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CONSTRUCTIVE APPROXIMATION
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On a Conjecture of Carl de Boor Regarding the Limits of Lagrange Interpolants

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Abstract. The purpose of this paper is to provide a counterexample to a conjecture of Carl de Boor [2], that every ideal projector is a limit of Lagrange projectors. The counterexample is based on a construction of A. Iarrobino [9] pointed to in this context by G. Ellingsrud (as mentioned in de Boor's paper [2]). We also show that the conjecture is true for polynomials in two variables.

1. Introduction

Let $\mathbb{C}[x_1, \dots, x_d] = \mathbb{C}[\mathbf{x}]$ be the ring of polynomials in d complex variables and let $\mathbb{C}_{<m}[\mathbf{x}]$ be the space of polynomials of degree less than m.

Definition 1.1 (See [1]). Let E be a finite-dimensional subspace of $\mathbb{C}[\mathbf{x}]$. A (linear) projector P from $\mathbb{C}[\mathbf{x}]$ onto E is called ideal if $\ker P$ is an ideal in $\mathbb{C}[\mathbf{x}]$.

The standard example of an ideal projector is a *Lagrange* projector, i.e., a linear projector P for which Pf is the unique element in its range that agrees with f at a certain finite set Z in $\mathbb{C}^d$. Its kernel consists of exactly those polynomials that vanish on Z, i.e., it is the radical ideal whose variety is Z.

To put it another way, with every ideal $I \subset \mathbb{C}[\mathbf{x}]$ one associates its variety

$$Z(I) := \{\mathbf{x} \in \mathbb{C}^d : f(x) = 0 \text{ for all } f \in I\}.$$

It is well known and easy to see (see, e.g., [3, p. 143]) that, for an ideal projector P, the cardinality $\#Z(\ker P)$ of its associated variety is bounded by the dimension of the range of P, i.e.,

$$\#Z(\ker P) \le \dim \operatorname{ran} P, \tag{1.1}$$

with equality if and only if P is a Lagrange projector.

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In [2] Carl de Boor made the following conjecture.

Conjecture 1.2. *Let P be an ideal projector. Then there exists a sequence of Lagrange projectors Q_n onto* ran P *such that*

$$Q_n f \to Pf \quad \text{for every} \quad f \in \mathbb{C}[\mathbf{x}]. \tag{1.2}$$

He also mentioned a suggestion by Geir Ellingsrud that for $d \geq 3$ there exists a counterexample based on the construction of Iarrobino [9]. We will follow that suggestion. We construct a space E and a family of ideal projectors $\{P_C : C \in \mathbb{C}^K\}$ onto E that depend on the (matrix) parameter $C \in \mathbb{C}^K$ with $K > d \times \dim E + 1$. Since every Lagrange projector on E depends on $d \times \dim E$ parameters (the coordinates of its interpolation sites), hence the family of all Lagrange projectors depends on fewer parameters. We use the resulting simple dimensional calculation to show that "most" and, therefore, at least one, of the projectors from the family P_C cannot be approximated by Lagrange projectors.

As pointed out in [2], the conjecture is trivially true for $d = 1$. In the last section of this paper we verify the above conjecture for $d = 2$. The verification is based on a highly nontrivial and remarkable theorem of Fogarty [6] that asserts that the Hilbert scheme $Hilb_N(\mathbb{C}^2)$ is an irreducible variety of dimension $2N$.

2. Counterexample for $d \geq 3$

We now proceed with the construction of a counterexample.

Theorem 2.1. *For every $d \geq 3$, there exists an ideal projector P on $\mathbb{C}[x_1, \ldots, x_d]$ that is not the limit of Lagrange projectors.*

Proof. We fix an integer m, and let $U \cup V = W_m$ be a nontrivial partition of the set W_m of all monomials of degree m in $\mathbb{C}[x_1, \ldots, x_d] = \mathbb{C}[\mathbf{x}]$. Let E be the subspace of $\mathbb{C}[\mathbf{x}]$ spanned by $\mathbb{C}_{<m}[\mathbf{x}]$ and V. With each linear projector P from $\mathbb{C}[\mathbf{x}]$ onto E we associate the matrix $C_P \in \mathbb{C}^{U \times V}$ defined by the equation

$$Pu \in \sum_{v \in V} C_P(u, v)v + \mathbb{C}_{<m}[\mathbf{x}], \qquad u \in U. \tag{2.1}$$

