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CONSTRUCTIVE APPROXIMATION

Discrete Approximating Operators on Function Algebras

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Abstract. We give a new presentation and various extensions of one theorem of Somorjai. For any sequence of operators L_n, given by $L_n f = \sum_{k=1}^{n} f(z_{n,k}) l_{n,k}$ with $z_{n,k} \in \mathbf{T}$ and $l_{n,k} \in A(\mathbf{T})$, there exists a function $f \in A(\mathbf{T})$ such that $L_n f$ does not converge to f.

1. Introduction

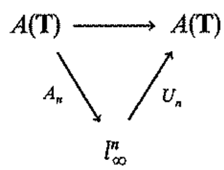
The main purpose of this paper is to give a new presentation, as well as some extensions, of the result obtained in Somorjai's paper [So].

Let $\mathbf{T} = \{z \in \mathbf{C}: |z| = 1\}$ be the unit circle and let $A(\mathbf{T})$ be the disk algebra. Linear operators $L_n: A(\mathbf{T}) \to A(\mathbf{T})$ are called discrete if they are of the form

$$L_n f = \sum_{k=1}^{n} f(z_{n,k}) l_{n,k}, \tag{1.1}$$

where $z_{n,k} \in \mathbf{T}$; $l_{n,k} \in A(\mathbf{T})$. Somorjai [So] gave an elegant proof that for any sequence $\{L_n\}$ of discrete operators, there exists a function $f \in A(\mathbf{T})$ such that $L_n f$ does not converge to f in the topology of $A(\mathbf{T})$. The proof uses the translation-invariant property of $A(\mathbf{T})$.

Our analysis traces this result to some Banach space properties of $A(\mathbf{T})$, hence lends itself to extensions of the theorem to more general domains and more general function algebras. We consider (using the Rudin–Carleson theorem) L_n as a composition of two maps



where

$$A_n f = \{f(z_{n,k})\} \in l_\infty^n; \qquad U_n\{\zeta_k\} = \sum_{k=1}^{n} \zeta_k l_{n,k} \in A(\mathbf{T}).$$

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Thus L_n has a natural factorization through l_∞ space. An easy proposition shows that if a sequence of operators L_n on a Banach space $\mathbf{X}$ factors through l_∞ and serves as a nice approximation on $\mathbf{X}$ (i.e., $L_n x \to x$ for all $x \in \mathbf{X}$), then $\mathbf{X}$ must inherit certain properties of l_∞, namely $\mathbf{X}$ must be an $\mathscr{L}_\infty$ space. The Somorjai result follows from the fact that $A(\mathbf{T})$ is not an $\mathscr{L}_\infty$ space.

In the next section we observe some simple facts related to approximation by operators that factor through $\mathscr{L}_\infty$ spaces. In Section 3 we consider approximation on various subspaces $\mathbf{X} \subset C(K)$ for which some analogs of the F. and M. Riesz theorem holds. On the one hand, this theorem implies the Rudin–Carleson theorem (see [B2]) and hence gives us the factorization of operators. On the other hand, it implies (see [P2], Corollary 5.1) that the space $\mathbf{X}$ is not an $\mathscr{L}_\infty$ space.

We use the rest of this section to state the theorem of Somorjai in full strength.

Definition 1. Let $\mathbf{X}$ be a subspace of $C(K)$. Let L be a linear operator on $\mathbf{X}$ and let $H \subset K$ be a subset of K. We say that L is determined on H if $Lf = Lg$ for all $f, g \in \mathbf{X}$, such that $f(k) = g(k)$ for all $k \in H$ (*i.e.*, $f|H = g|H$).

Theorem 1 (see [So]). *Let H_n be closed subsets of $\mathbf{T}$ of Lebesgue measure zero. Let L_n: $A(\mathbf{T}) \to A(\mathbf{T})$ be linear operators that are determined on H_n. Then there exists a function $f \in A(\mathbf{T})$ such that $L_n f$ does not converge to f.*

2. $\mathscr{L}_\infty$ Spaces

We use l_∞^n to denote $\mathbf{C}^n$ equipped with the norm $\|(x_j)\|_\infty = \max|x_j|$. We define the Banach–Mazur distance from an arbitrary n-dimensional Banach space E to l_∞^n as

$$d(E, l_\infty^n) = \inf\{\|T\|\,\|T^{-1}\|: T \text{ is an isomorphism from } E \text{ onto } l_\infty^n\}.$$

It is well known (see [LT]) that

$$d(E, l_\infty^n) \le n \tag{2.1}$$

for all E.

The next proposition is also well known (see [LT]).

