. *For all fields F if $1 \leq k \leq n$ then*

$$\gamma(k, n+1, F) \leq \gamma(k, n, F) + \gamma(k-1, n, F)$$

Proof. Let $a = \gamma(k,n)$ and $b = \gamma(k-1,n)$. Then there are k-subspaces $U_1, \ldots, U_a$ and $(k-1)$-subspaces $V_1, \ldots, V_b$ such that every $(n-k)$-subspace of F^n has one of the U_i's as a complement and every $(n-k+1)$-subspace of F^n has one of the V_i's as a complement. We identify F^n with the elements in F^{n+1} with last coordinate equal to 0. Then $F^{n+1} = F^n \oplus Fe$ where $e = (0, \ldots, 0, 1)$. Expand each V_i to $V_i^* = V_i \oplus Fe$.

Now we claim that every $(n+1-k)$-subspace W of F^{n+1} has one of the subspaces $U_1, \ldots, U_a, V_1, \ldots, V_b$ as a complement. There are 2 cases (1) $W \subseteq F^n$ and (2) $W \nsubseteq F^n$.

If the first case holds then $W \oplus V_i = F^n$ for some i. and since $e \notin F^n$ we have $W \oplus V_i^* = F^{n+1}$.

If the second case holds and $W \nsubseteq F^n$ then $F^{n+1} = W + F^n$. It follows that $dim(W \cap F^n) = n - k$ and $W = Fx \oplus (W \cap F^n)$ for some $x \in W - F^n$. Also since $W \cap F^n$ is an $(n-k)$-subspace of F^n for some i $(W \cap F^n) \oplus U_i = F^n$ and it follows that

$$W \oplus U_i = Fx \oplus (W \cap F^n) \oplus U_i = Fx. \oplus F^n = F^{n+1},$$

which completes the proof. $\square$

Corollary. *For all fields F of characteristic not equal to 2*

$$\gamma(2,n,F) \leq \binom{n}{2} - 3, \qquad \text{if} \quad n \geq 5$$

.

Proof. We know from Proposition 7 that $\gamma(2,5,F) \le 7 = \binom{5}{2} - 3$. If it holds for n then

$$\gamma(2,n+1,F) \le \gamma(2,n) + \gamma(1,n) \le \binom{n}{2} - 3 + \binom{n}{1} = \binom{n+1}{2} - 3$$

so it holds for $n+1$. $\square$

Similarly, we obtain:

Corollary. *For all fields F of characteristic not equal to 2*

$$\gamma(3,n,F) \le \binom{n}{2} - 3(n-5), \qquad \textit{if} \quad n \ge 6. \quad \square$$

Clearly, using Propostion 6, 7 and 8 we may obtain improvements to the bound in Proposition 1 similar to those in the above corollaries. However, since we hope that better results will be forthcoming we omit the details. In fact we put forward the following conjecture:

Conjecture. *If F is algebraically closed then $\gamma(k,n,F) = k(n-k)+1$.*

Acknowledgements. We wish to express our thanks to Ted Ryan for suggesting Proposition 8 and to the program Maple V for assisting us in some of the tedious computations in the proofs of Propositions 6 and 7.

References

1. G. E. Andrews, *The theory of partitions, Encyclopedia of mathematics and its applications*, Addison-Wesley Pub. Co., 1976.
2. D. M. Bloom, *Linear Algebra and Geometry*, Cambridge University Press, 1979.
3. A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-regular graphs*, Springer-Verlag, 1989.
4. G. Chartrand and L. Lesniak, *Graphs and Diagraphs (2nd edition)*, Wadsworth, Inc, 1986.
5. W. E. Clark, *Matching Subspaces with Complements in Finite Vector Spaces*, Bulletin of the Inst. of Combinatorics and its Applications **8** (1992), 33-38.
6. M. E. H. Ismail, Dennis Stanton and Gérard Vienot, *The Combinatorics of Q-Hermite polynomials and the Askey-Wilson Integral*, Europ.J. Combinatorics **8** (1987), 379-392.
7. N. Jacobson, *Basic Algebra I (2nd edition)*, W. H. Freeman and Co., 1985.
8. J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, Cambridge University Press, 1992.

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