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SOME IDEMPOTENT MATRICES OF LARGE RANK

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Abstract We investigate the entries of idempotent matrices of large rank. In particular, we construct an example of a sequence of such matrices so that all the entries in most of the rows tend to zero. That answers a question previously raised by the author.

1. Motivation

Let $\tilde{P}$ be a projection from $C(-\pi,\pi)$ into itself such that the range of $\tilde{P}$ is an n-dimensional subspace of span$[e^{i\lambda_1\theta},\dots,e^{i\lambda_m\theta}]$, where $\lambda_1 < \lambda_2 < \dots < \lambda_m$ are arbitrary integers. Throughout this note we will assume that

$$m := m(n) = n + q(n),$$

where $q(n)/n \to 0$ as $n \to \infty$.

We wish to estimate the norm of $\tilde{P}$ from below. The desired conclusion (cf. [2,5,4])

$$\|\tilde{P}\| \geq c\log\frac{n}{q(n)}.$$

Each $\tilde{P}$ can be written as

$$\tilde{P}f = \sum_{j=1}^{m}\left(\int f d\mu_j\right)e^{i\lambda_j\theta}$$

where $\mu_j - s$ are regular Borel measures on $(-\pi,\pi)$.

Using the Littlewood inequality (cf. [3]) we obtain

$$\|\tilde{P}\| \geq c\sum_{j=1}^{m}\|\mu_j\|/j.$$

Hence further estimates depend on the bounds for the norms $\|\mu_j\|$.

It is easy to observe that the $m \times m$ matrix $P = (p_{k,j})$; $k, j = 1, \ldots, m$, with $p_{k,j} = \mu_k(e^{i\lambda_j\theta})$ is an idempotent matrix of rank n.

The estimate (1.1) implies

$$\|\tilde{P}\| \geq c \sum_{k=1}^{m} \frac{1}{k} \max_j |p_{k,j}|. \tag{1.2}$$

Some time ago we conjectured (based on a very closely related Proposition 2 of [4]) that many rows of P contain an element of large size.

CONJECTURE 1. *Let $m(n) = n + q(n)$; $q(n)/n \to 0$. Then there exists a constant $c_0 > 0$ such that for any $m \times m$ idempotent matrix P of* rank n *there exist $n - q(n)$ rows $k_1, \ldots, k_{n-q(n)}$ such that*

$$\max \{|p_{k_\ell,j}| : j = 1, \ldots, m\} \geq c_0 \ .$$

In the next section we give a counterexample to this conjecture and its consequences. In section three, we provide some positive results in this direction. First we need some notation.

Let $A = (a_{ij})$ be an $m \times n$ matrix. We use $\|A\|_{p,q}$ to denote the norm of an operator $A : \ell_p^{(n)} \to \ell_q^{(m)}$ induced by the matrix A. In particular

$$\|A\|_{1,\infty} = \max\{|a_{ij}| : i = 1, \ldots, m, j = 1, \ldots, n\}.$$

If $p = q$ we use $\|A\|_p := \|A\|_{p,p}$.

We use $\nu_{p,q}(A)$ to denote the nuclear norm (cf. [1], section 6.3.1) of an operator $A : \ell_p^{(n)} \to \ell_q^{(m)}$. Also $\nu_p(A) := \nu_{p,p}(A)$. In particular,

$$\nu_1(A) = \sum_{i=1}^{m} \max_j |a_{ij}| \ ; \qquad \nu_\infty(A) = \sum_{j=1}^{n} \max_i |a_{ij}| \ .$$

Further properties of the nuclear norm can be found in [1].

2. The Main Example

We will need the following:

PROPOSITION 1 (cf. [1], section 11.11.11). *There exists an $n \times n$ matrix A such that* rank $A = q(n)$ *and*

$$\|I - A\|_{1,\infty} \leq 3 \left[\frac{\log(n+1)}{q(n)} \right]^{\frac{1}{2}} .$$

With the aid of this proposition we can prove:

PROPOSITION 2. *There exists an idempotent $m \times m$ matrix P with* rank $P = n$ *such that the first n rows in P have entries*

$$|p_{ij}| \leq 3 \left[\frac{\log(n+1)}{q(n)} \right]^{\frac{1}{2}}.$$

PROOF. Let A be the matrix of the Proposition 1. Then there exist a $q(n) \times n$ matrix B and an $n \times q(n)$ matrix C such that $A = CB$. For arbitrary $\epsilon > 0$ consider a block-matrix

$$P = \begin{bmatrix} I - CB & \epsilon C \\ \frac{1}{\epsilon} B(I - CB) & BC \end{bmatrix}.$$

It is easy to see that P is an $m \times m$ idempotent matrix with rank $P = n$. The first n rows of this matrix consist of the entries of the matrices $I - CB$ and ϵC. By Proposition 1, every entry of $I - CB$ satisfies the conclusion of the Proposition 2. Choosing ϵ sufficiently small we obtain the desired result. ■

COROLLARY 3. *The Conjecture* 1 *is false.*

PROOF. It suffices to choose P as in Proposition 2 with $q(n) = \log^2(n+1)$.

