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ON THE NORMS OF INTERPOLATING OPERATORS

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ABSTRACT
In this paper we estimate the norms of linear interpolating operators from the space of continuous functions onto polynomials. The estimate eliminates the gap between classical results of Faber and Bernstein. It also provides an affirmative answer to a question recently raised by J. Szabados.

I. Introduction

In this paper we study the norms of certain interpolating operators on the space of continuous functions $C_{[-1,1]}$.

Let $\Delta_n = \{t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)}\}$ be a given partition of the interval $[-1, 1]$. Let $q : \mathbf{N} \to \mathbf{N}$ be a function that maps non-negative integers into itself and let $\mathscr{P}_k \in C_{[-1,1]}$ be the space of polynomials of degree $k-1$.

We will say that a map

$$F(\Delta_n) : C_{[-1,1]} \to \mathscr{P}_{n+q(n)}$$

is an interpolating operator if for any $f \in C_{[-1,1]}$ and any $\tau \in \Delta_n$ we have

$$(F(\Delta_n) \cdot f)(\tau) = f(\tau).$$

Notice that operators $F(\Delta_n)$ may depend on $q(n)$. As usual we define the norm of such operator as

$$\| F(\Delta_n) \| = \sup(\| F(\Delta_n) f \| \, ; \, \| f \| \leqq 1).$$

In case $q(n) = 0$ the operator $F(\Delta_n)$ is uniquely defined and the classical result of Faber (cf. [3]) states that there exists a constant $C > 0$ such that

$$\| F(\Delta_n) \| \geqq C \cdot \log n.$$

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In the opposite direction Fejer proved (cf. [3]) that for $q(n) = n$ there exists a partition Δ_n and an operator $F(\Delta_n)$ so that

$$\| F(\Delta_n) \| = 1.$$

Finally Bernstein (cf. [1]) has shown that for an arbitrary $\alpha > 0$ and $q(n) = [\alpha n]$ there exists a partition Δ_n and operators $F(\Delta_n)$ so that

$$\| F(\Delta_n) \| \leqq O(1).$$

It is worth mentioning that in all of the above-stated results the optimal rate of convergence was obtained for Δ_n to be Chebyshev points on $[-1, 1]$ and for the *linear* operators $F(\Delta_n)$. The gap between $q(n) = 0$ and $q(n) = \alpha n$ remained open until recently Szabados [5] demonstrated that for Δ_n Chebyshev and $q(n)$ arbitrary we have

$$\limsup \left[\frac{\| F(\Delta_n) \|}{\log(n/q(n))}\right] > 0 \tag{1.1}$$

and moreover the optimal rate of convergence is obtained on linear operators $F(\Delta_n)$.

In the same paper [5], Szabados conjectured that (1.1) holds for arbitrary choice of Δ_n.

In this paper we give an affirmative solution to the conjecture of Szabados for linear operators $F(\Delta_n)$ for an arbitrary choice of Δ_n.

In fact we will do a little bit more. We will show that (1.1) holds for a slightly more general choice of linear operators and with lim sup in (1.1) being replaced by lim, thus closing the gap between the Faber result and the Bernstein theorem.

Our method of proof is based on Functional Analysis and completely different from the one in [5].

It will be more convenient for us to work with trigonometric functions rather than algebraic polynomials. Hence, our operators will be the linear operators

$$F(\Delta_n) : \tilde{C}_{[-1,1]} \to \mathcal{T}_{n+q(n)}$$

where $\mathcal{T}_k$ is the cosine polynomials of degree k and $\tilde{C}_{[0,\pi]}$ is the space of continuous function on $[0, \pi]$ with $f(0) = f(\pi)$. That transformation is easily accomplished by the map

$$C_{[-1,1]} \xrightarrow{\mathcal{J}} \tilde{C}_{[0,\pi]}, \quad \mathcal{J} : f \to \varphi \quad \text{with } \varphi(0) = f(\cos \theta).$$

The isometry $\mathscr{J}$ maps algebraic polynomials into cosine polynomials and $\cos\theta$ maps partitions of $[0, \pi]$ into partions on $[-1, 1]$. Hence, the algebraic case is equivalent to the trigonometric case.

