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ON A PROBLEM OF G. G. LORENTZ REGARDING THE NORMS OF FOURIER PROJECTIONS

BORIS SHEKHTMAN

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ABSTRACT. For any $0 < \alpha < \frac{1}{2}$ we construct a sequence of integers $(\mu_1, \dots, \mu_n, \dots)$ such that the norms of Fourier projections

$$F_N = \sum_1^N e^{i\mu_j\theta} \otimes e^{i\mu_j t}: C_{[-\pi, \pi]} \rightarrow C_{[-\pi, \pi]}$$

grow as N^α . This answers a question of Prof. G. G. Lorentz.

I. INTRODUCTION

Let $M = (\mu_1, \dots, \mu_n) \subset \mathbf{N}$ be an increasing sequence of integers. Let $C_{[-\pi, \pi]}$ be the space of continuous functions f on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$. For every $f \in C_{[-\pi, \pi]}$ let $F_M f$ be the Fourier projection of f :

$$F_M f = \frac{1}{2\pi} \sum_{j=1}^n \left(\int_{-\pi}^{\pi} f(\theta) e^{i\mu_j \theta} d\theta \right) e^{i\mu_j \theta}.$$

The celebrated solution of the Littlewood conjecture (cf. [2]) implies (cf. [3])

$$(1) \quad C_1 \log(\#M) \leq \|F_M\| \leq C_2 \sqrt{\#M}.$$

We use $\#M$ to denote the cardinality of M . Here and throughout this paper we use the letter C with subscripts to denote positive constants that do not depend on M .

It is also well known (cf. [1], [3]) that the lower and upper bounds in (1) are attained with M being an arithmetic and lacunary sequence, respectively:

Proposition 1. *There exist constants C_3, C_4, C_5, C_6 such that*

- (a) *if M is an arithmetic sequence then $C_3 \log(\#M) \leq \|F_M\| \leq C_4 \log(\#M)$*
- (b) *if $M = (\lambda_1, \dots, \lambda_n)$ is such that $(\lambda_k + 1)/\lambda_k > 2$ then $C_5 \sqrt{\#M} \leq \|F_M\| \leq C_6 \sqrt{\#M}$.*

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In the 1986 Edmonton conference on constructive approximation, Professor G. G. Lorentz posed the following:

Problem. Let $0 < \alpha < \frac{1}{2}$ be a real number. Exhibit a sequence of integers $M = (\mu_1, \dots, \mu_n, \dots)$ so that

$$\|F_{M_n}\| \asymp n^\alpha \text{ as } n \rightarrow \infty,$$

where $M_n = (\mu_1, \dots, \mu_n)$.

In this note we solve the problem.

The idea of the construction is to combine the lacunary sequences and arithmetic progressions in such a way that the cardinality of the set of arithmetic sequences is sufficiently large to be a major factor in $\#M_n$, while the cardinality of the lacunary sequence determines the norms of $\{F_{M_n}\}$, thus achieving an arbitrary balance between the two.

For a positive number a we use $[a]$ to denote the integer part of a . We will need the following simple estimates:

Proposition 2. *Let $a > 1$. There exist constants $C_7(a)$, $C_8(a)$, $C_9(a)$, $C_{10}(a)$ such that*

$$(2) \quad C_7(a)n^{a+1} \leq \sum_{k=1}^n [k^a] \leq C_8(a)n^{a+1}$$

$$(3) \quad C_9(a)n \log n \leq \sum_{k=1}^n \log [k^a] \leq C_{10}(a)n \log n.$$

Proof. To prove (2) it suffices to observe

$$\sum_1^n [k^a] \leq \sum_1^n [n^a] = n[n^a] \leq n \cdot n^a.$$

On the other hand

$$\sum_1^n [k^a] \geq \sum_{[n/2]}^n [k^a] \geq \sum_{[n/2]}^n \left[\left(\frac{n^a}{2} \right) \right] \geq \left(\frac{1}{2} \right)^a \cdot \frac{1}{2} n \cdot n^a - 1.$$

The proof of (3) is identical:

$$\begin{aligned} \sum_1^n \log [k^a] &\leq \sum_1^n \log = [n^a] \leq n \log(n^a) = an \log n. \\ \sum_1^n \log [k^a] &\geq \sum_{[n/2]}^n \log \left(\frac{n}{2} \right)^2 \geq C_9(a)n \log n. \square \end{aligned}$$

II. CONSTRUCTION

Given an $\alpha: 0 < \alpha < \frac{1}{2}$ let

$$(4) \quad a = 3/2\alpha - 1: a > 2.$$

We start with a lacunary sequence of integers

$$\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n, \dots)$$

that satisfies

$$(5) \quad \lambda_{m+1} - \lambda_m > \lfloor n^a \rfloor \quad \text{for all } m \geq n.$$

(For instance $\lambda_m = m^m$ will do).

We now construct the set M in an inductive manner.

