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ON ARCHIMEDEAN ORDERED VECTOR SPACES AND A CHARACTERIZATION OF SIMPLICES

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ABSTRACT. We show that a convex subset K of a linear space is a simplex if and only if it is line compact and every nonempty intersection of two translates of K is a homothet of K. This answers a problem posed by Rosenthal. The proof uses a reformulation of this problem in terms of Archimedean ordered spaces

INTRODUCTION

Let K be a convex subset of a linear space E. If $K \times \{1\}$ is the base for a lattice cone in $X \times \Re$, then K is called a simplex (see [7] for the definitions). A remarkable result of Kendall [4] shows that K is a simplex if and only if K is line-compact[1] and the nonempty intersection of two homothets[2] of K is a homothet of K. Moreover, every simplex has the property that it is line compact and every nonempty intersection of two translates of K is a homothet of K. It is an open problem whether this last condition fully characterizes simplices, but a result of Rosenthal [7] shows that this is the case at least for σ-convex subsets of topological vector spaces. Using Archimedean ordered spaces and their known relation to simplices, Rosenthal [7] reformulated this open problem as follows.

Let E be an ordered vector space. Given a positive function $\mu: E \to \Re$, we say that E has a μ-lattice structure if $\mu(u_1) = \mu(u_2)$ implies that $u_1 \vee u_2$ exists in E.

Note that in an ordered vector space the existence of $-u_1 \vee H - u_2$ implies that $u_1 \wedge u_2$ exists and is equal to $-(-u_1 \vee -u_2)$. Hence in a μ-lattice the infimum $u_1 \wedge u_2$ exists whenever $\mu(u_1) = \mu(u_2)$.

Recall that a positive functional μ is called strictly positive if $u \geq 0$ and $\mu(u) = 0$ implies that $u = 0$.

Rosenthal has shown that an affirmative answer to the following question leads to a positive solution of the open problem concerning simplices as stated above.

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[1] K is line-compact if the intersection of every line with K is compact.

[2] A homothet of K is a set of the form $a + r \cdot K$, where $r \geq 0$ is a positive constant.

Problem. Let E be an Archimedean ordered space. If E has a μ-lattice structure for a strictly positive linear functional μ on E, is it necessarily a vector lattice?

In this note we give an affirmative answer to this problem (Theorem 3.5).

For definitions and results concerning vector lattices and Banach lattices, see [5, 3].

2. μ-SUBLATTICES OF $C(K)$

In the following, let $E \subseteq C([0, 1])$ be a subspace such that (i) $1, x \in E$ and (ii) there is a positive measure μ on $[0, 1]$ such that $0, 1 \in \operatorname{supp}(\mu)$ and such that for all $f, g \in E$ with $\mu(f) = \mu(g)$ we have $\sup(f, g) \in E$. We would like to show that E contains all piecewise linear functions.

We need some notation. For a point $a \in [0, 1]$, define maps $\lambda_a, \rho_a \in C([0, 1])$ by

$$\lambda_a(x) = \begin{cases} a - x & \text{if } x \leq a, \\ 0 & \text{else} \end{cases}$$

and

$$\rho_a(x) = \begin{cases} x - a & \text{if } x \geq a, \\ 0 & \text{else.} \end{cases}$$

Since $\lambda_a(x) - \rho_a(x) = a - x = a \cdot 1 - x$, we have $\lambda_a \in E$ if and only if $\rho_a \in E$. Also, since $\lambda_a(x) = \sup\{(a \cdot 1 - x), 0)$, condition (ii) implies that there is a number a_0 with $0 < a_0 < 1$ such that $\lambda_{a_0}, \rho_{a_0} \in e$.

2.1. **Proposition.** *Let $E \subseteq C([0, 1])$ be a subspace such that*

(i) $1, x \in E$;

(ii) *there is a positive measure μ on $[0, 1]$ such that $0, 1 \in \operatorname{supp}(\mu)$ and such that E is a μ-lattice.*

Then E contains all piecewise linear functions.

Proof. Every piecewise linear function is a linear combination of functions of the forms λ_b and ρ_b, $0 < b < 1$. Hence, we have to show that $\lambda_b, \rho_b \in E$ for all b with $0 < b < 1$. Let a_0 be a number with $0 < a_0 < 1$ such that λ_{a_0}, $\rho_{a_0} \in E$, and let $0 < b < 1$ be given. We shall assume that $a_0 < b$; the case where $b < a_0$ is treated similarly.