Throughout this paper we will identify the dual space of $C_{[a,b]}$ with the space of regular Borel measures on $[a, b]$: $\mathscr{M}(a, b)$ and will treat the elements of this space as measures or as functionals without a warning. We also reserve a letter C to denote various constants.

In Section 2 we formulate some propositions and the main theorem. We also prove the main theorem with the help of the propositions.

We postpone the proof of the propositions to Section 3.

The last section of the paper is dedicated to some additional remarks and problems.

II. The Main Theorem

Let $\{\nu_1, \ldots, \nu_n\}$ be a set of positive normalized Borel measures on $[0, \pi)$ with pairwise disjoint supports.

THEOREM 1. *Let F_n be a linear operator from $\tilde{C}_{[0,\pi]}$ into $\mathscr{T}_{n+q(n)}$ such that $\nu_j(Ff) = \nu_j(f)$ for all $f \in \tilde{C}_{[0,\pi)}$. Then there exists a universal constant $C > 0$ such that*

$$\| F_n \| \geqq C \cdot \log\left(\frac{n}{q(n)}\right). \tag{2.0}$$

(*The constant C does not depend on the choice of ν_j.*)

Before proving this theorem we will need two propositions:

PROPOSITION 1 (cf. [4]). *For any sequence of reals $a_1, \ldots, a_k$*

$$\int_0^\pi \left|\sum_1^k a_j \cos j\theta\right| d\theta \geqq \frac{1}{\pi} \cdot \sum \frac{|a_j|}{k-j+1}.$$

PROPOSITION 2. *Let a and M be $n \times (n + q(n))$ and $(n + q(n)) \times n$ matrices respectively*:

$$A = \begin{bmatrix} a_{11}; & a_{12} & \ldots; & a_{1,n+q(n)} \\ a_{21}; & a_{22}; & \ldots; & a_{2,n+q(n)} \\ \vdots & \vdots & & \vdots \\ a_{n1}; & a_{n2}; & \ldots; & a_{n,n+q(n)} \end{bmatrix}; \quad M = \begin{bmatrix} \mu_{11}; & \mu_{12}; & \ldots; & \mu_{1,n} \\ \mu_{21}; & \mu_{22}; & \ldots; & \mu_{2,n} \\ \vdots & \vdots & & \vdots \\ \mu_{n+q(n),1}; & \mu_{n+q(n),2}; & \ldots; & \mu_{n+q(n),n} \end{bmatrix}$$

Suppose that $q(n) < n$; $|a_{ij}| \leqq 1$ $(i = 1, \ldots, n;\ j = 1, \ldots, n + q(n))$ *and* $A \cdot M = I$, *an identity matrix on* $\mathbf{R}_n$.

Then there are $n - q(n)$ *rows of the matrices* $M : k_1, \ldots, k_{n-q(n)}$ *so that*

$$\sum_{j=1}^{n} |\mu_{k_i j}| \geqq \tfrac{1}{2} \qquad \text{for } i = 1, \ldots, n - q(n). \tag{2.1}$$

We postpone the proofs of the propositions till the next section.

REMARK. The meaning of Proposition 2 becomes clear by considering the case $q(n) = 0$. Then A and M are square matrices with $AM = I$, hence $MA = I$ and for every row in M, say $(\mu_{i1}, \mu_{i2}, \mu_{in})$, we have

$$1 = \Sigma a_{1i}\mu_{i1} \leqq (\Sigma |\mu_{ia}|) \cdot \max |a_{1i}| \leqq \Sigma |\mu_{i1}|.$$

Hence there are $n - n(q) = n$ rows with the property (2.1).