Observe that the resulting mapping $P \mapsto C_P$ is continuous in the sense that if P converges pointwise to Q, then $C_P \to C_Q$. If such a P is a Lagrange projector, at the $N := \dim E$ distinct interpolation sites $\theta = (\theta_1, \ldots, \theta_N)$ in $\mathbb{C}^d$, then Cramer's rule provides the formula

$$C_P(u, v) = C_\theta(u, v) := \frac{\Delta_{u,v}(\theta)}{\Delta(\theta)}, \qquad (u, v) \in U \times V, \tag{2.2}$$

in which $\Delta(\theta)$ is the determinant of the Vandermonde involving a monomial basis of E evaluated at the points θ, and $\Delta_{u,v}(\theta)$ is the determinant of the matrix obtained from the Vandermonde by replacing, in its formulation, the monomial v by the monomial u.

Since the denominator in (2.2) does not depend on (u, v), it follows that the set of all such $U \times V$ matrices C_θ must lie in the range of the polynomial map

$$F : \mathbb{C}^{dN+1} \to \mathbb{C}^{U\times V}$$

defined by

$$F(\theta, z) = (\Delta_{u,v}(\theta) \cdot z : u \in U, v \in V), \tag{2.3}$$

hence, by a standard theorem from algebraic geometry (see [4, Theorem 2, p. 466] and Remark 2.2 below), must fail to be dense in $\mathbb{C}^{U\times V}$ as soon as

$$dN + 1 < \#U \cdot \#V. \tag{2.4}$$

On the other hand, as pointed out by Iarrobino in [8], for any choice of the matrix $C \in \mathbb{C}^{U\times V}$, the space I_C spanned by monomials of degree greater than m and the specific polynomials

$$p_u := u - \sum_{v\in V} C(u, v)v, \qquad u \in U,$$

complements E and is an ideal. The latter is so because each p_u is homogeneous, thus every product of a monomial with p_u is in I_C. Hence the linear projector P_C onto E with $\ker P_C = I_C$ determined by the decomposition

$$\mathbb{C}[\mathbf{x}] = E \oplus I_C$$

is an ideal projector. Since $C_{P_C} = C$, we conclude that P_C is not the limit of Lagrange projectors for every C in the interior of the complement of the range of F.

It remains to show that for every $d \geq 3$ one can choose an integer m and a partition $U \cup V = W_m$ satisfying (2.4). Iarrobino selects U and V of roughly the same size:

$$|\#U - \#V| \leq 1.$$

With this selection, direct computation yields (2.4) with $m = 7$ for $d = 3$, $m = 3$ for $d = 4$ or 5, and $m = 2$ for $d > 5$. ■

Remark 2.2. The above-mentioned Theorem 2 of [4] states that the dimension of an affine variety X equals the maximal number of elements of the coordinate ring $\mathbb{C}[X]$ which are algebraically independent.

In particular, choosing $X = \mathbb{C}^{dN+1}$, hence $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_{dN+1}]$, it implies that every $K > dN + 1$ algebraic polynomials

$$\varphi_1, \ldots, \varphi_K \in \mathbb{C}[x_1, \ldots, x_{dN+1}]$$

are algebraically dependent. That is, there exists a polynomial $g \in \mathbb{C}[x_1, \ldots, x_K]\backslash 0$ such that

$$g(\varphi_1, \ldots, \varphi_K) \equiv 0.$$

It follows that the range of a polynomial mapping $F := (\varphi_1, \ldots, \varphi_K) : \mathbb{C}^{dN+1} \to \mathbb{C}^K$ lies in a hypersurface $\{\mathbf{x} \in \mathbb{C}^K : g(\mathbf{x}) = 0\}$, hence cannot be dense in $\mathbb{C}^K$.

3. Bivariate Case

In this section we will use $\mathbb{C}[x, y]$ to denote the ring of polynomials in two variables and verify Conjecture 1.2 for this case ($d = 2$). Namely we will prove the following theorem.