COROLLARY 4. *There exists a subspace $E \subset \ell_\infty^{(m)}$ such that* codim $E \leq 2q(n)$ *and for every projection Q on $\ell_\infty^{(m)}$ with* Range $Q = E$ *we have*

$$\|Q\| \geq c \frac{\sqrt{q(n)}}{\log^{\frac{1}{2}}(m+1)}.$$

PROOF. Let $G \subset \ell_\infty^{(m)}$ be a subspace of all vectors in $\ell_\infty^{(m)}$ such that the last $q(n)$ coordinates of these vectors are zero. Then $\operatorname{codim} G = q(n)$. Let P be as in Proposition 2. Then

$$\operatorname{codim} \operatorname{Range} P^T < q(n).$$

Hence,

$$\operatorname{codim}(\operatorname{Range} P^T \cap G) \leq 2q(n).$$

Define $E := G \cap \operatorname{Range} P^T$ and let Q be an arbitrary projection from $\ell_\infty^{(m)}$ onto E.

Consider a matrix P^1 with the first n columns being the same as in P^T and the last $q(n)$ columns being zero. Then for every $e \in E$

$$P^T e = P^1 e = e.$$

Hence for every $x \in \ell_\infty^{(m)}$

$$P^T Q x = P^1 Q x = x.$$

By trace-duality (cf. [1])

$$\nu_\infty(P^1)\|Q\|_\infty \geq \operatorname{tr}(P^1 Q) = \operatorname{tr} Q \geq m - 2q(n).$$

Since the modulus of every element in P^1 is less than $4\sqrt{\log(n+1)}/\sqrt{q(n)}$ we obtain

$$\nu_\infty(P^1) \leq 4m\sqrt{\log(n+1)}/\sqrt{q(n)},$$

and from $m/n \to 1$

$$\|Q\|_\infty \geq C\sqrt{2q(m)}/\sqrt{\log(m+1)},$$

for some absolute constant C.

Remark. It is known that for every $E \subset \ell_\infty^{(m)}$ with $\operatorname{codim} E \leq 2q(n)$ there exists a projection Q onto E such that $\|Q\|_\infty \leq \sqrt{2q(n)} + 1$. It would be interesting to know if the factor $1/\sqrt{\log(m+1)}$ can be removed in Corollary 2.

3. Positive Results

While the Conjecture 1 does not hold, in general the situation changes if we restrict ourself to the case of symmetric matrices.

PROPOSITION 5. *Let P be a symmetric $m \times m$ idempotent matrix with* $\operatorname{rank} P = n$. *Then there exists $n - q(n)$ rows of $P : k_1, k_2, \ldots, k_{n-q(n)}$ such that*

$$\max\{p_{k_\ell,j} : j = 1,\ldots,m\} \geq \frac{1}{2}, \tag{3.1}$$

for $\ell = 1,\ldots,n-q(n)$.

PROOF. Let $K = \{k : 1 \leq k \leq m \text{ and } \max\{p_{k,j} : j = 1,\ldots,m\} \geq \frac{1}{2}\}$. Assume that

$$s := \#K < n - q(n), \tag{3.2}$$

and consider a new matrix $P^1 = (p^1_{k,j})$ where

$$p^1_{km} = \begin{cases} p_{k,j} & \text{if } k \notin K \text{ and } j \notin K, \\ 0 & \text{otherwise.} \end{cases}$$

Using (3.1) we obtain $|p^1_{k,j}| < \frac{1}{2}$ for all k, j. In order to prove the proposition we will contradict this inequality.

Observe that P is symmetric and idempotent, hence positive. Therefore, P^1 is also positive. Let $G = \{x = (x_1, \dots, x_m) \in \mathbf{C}_m : x_k = 0 \text{ for all } k \notin K\}$. By assumption (3.2)

$$\operatorname{codim} G = s < n - q(n),$$

while $\dim \operatorname{Range} P = n$. Hence for $E := \operatorname{Range} P \cap G$

$$\dim E > n - s.$$

Since P is an identity on E so is P^1. Hence P^1 has at least $n - s$ eigenvalues equal to 1.

Since P^1 is positive, the rest of its eigenvalues are positive and hence

$$n - s \leq \operatorname{tr} P^1 = \sum_{j=0}^{m} p^1_{ij} \leq (n + q(n) - s) \max_{j \notin K} p^1_{ij}.$$

Thus by (3.2)

$$\max_{j \notin K} p^1_{jj} \geq \frac{n - s}{n + q(n) - s} \geq \frac{1}{2}. \qquad \blacksquare$$

Let $D = [1, \frac{1}{2}, \dots, \frac{1}{m}]$ be a diagonal $m \times m$ matrix. It is clear that the expression (1.2) can be rewritten as

$$\|\tilde{P}\| \geq c \cdot \nu_1(DP).$$

While we cannot evaluate this last expression from below, we finish this note with the following formally weaker inequality.

PROPOSITION 6. *Let P be an $m \times m$ idempotent matrix with* $\operatorname{rank} P = n$. *Then*

$$\nu_2(DP) \geq c \cdot \log \frac{n}{q(n)},$$

for some absolute constant c.

PROOF. Let Q be an orthogonal projection from $\mathbf{C}_m$ onto the $\operatorname{Range} P$. Then $PQ = Q$ and $\|Q\|_2 = 1$. Using the ideal property of the nuclear norm we have

$$\nu_2(DQ) = \nu_2(DPQ) \leq \nu_2(DP)\|Q\|_2.$$

Thus by Proposition 5

$$\nu_2(DP) \geq \nu_2(DQ) \geq \frac{1}{2} \sum_{j=1}^{n-q(n)} \frac{1}{k_j}.$$

Since $1 < k_j < n + q(n)$ we obtain the result. ■

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