Consider the function f given by $f(x) = x - b$ and let $r = \int f(x)\,d\mu$. Then, since μ is positive and $0 \in \operatorname{supp}(\mu)$, it follows that $\int \lambda_{a_0}(x)\,d\mu > 0$, and we can find a number s such that

$$r = s \int \lambda_{a_0}(x)\,d\mu .$$

From our assumptions on E it follows that

$$g_1 = \sup(f, s \cdot \lambda_{a_0}) - s \cdot \lambda_{a_0} \in E .$$

We will consider two cases:

(i) $-b/a_0 \leq s$. In this case, $\rho_b = g_1 \in E$.

(ii) $s < -b/a_0$. In this case, $g_1 = r_1 \cdot \lambda_{a_1} + \rho_b$ where

$$r_1 = -(1 + s) \quad \text{and} \quad a_1 = \frac{b + s a_0}{1 + s} \leq a_0 .$$

Let

$$R = \int \rho_b \, d\mu, \qquad S = \int \lambda_{a_0} \, d\mu.$$

Since $\rho_b \leq g_1$, we have $\int g_1 \, d\mu \geq R$. Again, we can find a number s_1 so that

$$s_1 \int \lambda_{a_0} \, d\mu = \int g_1 \, d\mu,$$

and it follows that $s_1 \geq R/S$. It follows that $a_0 \cdot R/S \leq s_1 \cdot a_0 = s_1 \cdot \lambda_{a_0}(0)$. Define

$$g_2 = \sup(g_1, s_1 \cdot \lambda_{a_0}) - s_1 \cdot \lambda_{a_0}.$$

Then $g_2 \in E$, and either $g_2 = \rho_b \in E$ (in the case where $s_1 \cdot a_0 \geq r_1 \cdot a_1$) or $g_2 = r_2 \cdot \lambda_{a_2} + \rho_b$ where $a_2 \leq a_1 \leq a_0$ and

$$r_2 \cdot a_2 = r_1 \cdot a_1 - s_1 \cdot a_0 \leq r_1 \cdot a_1 - (R/S) \cdot a_0.$$

We continue in this way with r_2 and a_2 in the place of r_1 and a_2 until we finally find an index n such that $s_n \cdot a_0 \geq r_n \cdot a_n$, and therefore $g_{n+1} = \rho_b \in E$. Note that this procedure has to terminate after finitely many steps since

$$r_n \cdot a_n \leq r_{n-1} \cdot a_{n-1} - (R/S) \cdot a_0 \quad \text{and} \quad s_n \geq R/S. \quad \square$$

In the following result, we generalize the domain of the functions slightly. Before we do this, let us make some remarks concerning positive measures. Let K and K' be compact Hausdorff spaces, and let $\phi: K \to K'$ be a continuous map. If ν is a measure on K, then define a measure ν' on K' by

$$\int d \, d\nu' = \int (f \circ \phi) \, d\nu.$$

Then $\nu' = T_\phi^*(\nu)$, where T_ϕ^* is the adjoint to the operator

$$T_\phi: C(K') \to C(K), \qquad f \mapsto f \circ \phi.$$

The supports of ν and ν' are related by the equation

$$\operatorname{supp}(\nu') = \phi(\operatorname{supp}(\nu)).$$

2.2. **Proposition.** *Let $E \subseteq C([a, b])$ be a subspace such that*

(i) $1, x \in E$;
(ii) *there is a positive measure μ on $[0, 1]$ such that $a, b \in \operatorname{supp}(\mu)$ and such that E is a μ-lattice.*

Then E contains all piecewise linear functions.

Proof. Define a map

$$\phi: [0, 1] \to [a, b], \qquad x \mapsto a + (b - a) \cdot x.$$

Then the map

$$T_\phi: C([a, b]) \to C([0, 1]), \qquad f \mapsto f \circ \phi$$

is a (norm-preserving) linear bijection that respects the lattice structure of $C([a, b])$ and that also sends the space of all piecewise linear functions on $C([a, b])$ onto the space of all piecewise linear functions on $C([0, 1])$. Moreover, μ is a positive measure on $C([0, 1])$ if and only if it is of the form

$\mu = T^*_\phi(\nu)$, where ν is a positive measure on $[a, b]$. Furthermore, since $\operatorname{supp}(\mu) = \phi(\operatorname{supp}(\nu))$, we have $0, 1 \in \operatorname{supp}(\mu)$ if and only if $a, b \in \operatorname{supp}(\nu)$. Hence Proposition 2.2 follows from 2.1. □

2.3. **Theorem.** *Let K be a compact Hausdorff space, let $E \subseteq C(K)$ be a linear subspace containing a strictly positive function e, and let μ be a strictly positive measure on K such that E is a μ-lattice. Then E is a sublattice of $C(K)$.*

Proof. First, consider the map

$$T_{e^{-1}}: C(K) \to C(K), \qquad f \mapsto e^{-1} \cdot f.$$

This map is linear, bijective, and preserves the lattice structure. Hence, we may replace E by $T_{e^{-1}}(E)$, e by 1, and μ by $e \cdot \mu$ in the statement of the theorem. It follows that we may assume w.l.o.g. that $1 \in E$.