Proposition 1. *Let E be an n-dimensional subspace of a Banach space $\mathbf{X}$. Then there exists a projection P from $\mathbf{X}$ onto E such that*

$$\|P\| \le d(E, l_\infty^n). \tag{2.2}$$

Definition 2. Let $1 \le \lambda < \infty$. A Banach space $\mathbf{X}$ is said to be an $\mathscr{L}_{\infty,\lambda}$ space if for every finite dimensional subspace $E \subset \mathbf{X}$ there exists a finite dimensional subspace $F \subset \mathbf{X}$ such that $E \subset F$ and

$$d(F, l_\infty^m) \le \lambda \qquad \text{where} \quad m = \dim F.$$

A Banach space $\mathbf{X}$ is an $\mathscr{L}_\infty$ space if $\mathbf{X}$ is an $\mathscr{L}_{\infty,\lambda}$ space for some $\lambda < \infty$.

Remark 1 (see [LR]). For every $\varepsilon > 0$ the spaces l_∞, $C(K)$, $L_\infty(\mu)$ are $\mathscr{L}_{\infty,1+\varepsilon}$ spaces.

Remark 2 (see [LT], II.3.1). Let $\mathbf{X}$ be a separable Banach space. Then $\mathbf{X}$ is an $\mathscr{L}_{\infty,\lambda}$ space if and only if $\mathbf{X} = \overline{UE_n}$ where $\dim E_n = n$, $E_n \subset E_{n+1} \subset \mathbf{X}$, and $d(E_n, l_\infty^n) < \lambda$.

Theorem 2 (see [LR], Theorem 4.3). *A Banach space* $\mathbf{X}$ *is an* $\mathscr{L}_\infty$ *space if and only if there exist constants* $\lambda, K \geq 1$, *such that for every finite dimensional subspace* $E \subset \mathbf{X}$ *there exists an* $\mathscr{L}_{\infty,\lambda}$ *space* Y *and operators* $A\colon E \to Y$, $B\colon Y \to \mathbf{X}$ *such that* $\|A\|\|B\| \leq K$ *and* $BAe = e$ *for all* $e \in E$.

The main tool in our investigation is the following:

Theorem 3. *A Banach space* $\mathbf{X}$ *is an* $\mathscr{L}_\infty$ *space if and only if there exists* $\lambda \geq 1$, $K \geq 1$, *a sequence of* $\mathscr{L}_{\infty,\lambda}$ *spaces* Y_n, *and a sequence of linear operators* $A_n\colon \mathbf{X} \to Y_n$ *and* $U_n\colon Y_n \to \mathbf{X}$ *such that*

$$U_n A_n x \to x \qquad \textit{for all} \quad x \in \mathbf{X}$$

and $\|U_n\|\|A_n\| \leq K$.

Proof. If $\mathbf{X}$ is an $\mathscr{L}_\infty$ space we choose $Y_n := \mathbf{X}$; $A_n = U_n = I$. Conversely, let E be a finite dimensional subspace of $\mathbf{X}$ with $\dim E = N$. We use a standard perturbation argument (see [LT], p. 198). By (2.2) there exists a basis $e_1, \ldots, e_N \in E$ so that

$$\frac{1}{\sqrt{N}} \max|\lambda_j| \leq \|\textstyle\sum \lambda_j e_j\| \leq \sqrt{N} \max|\lambda_j| \tag{2.3}$$

for all choices of $\lambda_1, \ldots, \lambda_N \in \mathbf{C}$. Let $1 > \delta > 0$. Pick $\varepsilon = \delta/2N^{3/2}$ and choose n so large that for $f_j := U_n A_n e_j$ we have $\|f_j - e_j\| < \varepsilon$. From

$$\|\textstyle\sum \lambda_j(f_j - e_j)\| \leq \dfrac{1}{2\sqrt{N}} \max|\lambda_j| \leq \|\textstyle\sum \lambda_j e_j\|$$

it now follows that

$$\frac{1}{2\sqrt{N}} \max|\lambda_j| \leq \|\textstyle\sum \lambda_j f_j\| \leq 2\sqrt{N} \max|\lambda_j|. \tag{2.4}$$

Let $F = \operatorname{span}\{f_j\} \subset \mathbf{X}$. Define functionals $\tilde{\mu}_k$ on F by $\tilde{\mu}_k(f_j) = \delta_{kj}$, $k = 1, \ldots, N$, $j = 1, \ldots, N$. By (2.4)

$$\|\tilde{\mu}_k\| \leq 2\sqrt{N}. \tag{2.5}$$

Let μ_k be Hahn–Banach extensions of $\tilde{\mu}_k$ onto $\mathbf{X}$. Define $T\colon \mathbf{X} \to \mathbf{X}$ by

$$Tx = x + \textstyle\sum \mu_k(x)(e_k - f_k). \tag{2.6}$$

Observe that $Tf_k = e_k$ for all $k = 1, \ldots, N$. It also follows from (2.6) that

$$\|Tx\| \le (1 + \delta)\|x\|.$$

Hence the operators $A := A_n$ and $B := TU_n$ satisfy the condition of Theorem 2 and $\mathbf{X}$ is an $\mathscr{L}_\infty$ space. ■

