Ideal Projections onto Planes

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Abstract. In this paper we classify all ideal projections from $\mathbb{C}[x, y]$ onto the linear span of $1, x, y$. In particular we show that every such projection is a limit of interpolating projections. That verifies one particular case of a conjecture of C. deBoor.

§1. Introduction

This work was motivated by a question asked by Carl deBoor (see Conjecture below) at this conference. Thus we study finite-dimensional projections whose kernels are polynomial ideals of finite codimension. The general theory of polynomial ideals is well investigated in Algebraic Geometry (cf.[5]). The relationship between polynomial ideals and approximation theory had been emphasized, among others, in papers[2],[3] and [4]. In this note,we specify not only the codimension of an ideal, but also its complement i.e. the range of a projection. That restricts the number of ideals in question, and in some exceptional cases allows for a complete classification of all such ideals. In this paper we give a complete description of ideals that are complemented to the 3-dimensional space of polynomials of degree one.

Let $\mathbb{C}[x, y]$ stand for the ring of polynomials of two variables with complex coefficients and let $M[x, y] \subset \mathbb{C}[x, y]$ be the set of monomials. The elements of $\mathbb{C}[x, y]$ will be written as a finite sum $\sum a_{j,k} x^j y^k$ or $\sum a_J \mathbf{u}^J$ with an understanding that the vector $\mathbf{u} = (x, y) \in \mathbb{C}_n$ and the multi-index $J = (j, k) \in \mathbb{Z}_+^2$, where $\mathbb{Z}_+$ stands for non-negative integers. Similarly $M[x, y] = \{x^j y^k = \mathbf{u}^J : J = (j, k) \in \mathbb{Z}_+^2\}$

The symbol $\mathbb{C}_n[x, y]$ denotes the polynomials of degree at most n. Hence $f \in \mathbb{C}_n[x, y]$ implies $f = \sum_{j+k \leq n} a_{j,k} x^j y^k = \sum_{|J| \leq n} a_J \mathbf{u}^J$. Finally we let $H_n[x, y]$ stand for homogeneous polynomials of degree n and

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$M_n[x,y] \subset H_n[x,y]$ is the set of all monomials of degree n. Therefore $f \in H_n[x,y]$ if and only if $f = \sum_{j+k=n} a_{j,k} x^j y^k = \sum_{|J|=n} a_J \mathbf{u}^J$ and $f \in M_n[x,y]$ if and only if $f = x^j y^{n-j} = \mathbf{u}^J$ with $|J| = n$

Definition 1. *Let P be a projection from $\mathbb{C}[x,y]$ onto a finite-dimensional subspace.*

1) P is called an ideal projection if

$$P(f \cdot g) = P(f \cdot Pg) \ \forall f, g \in \mathbb{C}[x,y] \tag{1.1}$$

2) P is called an interpolating projection if there exists a set $\Delta \subset \mathbb{C}$ such that

$$Pf = 0 \text{ iff } f(u) = 0 \text{ for all } u \in \Delta.$$

In this case we denote the projection by P_Δ.

3) P is called a "lim-interpolating projection" (or LIP) if there exists a sequence P_k of interpolating projections such that $P_k f \to Pf$ for every $f \in \mathbb{C}[x,y]$.

Clearly if P is an interpolating projection then P is an ideal projection. Hence every lim-interpolating projection is ideal. It is well-known and easy to see that, for polynomials of one complex variable, the converse is also true.

Theorem 1. *Every ideal projection on $\mathbb{C}[x]$ is a Hermite interpolating projection, and therefore LIP.*

Proof. Let P be an ideal projection on $\mathbb{C}[x]$. Then $\ker P$ is an ideal in $\mathbb{C}[x]$, which is a principle ideal domain. Hence $\ker P$ is generated by one polynomial $f(x) = \prod_{j=0}^{m} (x - x_j)^{n(j)}$. Hence P interpolates the values at the points x_j together with the values of the derivatives at these points up to the order $n(j)$. $\blacksquare$

At the conference to which these proceedings are dedicated, Carl de-Boor proposed the following:

Conjecture 1. *([1]) Every ideal projection is LIP.*

In this paper we will give a complete description of ideal projections $P : \mathbb{C}[x,y] \to \mathbb{C}_1[x,y]$ and use it to verify the above-mentioned conjecture for this special case as well as for all three dimensional projections in $\mathbb{C}[x,y]$

§2. Ideal Projections.