Choose $\mu_1 = \lambda_1$. On the n th step we introduce the first n^2 available integers out of the set Λ (say $\lambda_{m+1}, \dots, \lambda_{m+n^2}$) and include in addition

$$\lambda_{m+n^2} + 1, \lambda_{m+n^2} + 2, \dots, \lambda_{m+n^2} + \lfloor n^a \rfloor.$$

By (5) we have

$$\lambda_{m+n^2} + \lfloor n^a \rfloor < \lambda_{m+n^2+1}.$$

Now our set consists of a lacunary set

$$(6) \quad \Lambda_n = (\lambda_1, \dots, \lambda_{m+n^2})$$

of length

$$(7) \quad \#\Lambda_n = \sum_{k=1}^n k^2$$

and of a set of arithmetic sequences $\mathcal{A}_1, \dots, \mathcal{A}_n$ of length

$$\#\mathcal{A}_j = \lfloor j^a \rfloor.$$

Theorem. For the set M constructed above let $F_N := F_{M_N}$. There exist constants $C, C_0 > 0$ such that

$$C_0 N^\alpha \leq \|F_N\| \leq C N^\alpha.$$

Proof. Let N be an integer and let us assume that the number μ_N had been added to the set M during the $(N+1)$ st step of the construction of the set M . Then

$$(8) \quad \#\Lambda_n + \sum_{k=1}^n (\#\mathcal{A}_k) \leq N \leq \#\Lambda_{n+1} + \sum_{k=1}^{n+1} (\#\mathcal{A}_k).$$

Using (6), (7), and (2) we have

$$\begin{aligned} N &\geq \sum_{k=1}^n k^2 + \sum_{j=1}^n \lfloor j^a \rfloor \\ &\geq C_7(2)n^3 + C_7(a)n^{a+1} \\ &\geq C_{11}n^{a+1} \quad (\text{since } a > 2). \end{aligned}$$

On the other hand

$$\begin{aligned} N &\leq C_8(2)(n+1)^3 + \sum_{j=1}^{n+1} \lfloor j^a \rfloor \\ &\leq C_8(2)(n+1)^3 + C_8(a)(n+1)^{a+1} \\ &\leq C_{12}n^{a+1}. \end{aligned}$$

Hence we have

$$(9) \quad C_{11}n^{a+1} \leq N \leq C_{12}n^{a+1}.$$

Next we estimate the norms $\|F_N\|$.

Notice that $M_N = \Lambda_n \cup (\bigcup_{k=1}^n \mathcal{A}_k) \cup \Lambda^p \cup \mathcal{A}^q$ where Λ^p is a lacunary sequence of length $p \leq (n+1)^2$ and \mathcal{A}^q is an arithmetic sequence of length $q \leq \lfloor (n+1)^a \rfloor$. Hence

$$\|F_N\| = \|F_{\Lambda_n \cup \Lambda^p} + \sum_{k=1}^n F_{\mathcal{A}_k} + F_{\mathcal{A}^q}\|,$$

Since $\Lambda_n \cup \Lambda^p$ (is a lacunary set) we have (by Proposition 1(a))

$$\begin{aligned} \|F_N\| &\geq \|F_{\Lambda_n \cup \Lambda^p}\| - \left(\sum_{k=1}^n \|F_{\mathcal{A}_k}\| \right) - \|F_{\mathcal{A}^q}\| \\ &\geq C_5 \left[\#(\Lambda_n \cup \Lambda^p) \right]^{1/2} - C_4 \sum_{k=1}^n \log(\#\mathcal{A}_k) - C_4 \log(\#\mathcal{A}^q) \\ &\geq C_5 \left(\left(\sum_{k=1}^n k^2 \right) + p \right)^{1/2} - C_4 \sum_{k=1}^n \log[k^a] - C_4 \log[q^a] \\ &\geq \left[C_5 C_7 (2)n^3 \right]^{1/2} - C_4 C_{10}(a)n \log n - C_4 \cdot a^2 \log(n+1) \\ &\geq C_{14}n^{3/2}. \end{aligned}$$

Similarly

$$\|F_N\| \leq \|F_{\Lambda_n \cup \Lambda^p}\| + \sum_{k=1}^n \|F_{\mathcal{A}_k}\| + \|F_{\mathcal{A}^q}\| \leq C_{13}n^{3/2}.$$

Hence

$$(10) \quad C_{14}n^{3/2} \leq \|F_N\| \leq C_{13}n^{3/2}.$$

Combining (9) and (10) we have

$$\|F_N\| \leq \left(C_{13}n^{3/2} \right) \leq C_{13} \left(\frac{N}{C_{11}} \right)^{3/2(a+1)} \leq CN^{3/2(a+1)} = CN^\alpha$$

as well as

$$\|F_N\| \geq C_{14}n^{3/2} \geq C_{14} \left(\frac{n}{C_{12}} \right)^{3/2(a+1)} \geq C_0 N^\alpha. \quad \square$$

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