PROOF OF THE THEOREM 1. Let $F_n : \tilde{C}_{[0,\pi]} \to \mathcal{T}_{n+q(n)}$ be an arbitrary linear operator. Then it can be represented in the form

$$F_n(f) = \sum_{j=0}^{n+q(n)} \mu_j(f) \cos j\theta = \sum_{j=0}^{n+q(n)} \int_0^{\infty} f(s) \cos j\theta \, d\mu_j(s) \tag{2.2}$$

where μ_j are regular Borel measure on $[0, \pi)$. Let $\sigma = \Sigma_{\sigma}^{n+q(n)} |\mu_j|$. Then each μ_j is absolutely continuous with respect to σ and by the Radon–Nikodym theorem there exist functions $\varphi_j \in L_1(\sigma)$ so that

$$d\mu_j = \varphi_j d\sigma \quad \text{and} \quad \| \varphi_j \|_{L_1(\sigma)} = \| \mu_j \|.$$

Now we can rewrite (2.2) as

$$F_n(f) = \sum_{j=0}^{n+q(n)} \int f(s)\varphi_j(s) \cos j\theta \, d\sigma(s) = \int_0^{\pi} K(s, \theta) f(s) d\sigma(s)$$

where $K(s, \theta) = \Sigma_{j=0}^{n+q(n)} \varphi_j(s) \cos j\theta$.

Hence, F_n is an integral operator from $C_{[0,\pi]} \to C_{[0,\pi]}$ and we have

$$\| F_n \| = \sup_{\theta \in [0,\pi]} \int_0^{\pi} |K(s, \theta)| d\sigma(s)$$

$$\geqq \int_0^{\pi} \int_0^{\pi} |K(s, \theta)| d\sigma(s) d\theta$$

$$= \int_0^\pi \int_0^\pi |K(s,\theta)|\,d\theta\,d\sigma(s)$$

$$= \int_0^\pi \left[\int_0^\pi \left| \sum_0^{n+q(n)} \varphi_j(s)\cos j\theta \right| d\theta \right] d\sigma(s).$$

We can now apply Proposition 1 to the inner integral:

$$\| F_n \| \geqq \frac{1}{\pi} \int_0^\pi \left(\sum_{j=0}^{n+q(n)} \frac{|\varphi_j(s)|}{n+q(n)-j+1} \right) d\sigma(s)$$

$$(2.3) \qquad = \frac{1}{\pi} \sum_{j=0}^{n+q(n)} \frac{1}{n+q(n)-j+1} \int |\varphi_j(s)|\,d\sigma(s)$$

$$= \frac{1}{\pi} \sum_{j=0}^{n+q(n)} \frac{\| \mu_j \|}{n+q(n)-j+1}.$$

To estimate the norms $\| \mu_j \|$ we use the fact that

$$(2.4) \qquad \nu_i(F_n f) = \nu_i(f) \qquad \forall f \in \tilde{C}_{[0,\pi]}, \qquad i = 1, \dots, n.$$

From (2.2) we have

$$(2.5) \qquad \nu_i(f) = \nu_i \left(\sum_{j=0}^{n+q(n)} \mu_j(f)\cos j\theta \right) = \sum_{j=0}^{n+q(n)} \mu_j(f) \cdot \nu_i(\cos j\theta).$$

Since $\| \nu_i \| = 1$ we have $|a_{ij}| \leqq 1$ where

$$a_{ij} := \nu_i(\cos j\theta).$$

Hence (2.5) can be written as

$$\nu_i(f) = \left(\sum_{j=0}^{n+q(n)} a_{ij}\mu_j \right)(f)$$

and since the latter identity holds for all $f \in C_{[0,\pi]}$

$$(2.6) \qquad \nu_i = \sum_{j=0}^{n+q(n)} a_{ij}\mu_j.$$

Let S_k be the supp ν_k. Since ν_i are normalized positive and disjointly supported, we have

$$\nu_i(S_k) = \delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

Evaluating the left and right side of (2.6) on the sets S_k we have

$$\delta_{ik} = \sum_{j=0}^{n+q(n)} a_{ij}\mu_j(S_k) = \sum_{j=0}^{n+q(n)} a_{ij}\mu_{jk} \qquad \text{for } 1 \leqq i, k \leqq n, \tag{2.7}$$

where

$$\mu_{jk} = \mu_j(S_k). \tag{2.8}$$

Notice that

$$\| \mu_j \| \geqq \sum_{k=1}^{n} |\mu_j(S_k)| = \sum_{k=1}^{n} |\mu_{jk}|. \tag{2.9}$$

Equations (2.7) can be seen as a matrix equation $AM = I$ where A and M satisfy the conditions of the Proposition 2.