The following proposition describes some obvious properties of the ideal projections:

Proposition 2. *Let P be a projection on $\mathbb{C}[x,y]$. Then*

1). P is ideal iff $\ker P$ is an ideal in $\mathbb{C}[x,y]$.

2). P is ideal projection iff

$$f_1, g_1, f_2, g_2 \in \mathbb{C}[x,y],\ f_1g_1 = f_2g_2 \Longrightarrow f_1Pg_1 - f_2Pg_2 \in \ker P. \tag{2.1}$$

3). If P is an ideal projection and $f_1Pg_1 - f_2Pg_2 \in \ker P$ then $f_1g_1 - f_2g_2 \in \ker P$.

4). P is ideal projection iff

$$f_1, g_1, f_2, g_2 \in M[x,y],\ f_1g_1 = f_2g_2 \Longrightarrow f_1Pg_1 - f_2Pg_2 \in \ker P. \tag{2.2}$$

Proof. The statements 1),2),3) are trivial. To verify 4) assume $f = \sum_J a_J\mathbf{u}^J$ and $g = \sum_K b_K\mathbf{u}^K$. Then
$P(fg) = P(\sum_J\sum_K a_Jb_K\mathbf{u}^J\mathbf{u}^K) = \sum_J\sum_K a_Jb_KP(\mathbf{u}^J\mathbf{u}^K)$
$= \sum_J\sum_K a_Jb_KP(\mathbf{u}^JP\mathbf{u}^K) = P(\sum_J\sum_K a_Jb_K\mathbf{u}^JP\mathbf{u}^K) =$
$P((\sum_J a_J\mathbf{u}^J)\cdot P(\sum_K b_K\mathbf{u}^K)) = P(fPg)$. ∎

Motivated by (2.2) we will localize the definition of an ideal projection.

Definition 2. *Let P be a projection on $\mathbb{C}_m[x,y]$. We say that P is m-ideal if*

$$f_1, g_1, f_2, g_2 \in \mathbb{C}_m[x,y],\ f_1g_1 = f_2g_2 \Longrightarrow f_1Pg_1 - f_2Pg_2 \in \ker P. \tag{2.3}$$

Let P be an $(m+k)$-ideal projection on $\mathbb{C}_{m+k}[x,y]$ with Im$P \subset \mathbb{C}_m[x,y]$. Then its restriction to $\mathbb{C}_m[x,y]$ is m-ideal. The converse is not necessarily true. It is possible that an m-ideal projection on $\mathbb{C}_m[x,y]$ does not have an $(m+k)$-ideal extension to $\mathbb{C}_{m+k}[x,y]$ for some k. However if such an extension exists, then the ideal property clearly guaranties its uniqueness.

The next Theorem shows that if a projection described above has an $(m+2)$-ideal extension, then it has an $(m+k)$-ideal extension for all k and hence has an ideal extension to $\mathbb{C}[x,y]$ with the same range.

Theorem 3. *Let P be an $(m+2)$-ideal projection with Im$P \subset \mathbb{C}_m[x,y]$. Then P has an ideal extension to $\mathbb{C}_n[x,y]$ for all $n \geq m+2$.*

Proof. The proof proceeds by induction. Let P be an ideal projection from $\mathbb{C}_n[x,y]$ onto Im$P \subset \mathbb{C}_m[x,y]$ and $n \geq m+2$. Define

$$Q(x^jy^k) = \begin{cases} P(x^jy^k) & \text{if } j+k \leq n \\ P(y\cdot P(x^jy^{k-1})) & \text{if } j+k = n+1;\ k \geq 1 \\ P(x\cdot P(x^n)) & \text{if } j = n+1, k = 0 \end{cases} \tag{2.4}$$

We need to show that for all $f_1, g_1, f_2, g_2 \in M_{n+1}[x,y]$,

$$f_1 g_1 = f_2 g_2 \Longrightarrow f_1 Q g_1 - f_2 Q g_2 \in \ker Q. \tag{2.5}$$

