

Interpolation by Polynomials in Several Variables

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Abstract. A subspace $H \subset C(\mathbb{R}_n)$ is called k -interpolating if for any choice of distinct points $x_1, \dots, x_k \in \mathbb{R}_n$ and for any choice $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ there exists $h \in H$ such that $h(x_j) = \alpha_j, j = 1, \dots, k$. Let Π_{k-1}^n be the space of polynomials of n variables of degree at most $k-1$. We show that for almost all choices $h_1, \dots, h_N \in \Pi_{k-1}^n$ with $N = nk + k$ the span $[h_1, \dots, h_N]$ is k -interpolating.

§1. Preliminaries

Let $C(\mathbb{R}_n)$ be the space of continuous functions on \mathbb{R}_n . Let $H \subset C(\mathbb{R}_n)$ be a linear subspace.

Definition 1. The space H is called k -interpolating if for any k distinct points $x_1, x_2, \dots, x_k \in \mathbb{R}_n$ and for any k values $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ there exists (not necessarily unique) $h \in H$ such that

$$h(x_j) = \alpha_j, \quad j = 1, \dots, k.$$

Definition 2. Define

$$i(n, k) = \min\{\dim H : H \text{ is } k \text{ interpolating}\}.$$

Examples. Obviously, we have $i(n, k) \geq k$, and it follows from Mairhuber's Theorem that $i(n, k) > k$ if $n > 1$.

- 1) Let Π_{k-1} be the space of polynomials of degree at most $k-1$. Then Π_{k-1} is a k -interpolating subspace of $C(\mathbb{R})$ and $i(n, k) = k$.

- 2) Let $H = \text{span}\{1, \text{Re}z^j, \text{Im}z^j\}_{j=1}^{k-1}$. Then $\dim H = 2k - 1$ and H is k -interpolating in $C(\mathbb{R}_2)$. Hence $i(2, k) \leq 2k - 1$. An amazing result discovered by D. Handel and F. Cohen [1] and then rediscovered by V. Vasiliev [5] states that

$$2k - \eta(k) \leq i(2, k) \leq 2k - 1,$$

where $\eta(k)$ is the number of ones in the binary expansion of the integer k .

- 3) Let $x = (t_1, \dots, t_n) \in \mathbb{R}_n$ and let $\pi_j(x) = t_j$. Then $[1] \subset C(\mathbb{R}_n)$ is a one-interpolating subspace of $C(\mathbb{R}_n)$; $\text{span}\{1, \pi_1, \pi_2, \dots, \pi_n\}$ is 2-interpolating in $C(\mathbb{R}_n)$; and $\text{span}\left\{1, \pi_1, \pi_2, \dots, \pi_n, \sum_{j=1}^n \pi_j^2\right\}$ is 3-interpolating in $C(\mathbb{R}_n)$. As was shown by the author [4] and by D. Wulbert [6] these are the "extremal" subspaces in the sense that

$$i(n, 1) = 1; \quad i(n, 2) = n + 1; \quad i(n, 3) = n + 2.$$

The purpose of this note is to obtain an upper bound on the quantity $i(n, k)$. The following conjecture is reasonable.

Conjecture. $i(n, k) \leq n(k - 1) + 1$.

Let Π_{k-1}^n be the space of polynomials of n variables of degree at most $k - 1$. It is easy to see that Π_{k-1}^n is k -interpolating in $C(\mathbb{R}_n)$ and $\dim \Pi_{k-1}^n = \binom{k-1+n}{n}$.

Theorem 1. *Let $N = nk + k$. Then generic (almost all) subspaces $H \subset \Pi_{k-1}^n$ with $\dim H = N$ are k -interpolating. In particular $i(n, k) \leq N$.*

The proof of this theorem is based on the notion of transversality. We will finish this section with the discussion of transversality. All the relevant details can be found in [2].

Definition 3. *Let A and B be two differentiable manifolds and F be a differentiable map*

$$F : A \rightarrow B.$$

Let C be a submanifold of B and $x \in A$. We say that F is transversal to C at the point x if

$$F(x) \notin C$$

or

F is transversal to C if it is transversal to C at every point $x \in A$. We use the symbol

$$F \pitchfork C$$

to denote the transversality of F to C .

Here $T_z D$ denotes the tangent plane to the manifold D at the point z ; and $(DF)_x$ is the derivative of the function F at the point x .

Let G be another differential manifold and Φ be a differentiable mapping

$$\Phi : G \times A \rightarrow B.$$

For any $g \in G$,

$$\Phi(g) : A \rightarrow B$$

is defined by

$$\Phi(g)(a) = \Phi(g, a).$$

The main ingredient of our proof is the following theorem.