Now let $\phi \in E$ be arbitrary. We have to show that $|\phi| \in E$. Let

$$a = \min_{0 \leq \max \leq 1} \phi(x), \qquad b = \max_{0 \leq x \leq 1} \phi(x).$$

Consider the linear operator

$$T_\phi: C([a, b]) \to C(K)$$
$$f \mapsto f \circ \phi.$$

Again, this operator is linear and a lattice homomorphism (in the sense that $T_\phi(|f|) = |T_\phi(f)|$ for all continuous $f \in C([a, b])$). Let $F = T_\phi^{-1}(E)$ and let $\mu' = T^*_\phi(\mu)$. Then F is a linear subspace of $C([a, b])$ with $1 \in F$ (since $T_\phi(1) = 1 \circ \phi = 1$), and we have $x \in F$; indeed, $x \in F$ means that $i \in F$, where $i(x) = x$. But $i \in F$ is equivalent to $T_\phi(i) = i \circ \phi = \phi \in E$, and the last statement is true by our assumptions. Moreover, μ' is positive; if $f \in C[a, b]$ is positive, then $f \circ \phi$ is positive, and we obtain $\mu'(f) = \mu(f \circ \phi) \geq 0$. Since μ is strictly positive, $\operatorname{supp}(\mu) = K$, hence $\operatorname{supp}(\mu') = \phi(K)$. It follows that a, $b \in \operatorname{supp}(\mu')$. Lastly, assume that $f, g \in F$ and that $\mu'(f) = \mu'(g)$. Then $\mu(T_\phi(f)) = \mu(T_\phi(g))$ and $T_\phi(f), T_\phi(g) \in E$. Hence, by our assumptions on E, we conclude that $T_\phi(\sup(f, g)) = \sup(T_\phi(f), T_\phi(g)) \in E$, and therefore $\sup(f, g) \in F$. It follows from Proposition 2.2 that F contains all piecewise linear functions. In particular, $|i| \in F$ (where $|i|(x) = |x|$ for all $a \leq x \leq b$). By the definition of F, we conclude that $|\phi| = |T_\phi(i)| = T_\phi(|i|) \in E$. □

2.4. **Proposition.** *Let $E \subseteq C(K)$ be a subspace such that $1 \in E$. Assume that E separates the points of K. Further, let $\phi: E \to \mathfrak{R}$ be a strictly positive functional on E such that E is a ϕ-lattice. Then ϕ can be extended to a strictly positive linear functional $\phi': C(K) \to \mathfrak{R}$.*

Proof. Firstly, we may assume that $\phi(1) = 1$. By the vector lattice version of the Hahn-Banach Theorem (see Proposition II.4.4 of [3] with $U = \{f \in C(K): -1 \leq f \leq 1\}$), we can extend ϕ to a positive functional $\phi': C(K) \to \mathfrak{R}$, and it suffices to show that each such extension is strictly positive. Actually, there is only one such extension, since it will follow later that E is dense in $C(K)$, and the density of E in $C(K)$ implies that this extension has to be strictly positive, but we cannot use the density of E at this point.

Since every positive functional is bounded and hence given by a measure on K, we find that $\phi'(f) = \int f \, d\mu$ for some positive measure μ on K. In order to show that ϕ' is strictly positive, we have to show that $\operatorname{supp}(\mu) = K$. Assume not, and let $x_0 \in K \setminus \operatorname{supp}(K)$. We will construct a positive element $f_0 \in E$ such that $f_0(x_0) > 0$ and $f_0(y) = 0$ for all $y \in \operatorname{supp}(\mu)$. This will lead to the contradiction $0 < \phi(f_0) = \phi'(f) = \int f_0 \, d\mu = \int_{\operatorname{supp}(\mu)} f_0 \, d\mu = 0$.

As a first step, we show that for every $x \in K$ there is an open neighborhood U of x and a positive element $f \in E$ such that $0 \neq f$ and $f(y) = 0$ for all $y \in U$. In order to construct f, pick any nonconstant function $g \in E$ and consider the element $g - \phi(g) \cdot 1$. There are two cases to consider.