First assume that $g_1$and $g_2 \notin \mathrm{Im} P$ and consider two cases:
Case1. Suppose that $f_1 = qh_1$ and $f_2 = qh_2$ with $\deg q \geq 1$. Then

$$Q(f_1 Q g_1 - f_2 Q g_2) = Q(q(h_1 P g_1 - h_2 P g_2))$$

Since

$$\deg(q(h_1 P g_1 - h_2 P g_2)) \leq m, \; h_1 g_1 = h_2 g_2 \in \mathbb{C}_{m+1}[x,y]$$

using the inductive assumption twice we have

$$Q(f_1 P g_1 - f_2 P g_2) = P(q(h_1 P g_1 - h_2 P g_2)) = P(qP(h_1 P g_1 - h_2 P g_2)) = 0,$$

since $h_1 P g_1 - h_2 P g_2 \in \ker P$
Case2: Suppose that g_1 and g_2 have no common divisors. Then $g_1 = x^s$ and $g_2 = y^t$. Therefore

$$f_1 = x^{n+1-j-s} y^j \text{ and } f_2 = x^{n+1-j} y^{j-t}.$$

From the previous case we have

$$(xyP(x^{m+1-j-1} y^j) - x^s P(x^{m+1-j-s} y^j)) \in \ker Q$$

and

$$(y^t P(x^{m+1-j} y^{j-t}) - xyP(x^{m+1-j-1} y^{j-1})) \in \ker Q.$$

Adding these two equations together we get the desired conclusion.
Now suppose that $g_1$and/or $g_2 \in \mathrm{Im} P$. Then choosing $f_0 = x$ or y we find $g_0 \notin \mathrm{Im} P$ such that

$$f_1 g_1 = f_0 g_0 = f_2 g_2.$$

We have

$$Q(f_1 P(g_1)) = Q(f_1 g_1) := Q(f_0 P(g_0)). \tag{2.6}$$

Similarly if $g_2 \in \mathrm{Im} P$, then

$$Q(f_2 P(g_2)) = Q(f_2 g_2) := Q(f_0 P(g_0)). \tag{2.7}$$

Otherwise (2.7) follows from the previous steps. Combining (2.6) and (2.7) we have the desired conclusion. ■

The importance of this Theorem can be illustrated by the following example.

Example 1. *To obtain all ideal projections from* $\mathbb{C}[x,y]$ *onto* $\mathbb{C}_m[x,y]$ *we start with arbitrary* $(m+1)$ *polynomials* $h_j \in \mathbb{C}_m[x,y]$, $j = 0,...,m$. *Define* $P : \mathbb{C}_{m+1}[x,y] \to \mathbb{C}_m[x,y]$ *by*

$$P(x^j y^{m+1-j}) = h_j \tag{2.8}$$

Obviously the projection P *is* $(m+1)$*-ideal. To extend this projection to an ideal projection on* $\mathbb{C}_{m+1}[x,y]$, *we have to make sure that the polynomials* h_j *satisfy the* m *"consistency equations":*

$$P(xh_{j-1}) = P(yh_j); j = 1,...,m. \tag{2.9}$$

That guaranties that P *has an ideal extension to* $\mathbb{C}_{m+2}[x,y]$ *and hence has an ideal extension to* $\mathbb{C}[x,y]$. *In other words every ideal projection onto* $\mathbb{C}[x,y]$ *is completely determined by* $(m+1)$ *polynomials satisfying the consistency equations.*

§3. Ideal varieties:

In this section we will switch our attention to the varieties determined by the kernels of ideal projections.

Definition 3. *Let* P *be an ideal projection. Define*

$$Z(P) := \{\mathbf{u} \in \mathbb{C}_2 : f(\mathbf{u}) = 0 \textit{ for all } f \in \ker P\}. \tag{3.1}$$

To parallel the previous section we will start with a few observations:

Proposition 4. *Let* P *be an ideal projection with a finite-dimensional range. Then*

1. $Z(P)$ *is a finite algebraic variety. Moreover the cardinality of* $Z(P)$

$$0 < \#Z(P) \le \textit{codim}(\textit{ker}P) = \dim(\textit{Im}P). \tag{3.2}$$

2. The projection P *is an interpolating projection if and only if*

$$\#Z(P) = \dim(\textit{Im}P).$$

3. The projection P *is an interpolating projection if and only if the ideal* $\ker P$ *is radical.*

Proof. Let $m = \textit{codim}(\textit{ker}P) = \dim(\text{Im}P)$. Since $\ker P$ is an ideal, the set $Z(P) = V(\ker P)$ is a variety generated by the ideal. The left-hand side of (3.2) is the "The Weak Nullstellensatz" (cf.[5]). For the right-hand side, observe that point-evaluation functionals $\delta_{\mathbf{u}}$ with $\mathbf{u} \in Z(P)$ are linearly independent and annihilate the ideal $\ker P$. Hence the number of such point-evaluations can not be greater then $codim(kerP)$.