Transversality Theorem. *Let*

$$\Phi : G \times A \rightarrow B$$

be as above. Let $C \subset B$ be a submanifold and

$$\Phi \pitchfork C.$$

Then for almost all (in the sense of category) $g \in G$

$$\Phi(g) \pitchfork C.$$

§2. Proof of Theorem 1

As before we let $N = (n + 1)k$. Further, we use $G = (\Pi_{k-1}^n)^N$ for a differential manifold (vector space) which is a direct product of N copies of Π_{k-1}^n . Let $A = A(n, k)$ be a configuration manifold, i.e., the set of all distinct k -tuples of points $x_1, \dots, x_k \in \mathbb{R}^n$. Finally, let $B = B(N, k)$ be the space of all k by N matrices and let $C \subset B$ be a submanifold of matrices of rank less than k .

We define a mapping

$$\Phi : G \times A \rightarrow B$$

by

$$\Phi(h_1, \dots, h_N, x_1, \dots, x_k) = [h_j(x_m)] \in B, \quad j = 1, \dots, N, \quad m = 1, \dots, k,$$

where

$$g = (h_j), \quad h_j \in \Pi_{k-1}^n$$

and

$$x_1, \dots, x_k \in \mathbb{R}_n$$

are distinct. Our immediate goal is to show that $\Phi \pitchfork C$.

In order to do so we need to verify that

$$T_{\Phi(g,x)}B = T_{\Phi(g,x)}C + (D\Phi)_{(g,x)}(G \times A). \quad (2.1)$$

Since B is a linear space, its tangent space is itself, and to prove (2.1) it is sufficient to show that

$$\text{rank} (D\Phi)_{(g,x)} = \dim (T_{\Phi(g,x)}B) = \dim B = kN. \quad (2.2)$$

The Jacobian $(D\Phi)_{(g,x)}$ is an augmented matrix

$$L = [L_1 \mid L_2],$$

where $L_1 = \frac{\partial \Phi}{\partial g}(h_1, \dots, h_N, x_1, \dots, x_k)$ and $L_2 = \frac{\partial \Phi}{\partial x}(g, x)$. To show (2.2) we will verify that $\text{rank} L_1 = kN$. Let p_1, \dots, p_d be a basis for Π_{k-1}^n where $d = \dim(\Pi_{k-1}^n)$. Then $g = (h_j)_{j=1}^N$ and

$$h_j = \sum_{m=1}^d a_m^{(j)} p_m.$$

Hence L_1 can be represented as

$$L_1 = \begin{bmatrix} \tilde{H} & 0 & \cdots & 0 \\ 0 & \tilde{H} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \tilde{H} \end{bmatrix}$$

where $\tilde{H} = \left[\frac{\partial}{\partial a_m^{(j)}} h_j \right] (x_1, \dots, x_k)$. This is so since $\frac{\partial}{\partial a_m^{(j)}} h_s = 0$ if $s \neq j$. Finally observe that

$$\tilde{H} = [p_s(x_m)], \quad s = 1, \dots, d \text{ and } m = 1, \dots, k.$$

Since Π_{k-1}^n is k -interpolating it follows that $\text{rank}(\tilde{H}) = k$ and since there are N of them we conclude that $\text{rank} L_1 = kN$ and $\Phi \pitchfork C$. By the Transversality Theorem we conclude that for almost all N -tuples of polynomials

$$h_1, \dots, h_N \in \Pi_{k-1}^n,$$

the mapping

$$\Phi(h_1, \dots, h_N) : A \rightarrow B$$

is transversal to C . Thus either

$$\Phi(h_1, \dots, h_N)(x_1, \dots, x_k) = [h_j(x_m)] \notin C \quad (2.3)$$

or

$$T_{\Phi(h)(x)}B = T_{\Phi(h)(x)}C + (D\Phi(h))_x A. \quad (2.4)$$

A simple dimension count shows that (2.4) is not possible for any choice of $x = (x_1, \dots, x_k)$. Indeed

$$\dim TB = \dim B = Nk$$

$$\dim(D\Phi(h))_x A \leq \dim A = kn$$

$$\dim TC = \dim C.$$

So if (2.4) were to be true, then

$$Nk \leq \dim C + kn$$

gives $\text{codim } C \leq kn$. But it is well known (see [3]) that

$$\text{codim } C = N - k + 1.$$

Since $N = nk + k$ we obtain a contradiction. The remaining viable alternative is (2.3); *i.e.*, we have

$$[h_j(x_m)] \notin C \text{ for all distinct } x_1, \dots, x_k \in \mathbb{R}_n.$$

Since $B \setminus C$ is the set of all matrices of full rank, we conclude that

$$H := \text{span } [h_1, \dots, h_N]$$

is k -interpolating. \square

Remark. The disparity between the conjectured bound of $n(k-1) + 1$ and the proven bound $(n+1)k$ can be partially explained by the "generic nature" of Theorem 1.

Indeed $i(1, 1) = 1$, yet the set of all polynomials (say of degree 10) which do not vanish at any point of the real line, are not dense among polynomials. However, the set of all pairs of polynomials (p_1, p_2) of degree 10 that do not vanish simultaneously at any point is dense in $\Pi_{10} \times \Pi_{10}$. It also suggests that generic subspaces of polynomials do not form interpolating spaces of minimal dimensions.

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