1. $g(x) \neq \phi(g)$. If $g(x) < \phi(g)$, then since $\phi(g - \phi(g) \cdot 1) = 0$, the element $f = (g - \phi(g) \cdot 1)_+ = (g - \phi(g) \cdot 1) \vee 0$ belongs to E and is 0 on a neighborhood of U. Moreover, since g is not constant, $g - \phi(g) \cdot 1 \neq 0$. Since $g - \phi(g) \cdot 1 < 0$ would imply that $\phi(g - \phi(g) \cdot 1) < 0$, we conclude that $g - \phi(g) \cdot 1$ is not negative, hence there is at least one point x_1 for which $f(x_1) = (g - \phi(g) \cdot 1)(x_1) > 0$. We conclude that $0 < f$. If $g(x) > \phi(g)$, then we replace g by $-g$.

2. $g(x) = \phi(g)$. In this case, the same construction as in case (1) delivers a function $f' > 0$ such that $f'(x) = 0$. We now replace g by f', and for this new g we have $g(x) = 0 < \phi(g)$. Hence case (1) applies to the new function g, and we also find a function $f > 0$ such that $f(y) = 0$ for all y in a neighborhood of x.

In the next step, we show that for every $x \in \operatorname{supp} \mu$ there is a function $f_x \in E$ such that $0 \leq f_x$, $f_x(x_0) > 0$ and such that f_x vanishes on a neighborhood U_x of x. In order to construct f_x, pick any function g such that $g(x_0) \neq g(x)$. Such a function exists since E separates the points of K. After subtracting a multiple of the constant function 1 and multiplying by -1, if necessary, we may assume that $g(x_0) > 0 > g(x)$. Let $r = \phi(g_0)$. If $r \geq 0$, then pick any positive function $0 \neq h \in E$ with $h(x_0) = 0$ and $\phi(h) = 1$. If $r \leq 0$, then pick a negative function $0 \neq h \in E$ that vanishes on a neighborhood of x and satisfies $\phi(h) = -1$. In both cases, the function h exists by the previous step. Now let $g' = g - |r| \cdot h$. Then $\phi(g') = 0$, $g'(x_0) > 0$, and g' is negative on a neighborhood of x. Hence, by our assumptions on E, $f_x = g' \vee 0 \in E$, $f_x(x_0) > 0$, and f_x vanishes on a neighborhood U_x of x.

Now we continue with a standard compactness argument. Finitely many of the neighborhoods U_x cover the compact set $\operatorname{supp} \mu$, say $\operatorname{supp} \mu \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. Clearly, there are strictly positive constants r_1 and r_2 such that $\mu(r_1 f_{x_1}) = \mu(r_2 f_{x_2})$, which implies that $r_1 f_{x_1} \wedge r_2 f_{x_2} \in E$. Continuing this process inductively yields constants $r_1, \ldots, r_n > 0$ such that $f_0 = r_1 f_{x_1} \wedge \cdots \wedge r_n f_{x_n} \in E$. This function f_0 vanishes on $\operatorname{supp} \mu$ and is strictly positive at x_0. □

2.5. **Corollary.** *Let K be a compact Hausdorff space, and let $E \subseteq C(K)$ be a linear subspace that separates the points of K. Assume that $1 \in E$. If there is a strictly positive functional $\phi: E \to \mathfrak{R}$ such that E is a ϕ-lattice, then E is a sublattice of $C(K)$.*

3. Archimedean ordered vector spaces

In this section, we will show that every Archimedean ordered vector space that is also a μ-lattice for a certain strictly positive functional μ is actually a vector lattice.

We will start our discussion with order unit spaces. This special case can be reduced to 2.5. The arguments are to a large extend standard in the theory of compact convex sets and boundary integrals, see also [1]. If (E, e) is an Archimedean ordered vector space with an order unit e, let $S(E)$ denote the state space of E, that is $S = \{\phi\colon E \to \Re | \phi$ is positive, and $\phi(e) = 1\}$. When equipped with the weak-*-topology, S is a compact convex set. The following result is a consequence of Herve's Theorem (see also Proposition 1.4.1 in [1]); the proof is a variation of the proof of Theorem II.1.9 in [1].