For convenience we introduce:

Definition 3. Let $\mathbf{X}$ be a subspace of a Banach space Y. We say that $\mathbf{X}$ is *near-complemented* in Y if there exists a sequence of operators $L_n: Y \to \mathbf{X}$ such that $\|L_n\|$ are uniformly bounded and $L_n x \to x$ for all $x \in \mathbf{X}$. We say that $\mathbf{X}$ is *locally complemented* in Y if there exists a sequence of finite dimensional operators $L_n: Y \to \mathbf{X}$ such that $\|L_n\|$ are uniformly bounded and $L_n x \to x$ for every $x \in \mathbf{X}$.

Theorem 4. *Let K be a compact metric space and let $\mathbf{X}$ be a subspace of $C(K)$. The following are equivalent*:

(a) $\mathbf{X}$ *is an* $\mathscr{L}_\infty$ *space*;
(b) $\mathbf{X}$ *is locally complemented in* $C(K)$;
(c) $\mathbf{X}$ *is near-complemented in* $C(K)$.

Proof. If $\mathbf{X}$ is an $\mathscr{L}_\infty$ space, then since $\mathbf{X}$ is separable there exists a sequence of spaces $E_n \subset E_{n+1} \subset \mathbf{X} \subset C(K)$ such that $\overline{\cup E_n} = \mathbf{X}$, $d(E_n, l_\infty^n) \le \lambda$. By Proposition 1 we can find a sequence of projections P_n from $C(K)$ onto E_n such that $\|P_n\| \le \lambda$. Clearly, $P_n x \to x$ for all $x \in \mathbf{X}$. The implication (b) $\Rightarrow$ (c) is trivial. To prove (c) $\Rightarrow$ (a) let $J: \mathbf{X} \to C(K)$ be a natural embedding. We now use Theorem 3 with $A_n = J$, $U_n = L_n$, and $Y_n = C(K)$. ■

Remark 3. If $\mathbf{X}$ is a complemented subspace of $C(K)$ then (see [LR], Theorem 3.2) it is an $\mathscr{L}_\infty$ space. The converse to that statement does not hold. Indeed we can find (see [LT], Proposition II.4.40) a subspace $\mathbf{X} \subset C_{[0,1]}$ which is an $\mathscr{L}_\infty$ space yet has no complement in $C_{[0,1]}$. Hence the near-complemented subspaces form a larger class of subspaces than the complemented subspaces.

The argument in [So] and the remarks after the proof seem to indicate that all that was needed is the fact that $A(\mathbf{T})$ is not complemented in $C(\mathbf{T})$. This inconsistency with Theorem 4 can be explained by translation-invariant properties of $A(\mathbf{T})$. Indeed, for any compact abelian group G a translation invariant subspace $\mathbf{X} \subset C(G)$ is complemented if and only if $\mathbf{X}$ is near-complemented if and only if $\mathbf{X}$ is an $\mathscr{L}_\infty$ space if and only if $\mathbf{X}$ is spanned by the characters in the dual group G from a coset ring in $\hat{G}$ (see [KP], pp. 311–312).

3. Extensions of Somorjai's Theorem

In this section we will extend Theorem 1 in several directions. The idea is to check that a given subspace $\mathbf{X} \subset C(K)$ is not an $\mathscr{L}_\infty$ space on the one hand, and $\mathbf{X}$ verifies some analog of the Rudin–Carleson theorem on the other. Fortunately, there are conditions that imply both statements. One such condition is an F. and M. Riesz theorem. Here is a direct generalization of Theorem 1.

Theorem 5. *Let K be the closure of a domain $D \subset \mathbf{C}$ whose boundary Γ consists of a finite number of nonintersecting analytic closed curves. Let $\mathbf{X}$ be a subspace of $C(\Gamma)$ that consists of all functions in $C(\Gamma)$ that have analytic continuation in D. Let $H_n \subset \Gamma$ be closed sets of Lebesgue measure zero. Finally, let $L_n: \mathbf{X} \to \mathbf{X}$ be determined on H_n. Then there exists a function $f \in \mathbf{X}$ such that $L_n f$ does not converge to f.*

Proof. Let μ be a regular Borel measure on $\mathbf{X}$ such that $\int f\, d\mu = 0$ for all $f \in \mathbf{X}$. Then (see [R1], Theorem 3) the measure μ is absolutely continuous with respect to Lebesgue measure. Now that implies (see [P2], Corollary 5.1) that $\mathbf{X}$ is not an $\mathscr{L}_\infty$ space. On the other hand, the absolute continuity of μ also implies (see [B2]) that for any function $g \in C(H_n)$ there exists a function $f \in \mathbf{X}$ such that $f(t) = g(t)$ for all $t \in H_n$ (i.e., $f|H_n = g|H_n$) and $\|f\| \le \|g\|$. (This is the Rudin–Carleson theorem.)