Hence there are $n - q(n)$ rows in $M: j_1, \ldots, j_{n-q(n)}$ (and consecutively $n - q(n)$ functionals $\mu_{j_1}, \ldots, \mu_{j_{n-q(n)}}$) so that

$$\| \mu_{j_i} \| \geqq \sum_{i=1}^{n} |\mu_{j_i i}| \geqq \frac{1}{2}. \tag{2.10}$$

Returning to the estimate (2.3) we see that the norm $\| F_n \|$ is least when $\| \mu_j \| \geqq \frac{1}{2}$ are accompanied by the largest denominators, i.e., for some universal $C_1 > 0$

$$\begin{aligned} \| F_n \| &\geqq \frac{1}{2\pi} \sum_{j=0}^{n-q(n)} \frac{1}{n + q(n) - j + 1} \\ &\geqq \frac{1}{2\pi} C_1[\log[n + q(n)] - \log[2q(n)]] \\ &= \frac{C_1}{2\pi} \log \frac{n + q(n)}{2q(n)}, \end{aligned}$$

which immediately implies (2.0) for some universal constant C. □

III. Proofs of the Propositions

Proposition 1 is well-known as a Sidon inequality (cf. [4]). It is also an easy consequence of the Hardy inequality (cf. [2])

$$\int_{-\pi}^{\pi} \left| \sum_{j=1}^{n} a_j e^{ij\theta} \right| d\theta \geqq \frac{1}{\pi} \sum_{1}^{n} \frac{|a_j|}{j}. \tag{3.0}$$

Indeed, writing $\cos j\theta = \frac{1}{2}(e^{ij\theta} + e^{-ij\theta})$ we have

$$\int_{-\pi}^{\pi}\left|\sum_{j=1}^{n} a_j \tfrac{1}{2}(e^{ij\theta}+e^{-ij\theta})\right| d\theta = \int_{0}^{\pi}\left|\sum_{j=1}^{n} a_j(e^{ij\theta}+e^{-ij\theta})\right| d\theta$$

$$= \int_{0}^{\pi}\left|e^{i(n+1)\theta}\sum_{j=1}^{n} a_j(e^{ij\theta}+e^{-ij\theta})\right| d\theta$$

$$= \int_{0}^{\pi}\left|\sum a_j e^{i(n-j+1)\theta}+\sum a_j e^{i(n+j+1)\theta}\right| d\theta$$

$$\geqq \frac{1}{\pi}\sum_{j=1}^{n}\frac{|a_j|}{n-j+1}.$$

To prove Proposition 2 we need some notation. For a Banach space $X_k = (\mathbf{R}_k, \|\ \|)$ we use $\|\ \|^*$ to denote the norm of the dual space $X_k^* = (\mathbf{R}_k, \|\ \|^*)$. Vectors $e_j \in \mathbf{R}_k$ $(j = 1, \dots, k)$ will be the standard basic vectors $e_j = (\delta_{ij})_{i=1}^k$. For an arbitrary $x \in \mathbf{R}_k$ we use $x(i)$ to denote the i-th coordinate at the vector x. If $x, y \in \mathbf{R}_k$ then $\langle x, y\rangle = \Sigma_{i=1}^k x(i)y(i)$ is the inner product of x and y. Hence $x(i) = \langle x, e_i\rangle$. Among all the different norms on $\mathbf{R}_k$ let

$$\| x \|_1 = \sum |x(i)|,$$

$$\| x \|_\infty = \sup(|x(i)|; 1 \leqq i \leqq k).$$

These norms give rise to the spaces

$$l_1^{(k)} = (\mathbf{R}_k, \|\ \|_1); \qquad l_\infty^{(k)} = (\mathbf{R}_k, \|\ \|_\infty).$$