3.1. **Theorem.** *Let (E, e) be an Archimedean ordered vector space with order unit e, and let $p \in \partial S(E)$ be an extreme point of $S(E)$. If a, $b \in E$ and $a \vee b$ exists, then $p(a \vee b) = \max\{p(a), p(b)\}$. Moreover, if ϕ belongs to the weak-*-closure of $\partial S(E)$, then $\phi(e) = 1$ and ϕ preserves all existing suprema.*

Proof. The proof of the first half of the theorem is an exact copy of the corresponding part of Theorem II.1.9 in [1]; one only has to remark that it is enough to postulate the existence of $a \vee b$ and that the full lattice structure is never used. For the second half, let $(p_i)_{i\in I}$ be a net of extreme points of $S(E)$ that converges to ϕ in the weak-*-topology. Then $\phi(e) = \lim p_i(e) = \lim 1 = 1$ and $\phi(a \vee b) = \lim p_i(a \vee b) = \lim \max\{p_i(a), p_i(b)\} = \max\{\lim p_i(a), \lim p_i(b)\} = \max\{\phi(a), \phi(b)\}$ whenever $a \vee b$ exists. □

3.2. **Theorem.** *Let (E, e) be an Archimedean ordered vector space with order unit e. Let K be the weak-*-closure of $\partial S(E)$. Then (E, e) is order isomorphic to a subspace F of $C(K)$ under an isomorphism $\Psi\colon E \to F$ such that*

(i) $\Psi(e) = 1$.
(ii) *If $a \vee b$ exists in E, then the pointwise supremum of $\Psi(u)$ and $\Psi(v)$ belongs to F and $\Psi(a \vee b) = \Psi(a) \vee \Psi(b)$.*
(iii) *F separates the points of K.*

In addition, if E admits a strictly positive functional μ such that E is a μ-lattice, then F is a $\mu \circ \Psi^{-1}$-lattice.

Proof. The proof of this result is an easy application of Kadison's Theorem (see also II.1.8 in [1]) in connection with 3.1. □

We now can show the following

3.3. **Corollary.** *Let (E, e) be an Archimedean ordered vector space with order unit e. If there is a strictly positive measure μ on E such that E is a μ-lattice, then E is a vector lattice.*

Proof. This statement follows from 2.5 and 3.2 and the observation that an ordered vector space that is order isomorphic to a vector lattice is a vector lattice in its own right. □

Before we prove our main result, we need one additional lemma.

3.4. **Lemma.** *Let E be an ordered vector space, and assume that the positive cone $C = \{x \in E\colon 0 \le x\}$ contains at least one nonzero element. If there is a strictly positive linear functional $\mu\colon E \to \Re$ such that E is a μ-lattice, then the positive cone is generating in the sense that $E = C - C$.*

Proof. Let $0 \neq u \in C$, and let $x \in E$ be arbitrary. If $\mu(x) \ge 0$, then 0, $x \le x \vee (\mu(x)/\mu(u)) \cdot u \in E$. It follows that

$$x = (\mu(x)/\mu(u)) \cdot u - ((\mu(x)/\mu(u)) \cdot u - x) \in C - C.$$

If $\mu(x) \leq 0$, then $\mu(-x) \geq 0$, hence $-x \in C - C$ by the previous argument, and thus $x \in C - C$, since $C - C$ is a linear subspace. □

We are now able to show

3.5. **Theorem.** *Let E be an Archimedean ordered vector space, and assume that the positive cone of E is generating. If there exists a strictly positive functional μ on E such that E is a μ-lattice, then E is a vector lattice.*

Proof. For each positive $u \in E$ let $E_u = \bigcup_{n>0}\{x \in E: nu \leq x \leq nu\}$ be the order ideal generated by u. Then (E_u, u) is an Archimedean ordered vector space with order unit u. Moreover, the restriction of μ to E_u defines a strictly positive functional on E_u, and E_u is a μ-lattice with respect to this functional. Hence each E_u is a vector lattice by 3.3. Since $E = \bigcup_{u>0} E_u$, it can be expected that E is also a vector lattice; we just have to show that the supremum of the element $x \in E$ with 0 does not depend on the order ideal E_u in which it is computed. Let us denote the supremum of x and 0 in E_u by x_u. Let u and v be two positive elements such that $x \in E_u \cap E_v$. If $u \leq v$, then $E_u \subseteq E_v$, and hence it follows that $x_v \leq x_u$ since $x_u \in E_v$ is an upper bound of x and 0. Because E_u is an order ideal, it follows that $x_v \in E_u$, and since x_u is the least upper bound of x and 0 in E_u, we conclude that $x_u \leq x_v$, i.e., $x_u = x_v$. If u and v are arbitrary positive elements such that $x \in E_u \cap E_v$, then also $x \in E_{u+v}$, and we have just argued that $x_u = x_{u+v} = x_v$. Therefore the supremum of x and 0 exists in E, and E is a vector lattice. □

In connection with Rosenthal's result in [7] we now obtain

3.6. **Corollary.** *Let K be a line-compact convex set such that the nonempty intersection of two translates of K is a homothet of K. Then K is a simplex.*

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