Let $L_n: \mathbf{X} \to \mathbf{X}$ be determined on H_n. Then for each $g \in C(\Gamma)$ we can define $\tilde{L}_n g$ to be $L_n f$ where $f \in \mathbf{X}$ is such that $g|H_n = f|H_n$. (Since L_n are determined on H_n the value $L_n f$ does not depend on the choice of f.) Hence $\|\tilde{L}_n\| = \|L_n\|$.

Suppose that $L_n f \to f$ for all $f \in \mathbf{X}$. Then $\|L_n\|$ are uniformly bounded. Then the norms of $\tilde{L}_n: C(\Gamma) \to \mathbf{X}$ are also uniformly bounded. If $\tilde{L}_n f \to f$ for all $f \in \mathbf{X}$ then $\mathbf{X}$ is near-complemented and is hence an $\mathscr{L}_\infty$ space. We have the desired contradiction. ∎

Remark 4. The cited result of Pelczynski actually states that $\mathbf{X}$ does not have local unconditional structure. That clearly implies that $\mathbf{X}$ is not an $\mathscr{L}_\infty$ space.

We now prove another generalization of Theorem 1 where the analyticity of the boundary is not required. Let $\bar{\mathbf{C}}$ denote the extended complex plane.

Theorem 6. *Let K be a compact set in $\mathbf{C}$ with nonempty connected interior and connected complement such that the boundary Γ of K is accessible from the complement $G := \bar{\mathbf{C}} \setminus K$ through Jordan curves, i.e., every point $z \in \Gamma$ is the endpoint of the Jordan curves contained in $G \cup \{z\}$. Let $A(\Gamma)$ be the subalgebra of $C(\Gamma)$ of functions analytic in the interior of K. Then for every sequence of operators L_n defined by*

$$L_n f = \sum_{k=1}^{n} f(z_{n,k}) l_{n,k}, \qquad z_{n,k} \in \Gamma, \quad l_{n,k} \in A(\Gamma), \tag{3.1}$$

there exists a function $f \in A(\Gamma)$ such that $L_n f$ does not converge to f.

Proof. It follows from the Rudin–Carleson theorem and the Carathéodory extension method (see [S–To], Proof of Lemma 1) that for any finite sequence of points $z_1, \ldots, z_n \in \Gamma$ and for any set of complex numbers $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ with $|\alpha_j| \le 1$, there exists a function $f \in A(\Gamma)$ such that $f(z_j) = \alpha_j$ for $j = 1, \ldots, n$ and $\|f\| \le 1$.

Since the operators L_n are determined on the sets $\{z_{n,1}, \dots, z_{n,n}\}$ they can be extended to operators $\tilde{L}_n$ on $C(\Gamma)$ such that $\|\tilde{L}_n\| \le \|L_n\|$. Hence, if $L_n f \to f$ for all $f \in A(\Gamma)$, then $A(\Gamma)$ is near-complemented in $C(\Gamma)$ and thus is an $\mathscr{L}_\infty$ space. On the other hand, Bishop (see [B1], Theorem 3) proved an analog of the F. and M. Riesz theorem for $A(\Gamma)$, and using the same result of Pelczynski (see [P2], Corollary 5.1) we learn that $A(\Gamma)$ is not an $\mathscr{L}_\infty$ space. ∎

Our final result extends Theorem 1 to several variables.

Theorem 7. *Let $U^N = \{(z_1, \dots, z_n) \in \mathbf{C}^N : |z_j| < 1\}$. Let $A(U^N)$ be the algebra of all functions which are holomorphic in the polydisk U^N and continuous on its closure $\bar{U}^N$.*

Let $H_n \subset H_1^n \times H_2^n \times \cdots \times H_N^n$ where H_j^n are closed subsets of $\mathbf{T}$ with Lebesgue measure zero. Let operators L_n: $A(U^N) \to A(U^N)$ be determined on H_n. Then there exists a function $f \in A(U^N)$ such that $L_n f$ do not converge to f.

Proof. An appropriate analog of the Rudin–Carleson theorem can be found in [R2], Example 6.3(8). The fact that $A(U^N)$ is not an $\mathscr{L}_\infty$ space is proved in [P2], Theorem 11.2. ∎

Remark 5. More exotic extensions of Theorem 1 can be obtained by combining the "Main Theorem" and its corollaries in [P1] with the results of Sections 5, 10, and 11 of [P2].

The extensions of Theorem 1 to certain translation-invariant subspaces on general compact Abelian groups can be obtained using the results in [KP], Section 2, in combination with the extensions of the F. and M. Riesz theorem (see [DLG]) as well as with Corollaries 1 and 2 of [P1].

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