We also need the following

Lemma 1. *$X_m = (\mathbf{R}_m, \|\ \|)$ be an m-dimensional space. Let U be a linear operator from $l_1^{(k)} \to X_m$ and V be a linear operator from $X_m \to l_1^{(k)}$ such that*

$$\| U \| \leqq 1 \quad \text{and} \quad UV = I_{X_m}.$$

Then

$$\max\{\| Vx \|_\infty, \| x \| \leqq 1, x \in X_m\} \geqq m/k.$$

Proof. We use matrix representations for U and V with respect to the standard bases in $\mathbf{R}_k$:

$$U = \begin{bmatrix} u_{11}, & u_{12}, & \dots, & u_{1k} \\ u_{21}, & u_{22}, & \dots, & u_{2k} \\ \vdots & \vdots & & \vdots \\ u_{m1}, & u_{m2}, & \dots, & u_{mk} \end{bmatrix}; \qquad V = \begin{bmatrix} v_{11}, & v_{12}, & \dots, & v_{1m} \\ v_{21}, & v_{22}, & \dots, & v_{2m} \\ \vdots & \vdots & & \vdots \\ v_{k1}, & v_{k2}, & \dots, & v_{km} \end{bmatrix}.$$

(It follows from the assumptions of the lemma that $k > m$.) Let

$$u_j = (u_{ij})_{i=1}^m \in \mathbf{R}_m; \qquad j = 1, \ldots, k$$

be the column vectors of the matrix U.

Let $v_i = (v_{ij})_{j=1}^m \in \mathbf{R}_m$ $(i = 1, \ldots, k)$ be the row vectors of the matrix V. Since $\| U \| \leqq 1$ we have

$$1 \geqq \| Ue_j \| = \| u_j \| \qquad \text{for all } j = 1, \ldots, k.$$

Since $UV = I_{X_m}$ we moreover have

$$m = \operatorname{tr}(UV) = \operatorname{tr}(VU) = \sum_{i=1}^k \langle v_i, u_i \rangle \leqq \sum_{i=1}^k \| v_i \|^* \| u_i \| \leqq \sum_{i=1}^k \| v_i \|^*.$$

Hence there exists an $i_0 \in \{1, \ldots, k\}$ so that

$$\| V_{i_0} \|^* \geqq m/k.$$

Let $x_0 \in X_m$ with $\| x_0 \| = 1$ so that

$$\langle x_0, v_{i_0} \rangle = \| v_{i_0} \|^* \geqq k/m.$$

Then

$$\max(\| Vx \|_\infty;\ \| x \| = 1\} \geqq |(Vx_0)(i_0)| = \langle v_{i_0}, x_0 \rangle \geqq m/k. \qquad \square$$

We are now in a position to give the

Proof of Proposition 2. Consider the diagram

$$l_\infty^{(n)} \xleftarrow{\ A\ } l^{(n+q(n))} \xleftarrow{\ M\ } l_\infty^{(n)}.$$

The assumption $|a_{ij}| \leqq 1$ means that $\| A \| \leqq 1$.

(3.1) Let $K = \{i : \max(|(Mx)(i)|; x \in l_\infty^{(n)}, \| x \|_\infty \leqq 1) \geqq \frac{1}{2}; i = 1, \ldots, n + q(n)\}$.

The conclusion of the proposition is equivalent to

$$\# K \geqq n - q(n).$$

Indeed if $i \in K$ then there exists $x = (x_1, \ldots, x_n) \in l_\infty^{(n)}$ such that $\| x \|_\infty = 1$ and $\Sigma_{j=1}^n \mu_{ij} x_j > \frac{1}{2}$. Thus

$$\frac{1}{2} \leqq \sum_{j=1}^n \mu_{ij} x_j \leqq \sum_{i=1}^n |\mu_{ij}|.$$

In particular it follows from Lemma 1 that there exists $i \in \{1, \ldots, n + q(n)\}$ such that

$$\max\{|(Mx)(i)|;\ \|x\|_\infty \leqq 1\} \geqq \frac{n}{n+q(n)} \geqq \frac{1}{2}$$

and $\#K \geqq 1$.