The second assertion is equally trivial. Indeed $(f - Pf)$ vanishes on $Z(P)$, hence Pf interpolates f at m points and thus P is interpolating. Conversely, if P is an interpolating projection, then its m point-evaluation functionals annihilate $kerP$ and hence $m \geq \#Z(P)$. Combined with (3.2) this proves 2.

Finally if P is interpolating and f vanishes on $Z(P)$ then $Pf = 0$ and $f \in \ker P$ which is precisely the definition of radical ideal (cf.[5]). On the other hand if $\#Z(P) < \dim(\mathrm{Im}P)$ then there exists $f \neq 0$ such that f restricted to $Z(P)$ is zero and $f \in \mathrm{Im}P$. Hence $Pf = f \neq 0$ and $f \notin \ker P$ which contradicts the fact that $\ker P$ is radical. ■

We will now show that the variety $Z(P)$ is completely determined by the zeroes of polynomials of lower order in $\ker P$.

Theorem 5. *Let P be an ideal projection from $\mathbb{C}[x, y]$ onto $ImP \subset \mathbb{C}_m[x, y]$. Then*

$$Z(\ker P) = \{(x, y) \in \mathbb{C}^2 : x^j y^k - P(x^j y^k) = 0 \textit{ for } j + k \leq m + 1\}.$$

Proof. Let $W = \{(x, y) \in \mathbb{C}^2 : x^j y^k - P(x^j y^k) = 0 \text{ for } j + k \leq m + 1\}$. To show that $W = Z(\ker P)$ it is clearly sufficient to prove that $(f - Pf)(x, y) = 0$ for every $(x, y) \in W$ and for every monomial f. We proceed once again by induction on the degree of the monomial f. Suppose the claim is proven for the monomials of degree $n \geq m + 1$, and let f be a monomial of degree $n + 1$. Then, without loss of generality, we assume that $f = xg$. Let $Pg = \sum_{j+k\leq m} a_{j,k} x^j y^k$. We have

$$\begin{aligned} f - Pf &= xg - xPg + xPg - P(xPg) \\ &= x(g - Pg) + x(\sum_{j+k\leq m} a_{j,k} x^j y^k) - P(x(\sum_{j+k\leq m} a_{j,k} x^j y^k)) \\ &= x(g - Pg) + \sum_{j+k\leq m} a_{j,k}(x^{j+1} y^k - P(x^{j+1} y^k)). \end{aligned}$$

Now if $(x, y) \in W$ then the first term vanishes by the inductive assumption and the rest vanish by definition of the set W. ■

§4. Description of Ideal Projections

We will now specify the results of the previous sections to the ideal projections P onto $\mathbb{C}_1[x, y]$. By Theorem 3 (see also Example 1) we conclude that all such projections can be described by three first degree polynomials: $h_{2,0}, h_{1,1}$and $h_{0,2}$ (i.e. nine coefficients) provided they satisfy the consistency equations (2.9). We use Theorem 5 to conclude that the variety $Z(P)$ is the set of zeroes of equations

$$x^j y^k - h_{j,k}(x,y) = 0 \text{ for } j+k=2. \tag{4.1}$$

Assume that the "determining polynomials" $h_{2,0}, h_{1,1}$and $h_{0,2}$ are given by

$$\begin{aligned} h_{2,0} &= b_{2,0} + c_{2,0}x + d_{2,0}y \\ h_{1,1} &= b_{1,1} + c_{1,1}x + d_{1,1}y \\ h_{0,2} &= b_{0,2} + c_{0,2}x + d_{0,2}y. \end{aligned} \tag{4.2}$$