Theorem 3.1. *Every ideal projector on* $\mathbb{C}[x, y]$ *is the limit of Lagrange projectors.*

To set up the argument, we need a few notations and some bits and pieces of algebraic geometry.

A polynomial $f \in \mathbb{C}[x, y]$ can be written as a finite sum

$$\sum_{i,j} \hat{f}(i, j)x^i y^j$$

with $\hat{f}(i, j) \in \mathbb{C}$ denoting the appropriate coefficient of f.

Let $\mathfrak{I}_N$ denote the family of all ideals $I \subset \mathbb{C}[\mathbf{x}]$ of codimension N, i.e., all ideals $I \subset \mathbb{C}[x, y]$ such that $\dim(\mathbb{C}[x, y]/I) = N$. By the Hilbert Basis Theorem (see [4, p. 74]), every ideal in $\mathfrak{I}_N$ is generated by a finite set of polynomials $f_1, \ldots, f_k \in \mathbb{C}[\mathbf{x}]$:

$$I = \langle f_1, \ldots, f_k\rangle \in \mathfrak{I}_N.$$

Since the validity of (1.2) depends only on the kernels of the projectors Q_n and P, and not on the range E (see [2], [11]), it is sufficient to prove that every ideal $I \in \mathfrak{I}_N$ is a "limit" of radical ideals $J_n \in \mathfrak{I}_N$. In other words, we have to prove that if an ideal

$$I = \langle f_1, \ldots, f_k\rangle \in \mathfrak{I}_N$$

is generated by polynomials $f_1, \ldots, f_k \in \mathbb{C}[\mathbf{x}]$, then there exist polynomials $f_1^{(n)}, \ldots, f_k^{(n)} \in \mathbb{C}[\mathbf{x}]$, such that the ideals $J_n := \langle f_1^{(n)}, \ldots, f_k^{(n)}\rangle \in \mathfrak{I}_N$ are radical ($\#Z(J_n) = N$ for all n) and $\hat{f}_s^{(n)}(i, j) \underset{n\to\infty}{\to} \hat{f}_s(i, j)$ for all $s = 1, \ldots, k$ and all (i, j).

In one variable, the (irreducible) algebraic variety $\mathbb{C}^N$ "parametrizes" $\mathfrak{I}_N$:

$$\mathbb{C}^N \ni (a_0, \ldots, a_{N-1}) \quad\mapsto\quad f := x^N + \sum_{\lambda=0}^{N-1} a_\lambda x^\lambda \quad\mapsto\quad f \cdot \mathbb{C}[\mathbf{x}] \in \mathfrak{I}_N.$$

Similarly, in two variables, there exists an algebraic variety, namely the Hilbert scheme $Hilb^N(\mathbb{C}^2)$, which *parametrizes* $\mathfrak{I}_N$.

The general description of the Hilbert scheme $Hilb^N(\mathbb{C}^2)$ can be found in [5, pp. 262–266] and [9]. In particular, $Hilb^N(\mathbb{C}^2)$ is a scheme over the algebraically closed field $\mathbb{C}$ and thus (see [10, Cor. 1, p. 195]) is an algebraic variety. As such, it is a topological space equipped with the Zariski topology as well as the usual strong topology (see [10, p. 81]). (A lengthy, but explicit glueing construction of $Hilb^N(\mathbb{C}^2)$ as an algebraic variety can be found in [7, pp. 266–274]).

The words "parametrizes $\mathfrak{I}_N$" mean (see [7, p. 266]) that there is a continuous bijection Φ between the variety $Hilb^N(\mathbb{C}^2)$ and the ideals in $\mathfrak{I}_N$. As the point t in $Hilb^N(\mathbb{C}^2)$ varies continuously (with respect to the strong topology), the coefficients of the appropriately chosen defining polynomials for the ideal $\Phi(t)$ in $\mathfrak{I}_N$ likewise vary continuously.