(3.2) Let $k = \#K$ and let $k < n - q(n)$.

We introduce a subspace $E \subset l_1^{(n+q(n))}$ to be

$$E = \{x \in l_1^{(n+q(n))} : x(i) = 0 \text{ if } i \in K\}.$$

The subspace E is clearly isometric to $l_1^{(n+q(n)-k)}$. Indeed let $(i_1, \ldots, i_{n+q(n)-k})$ be an ordered set of integers such that $i_m \in K$. Then the natural map

$$(Rx)(m) = x(i_m)$$

defines the isometry. Then $\dim E = n + q(n) - k$; $\dim(\text{Range } M) = n$ and

$$l := \dim(E \cap (\text{Range } M)) \geqq n - k.$$

Let Y_l be a subspace of $l_\infty^{(n)}$ such that

$$M(Y_l) = E \cap \text{Range } M.$$

There exists a space $X_l = (\mathbf{R}_l, \|\ \|)$ such that X_l is isometric to the space Y_l. We now consider the diagram

$$X_l \xleftarrow{J^{-1}} Y_l \xleftarrow{\tilde{A}} E \xleftarrow{R^{-1}} l_1^{(n+q(n)-k)} \xleftarrow{R} E \xleftarrow{\tilde{M}} Y_l \xleftarrow{J} X_l$$

where J is the isometry from X_l onto Y_l, $\tilde{M}$ is the restriction of M onto Y_l, R is a natural isometry from E onto $l_1^{(n+q(n)-k)}$, and $\tilde{A}$ is a restriction of A onto E.

Clearly $\|\tilde{A}\| \leqq \|A\| \leqq 1$ and for

$$U = J^{-1}\tilde{A}R^{-1} \quad \text{and} \quad V = R\tilde{M}J$$

we have

$$\|U\| \leqq 1; \qquad UV = I_{X_l}.$$

Hence by Lemma 1 there exists $x \in X_l$; $\|x\| \leqq 1$, $i \in (1, \ldots, n + q(n) - k)$ such that

$$|(R\tilde{M}Jx)(i)| \geqq \frac{l}{n+q(n)-k} \geqq \frac{n-k}{n+q(n)-k}$$

and consecutively there exists $i_0 \notin K$ so that

$$|(\tilde{M}(Jx))(i_0)| \geqq \frac{n-k}{n+q(n)-k} \geqq \frac{1}{2}.$$

(The last inequality follows from $k < n - q(n)$.) Since $\| Jx \|_\infty = \| x \|$ we have for $x_0 = Jx : \| x_0 \| \leqq 1$

$$|(Mx)(i_0)| = |(\tilde{M}Jx)(i_0)| \geqq \tfrac{1}{2}.$$

which contradicts (3.1) since $i_0 \notin K$. □

IV. Problems and Remarks

With a little finesse the assumption of positivity of the functionals ν_j in Theorem 1 can be dropped. Indeed evaluating (2.6) on the various subsets of supp ν_j would give us the same conclusion.

PROBLEM. Does Theorem 1 hold if we do not assume that ν_j are disjointly supported?

A theorem similar to the Theorem 1 can be proved for complex polynomials on the circle.

THEOREM 2. *Let* $\mathbf{T}$ *be the unit circle and let* $\nu_1 \dots, \nu_n$ *be normalized disjointly supported measures on* $\mathbf{T}$. *Let* F_n *be a linear operator from* $C(\mathbf{T})$ *onto* $\mathcal{P}_{n+q(n)}$ *such that* $\nu_j(F_n f) = \nu_j(f)$ *for all* $f \in C(\mathbf{T})$. *Then*

$$\| F_n \| \geqq C \cdot \log \frac{n}{q(n)}.$$

The proof is identical to the one in Theorem 1 if we use inequality (3.0) instead of Proposition 1.

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