On Simultaneous Interpolation of Two Functions

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Abstract. Given an arbitrary continuous function $f \in C([-1,1])$ and $g \in Lip\,\alpha$, we show that there exists $\Delta_n \subset [-1,1]$ such that

$$L(\Delta_n)\, f \to f \text{ and } L(\Delta_n)\, g \to g,$$

where $L(\Delta_n)$ is the usual Lagrange interpolating projection at the points Δ_n.

Let Δ_n be a set of n distinct points in the interval $[-1,1]$ and let P_n stand for the space of algebraic polynomials of degree $n-1$. We use $L(\Delta_n)$ to denote the Lagrange interpolation projection from $C([-1,1])$ onto P_n.

The following problem had been haunting me for several years and is still at large (cf. [2]):

Problem. *Given two functions $f,\ g \in C([-1,1])$. Does there exist a sequence $\{\Delta_n\}$; $\Delta_n \subset [-1,1]$ such that*

$$L(\Delta_n)\, f \to f \text{ and } L(\Delta_n)\, g \to g?$$

In this note I wish to present a partial result:

Theorem 1. *Let $f \in C([-1,1])$ and let $g \in Lip\,\alpha$ for some $\alpha > 0$. Then there exists a sequence $\{\Delta_n\}$ such that*

$$L(\Delta_n)\, f \to f \text{ and } L(\Delta_n)\, g \to g.$$

Here the convergence is in the topology of $C([-1,1])$.

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Proof: We first fix the notations. In what follows T_n will stand for the Chebyshev polynomial of degree n; $z_1, \dots, z_n$ are the zeroes of T_n. Let $b_n(f) \in \mathrm{P}_n$ be the best approximation from P_n to f, and let $\widetilde{f} = (f - b_n(f))/\left\| f - b_n(f) \right\|$.

Step 1. Consider the equation

$$T_n(x) - \sqrt{\|f - b_n(f)\|}\ \widetilde{f}(x) = 0. \tag{1}$$

This equation has at least n solutions since the function $T_n(x)$ has $(n+1)$ alternating extremas on the interval $[-1, 1]$. Let $x_1, \dots, x_n$ be the solutions of (1) so that x_j has the property

$$|z_j - x_j| = \min\left\{|z_j - x| : x \in \text{ zeroes of } (1)\right\}.$$

Choose $\nu_j, \eta_j \in [0, \pi)$ so that $x_j = \cos\eta_j$; and $z_j = \cos\nu_j$. Since $\sqrt{\|f - b_n(f)\|} \to 0$, we have

$$|\nu_j - \eta_j| = o\left(\frac{1}{n}\right). \tag{2}$$

Let $\widetilde{\Delta}_n$ be the collection of points $x_1, \dots, x_n$.

Step 2. I now claim that for every $\alpha > 0$

$$\left\| L\left(\widetilde{\Delta}_n\right) \right\| = o(n^{\alpha}). \tag{3}$$

This estimate follows from (2) and is fairly standard. I could not find the exact reference, but the technique goes at least as far back as [1]. Basically, we estimate the ratio

$$\left| \prod_{j \neq k} (x_k - x_j) \right| \Big/ \left| \prod_{j \neq k} (z_k - z_j) \right|$$

$$= \prod_{j \neq k} \frac{\left|\cos\left(\nu_k + o\left(\frac{1}{n}\right)\right) - \cos\left(\nu_j + o\left(\frac{1}{n}\right)\right)\right|}{\left|\cos\nu_k - \cos\nu_j\right|}$$

$$\sim \exp\left(o\left(\sum \frac{1}{|k-j|}\right)\right) \sim \exp\left(o\left(\log n\right)\right) \le o\left(n^{\beta}\right)$$

for every $\beta > 0$. Similarly, we estimate

$$\left| \prod_{j \neq k} (x - z_j) \right| \Big/ \left| \prod_{j \neq k} (x - y_j) \right|.$$

Hence for the Lebesgue functions we have

$$\left|\sum_k \prod_{j\neq k} (x-x_j)/(x_k-x_j)\right| \le o\left(n^\beta\right)\left|\sum_k \prod_{j\neq k} (x-z_j)/(z_k-z_j)\right|$$
$$\le o\left(n^\beta\right)\log n = o\left(n^\alpha\right).$$

Step 3. Let x^*_{n+1} be chosen so that

$$\left|\prod_{j=1}^n \left(x^*_{n+1}-x_j\right)\right| = \sup\left\{\left|\prod_{j=1}^n (x-x_j)\right|;\ x\in[-1,1]\right\}.$$

Then for the function

$$\varphi_{n+1}(x) := \prod_{j=1}^n (x-x_j) \Big/ \prod_{j=1}^n \left(x^*_{n+1}-x_j\right) \in \mathcal{P}_n$$

we have

$$\|\varphi_{n+1}\| = 1,\ \varphi_{n+1}\left(x^*_{n+1}\right) = 1;\ \varphi_{n+1}(x_j) = 0. \tag{4}$$

Let $\Delta_{n+1} = \widetilde{\Delta}_n \cup \left\{x^*_{n+1}\right\}$. I wish to show that

$$L\left(\Delta_{n+1}\right) g \to g \text{ for any } g \in Lip\,\alpha. \tag{5}$$

Observe that by (3)

$$L\left(\widetilde{\Delta}_n\right) g \to g. \tag{6}$$

Now by (4)

$$L\left(\widetilde{\Delta}_n\right) g - L\left(\Delta_{n+1}\right) g = L\left(\Delta_{n+1}\right)\left(L\left(\widetilde{\Delta}_n\right) g - g\right)$$
$$= \left(\left(L\left(\widetilde{\Delta}_n\right) g\right)\left(x^*_{n+1}\right) - g\left(x^*_{n+1}\right)\right)\varphi_{n+1}$$

and by (4) and (6) we obtain

$$L\left(\Delta_{n+1}\right) g - L\left(\widetilde{\Delta}_n\right) g \to 0$$

which implies (5).

Step 4. It remains to prove that

$$L\left(\Delta_{n+1}\right) f \to f. \tag{7}$$

Since $x_1, \ldots, x_n$ are the roots of (1) we have

$$T_n(x) - \sqrt{\|f - b_n(f)\|}\left(L(\Delta_{n+1})\widetilde{f}\right)(x)$$

$$= L(\Delta_{n+1})\left[T_n - \sqrt{\|f - b_n(f)\|}\,\widetilde{f}\right]$$

$$= \left[T_n\left(x^*_{n+1}\right) - \sqrt{\|f - b_n(f)\|}\,\widetilde{f}\left(x^*_{n+1}\right)\right]\varphi_{n+1}(x).$$

Thus

$$\left\|L(\Delta_{n+1})\,\widetilde{f}\right\| \le (2 + \|T_n\|)\Big/\sqrt{\|f - b_n(f)\|}$$

and

$$\frac{L(\Delta_{n+1})\,f - b_n(f)}{\|f - b_n(f)\|} = L(\Delta_{n+1})\,\widetilde{f}.$$

This implies

$$\|L(\Delta_{n+1})f - b_n(f)\| \le 3 \cdot \sqrt{\|f - b_n(f)\|} \longrightarrow 0$$

and thus

$$L(\Delta_{n+1})f \to f. \quad \blacksquare$$

Actually we have proved a more general

Theorem 2. *Let $f \in C([-1,1])$. Then there exists a sequence $\Delta_n \subset [-1,1]$ such that $L(\Delta_n)f \to f$ and $L(\Delta_n)g \to g$ for every $g \in Lip\,\alpha$ for every $\alpha > 0$.*

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References

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