With this terminology, we need to prove that the set

$$X := \{t \in Hilb^N(\mathbb{C}^2) : \#Z(\Phi(t)) = N\} \subset Hilb^N(\mathbb{C}^2) \tag{3.1}$$

is dense (in the strong topology) in $Hilb^N(\mathbb{C}^2)$.

The proof relies on a celebrated theorem of Fogarty.

Theorem 3.2 [6]. *The Hilbert scheme $Hilb^N(\mathbb{C}^2)$ which parametrizes the family of ideals in $\mathbb{C}[x, y]$ of codimension N is nonsingular and irreducible, of dimension $2 \cdot N$.*

The irreducibility aspect of this theorem means that the variety $Hilb^N(\mathbb{C}^2)$ is not a union of two of its proper subvarieties.

There is a map

$$\sigma : Hilb^N(\mathbb{C}^2) \to (\mathbb{C}^2)^N/S_N,$$

where S_N is the symmetric group of order N and $(\mathbb{C}^2)^N/S_N$ is the variety (see [7, Example 10.23, p. 126]) of *unordered* N-tuples $[(x_1, y_1), \dots, (x_N, y_N)]$ of points in $\mathbb{C}^2$. The map is defined by

$$\sigma(t) = [\underbrace{(x_1, y_1), \dots, (x_1, y_1)}_{\mu_1}, \dots, \underbrace{(x_r, y_r), \dots, (x_r, y_r)}_{\mu_r}],$$

where $Z(\Phi(t)) = \{(x_1, y_1), \dots, (x_r, y_r)\}$ (hence $r \le N$ by (1.1)) and

$$\mu_j := \dim(\mathbb{C}[x, y]/I)_{(x_j, y_j)}, \qquad j = 1, \dots, r,$$

is the multiplicity (see [3, Definition 2.1, p. 139]) of the point $(x_j, y_j) \in Z(\Phi(t))$. In other words, $\sigma(t)$ lists the zero locus of the ideal $\Phi(t)$ with every zero repeated as many times as its multiplicity.

Since for every $I \in \mathfrak{I}_N$ we have $\sum_{\mathbf{z} \in Z(I)} \mu(\mathbf{z}) = N$ (see [3, Theorem 2.2, p. 141]), the map σ indeed has $(\mathbb{C}^2)^N/S_N$ as its range. This map is called the Hilbert–Chow morphism, and is a morphism of algebraic varieties (see [6, p. 516]). In particular, it is continuous in Zariski topology. That is, the inverse image of every Zariski closed set (subvariety) in $(\mathbb{C}^2)^N/S_N$ is closed in $Hilb^N(\mathbb{C}^2)$. We are now ready to present the proof of Theorem 3.1.

Proof of Theorem 3.1. Let Y be the set of all unordered N-tuples in $(\mathbb{C}^2)^N/S_N$, for which at least two points coincide. This is a subvariety of $(\mathbb{C}^2)^N/S_N$, and therefore a Zariski closed proper subset of $(\mathbb{C}^2)^N/S_N$. Hence $\sigma^{-1}(Y) \subset Hilb^N(\mathbb{C}^2)$ is the Zariski closed and thus a proper subvariety of $Hilb^N(\mathbb{C}^2)$. On the one hand, observe that the complement $(\sigma^{-1}(Y))^c$ coincides with the set X defined by (3.1), the density of which we aim to prove. On the other hand, $X = (\sigma^{-1}(Y))^c$ is a nonempty, Zariski-open subset of $Hilb^N(\mathbb{C}^2)$. It follows that X is Zariski dense in $Hilb^N(\mathbb{C}^2)$, for otherwise its Zariski closure $\tilde{X}$ would be a proper subvariety of $Hilb^N(\mathbb{C}^2)$ and

$$Hilb^N(\mathbb{C}^2) = \tilde{X} \cup \sigma^{-1}(Y)$$

is the union of its proper subvarieties, which contradicts the irreducibility of $Hilb^N(\mathbb{C}^2)$. Since X is Zariski dense, it is dense in the strong topology (see [10, Theorem 1, p. 82]) which makes the set X that parametrizes all radical ideals in $\mathfrak{I}_N$ an open and dense set in the strong topology of $Hilb^N(\mathbb{C}^2)$, and thus proves the theorem. ∎

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