We now rewrite the consistency equations (2.8) in terms of the coefficients of polynomials $h_{j,k}$ and we have
$P(xP(y^2)) = P(xh_{0,2}) = P(b_{0,2}x + c_{0,2}x^2 + d_{0,2}xy) =$
$b_{0,2}x + c_{0,2}h_{2,0} + d_{0,2}h_{1,1} =$
$(b_{0,2} + c_{0,2}c_{2,0} + d_{0,2}c_{1,1})\,x + (c_{0,2}d_{2,0} + d_{0,2}d_{1,1})\,y + (c_{0,2}b_{2,0} + d_{0,2}b_{1,1}).$
$= P(yP(xy)) = P(yh_{1,1}) = (b_{1,1}y + c_{1,1}h_{1,1} + d_{1,1}h_{0,2}) =$
$\left(c_{1,1}^2 + d_{1,1}c_{0,2}\right)x + (b_{1,1} + c_{1,1}d_{1,1} + d_{0,2}d_{1,1})\,y + (c_{1,1}b_{1,1} + d_{1,1}b_{0,2}).$
Similarly resolving the equation $P(yP(x^2)) = P(xP(xy)$ we obtain
$P(yP(x^2)) =$
$(c_{2,0}c_{1,1} + c_{0,2}d_{2,0})\,x + (b_{2,0} + c_{2,0}d_{1,1} + d_{2,0}d_{0,2})\,y + (c_{2,0}b_{1,1} + d_{2,0}b_{0,2}) =$
$P(xP(xy)) =$
$(b_{1,1} + c_{2,0}c_{1,1} + c_{1,1}d_{1,1})\,x + \left(c_{1,1}d_{2,0} + d_{1,1}^2\right)y + (c_{1,1}b_{2,0} + d_{1,1}b_{1,1})$
Equating the coefficients leads to the following system of equations:

$$\left\{ \begin{array}{c} (b_{0,2} + c_{0,2}c_{2,0} + d_{0,2}c_{1,1}) = (c_{1,1}^2 + d_{1,1}c_{0,2}) \\ (c_{0,2}d_{2,0} + d_{0,2}d_{1,1}) = (b_{1,1} + c_{1,1}d_{1,1} + d_{0,2}d_{1,1}) \\ (c_{0,2}b_{2,0} + d_{0,2}b_{1,1}) = (c_{1,1}b_{1,1} + d_{1,1}b_{0,2}) \\ (c_{2,0}c_{1,1} + c_{0,2}d_{2,0}) = (b_{1,1} + c_{2,0}c_{1,1} + c_{1,1}d_{1,1}) \\ (b_{2,0} + c_{2,0}d_{1,1} + d_{2,0}d_{0,2}) = \left(c_{1,1}d_{2,0} + d_{1,1}^2\right) \\ (c_{2,0}b_{1,1} + d_{2,0}b_{0,2}) = (c_{1,1}b_{2,0} + d_{1,1}b_{1,1}). \end{array} \right. \tag{4.3}$$

Solving for b-s we obtain

$$\left\{ \begin{array}{c} b_{0,2} = (c_{1,1}^2 + d_{1,1}c_{0,2}) - (c_{0,2}c_{2,0} + d_{0,2}c_{1,1}) \\ b_{1,1} = (c_{0,2}d_{2,0} + d_{0,2}d_{1,1}) - (c_{1,1}d_{1,1} + d_{0,2}d_{1,1}) \\ b_{2,0} = \left(c_{1,1}d_{2,0} + d_{1,1}^2\right) - (c_{2,0}d_{1,1} + d_{2,0}d_{0,2}). \end{array} \right. \tag{4.4}$$

It is easy to check that the rest of the equations (4.3) hold with the values assigned by (4.2). This leads to a complete description of ideal projections onto $\mathbb{C}_1[x,y]$:

Theorem 6. *Every ideal projection P from $\mathbb{C}[x,y] \to \mathbb{C}_1[x,y]$ is determined by polynomials $h_{2,0}, h_{1,1}$ and $h_{0,2}$, with coefficients satisfying (4.4).*

More over if $c_{0,2} \neq 0$*, the corresponding variety is* $Z(P) = \{(x_j, y_j); j = 1, 2, 3\}$*, where*

$$x_j = -\frac{-y_j^2 + c_{1,1}^2 + d_{1,1}c_{0,2} - c_{0,2}c_{2,0} - d_{0,2}c_{1,1} + d_{0,2}y_j}{c_{0,2}}$$

and y_j *are the solutions of the cubic equation*

$$\begin{aligned} 0 &= y^3 - (d_{0,2} + c_{1,1})\, y^2 + \left(2d_{0,2}c_{1,1} - c_{1,1}^2 - 2d_{1,1}c_{0,2} + c_{0,2}c_{2,0}\right) y + \\ & \quad (2c_{1,1}d_{1,1}c_{0,2} - c_{1,1}c_{0,2}c_{2,0} - d_{0,2}c_{1,1}^2 + c_{1,1}^3 - c_{0,2}^2 d_{2,0}). \end{aligned}$$

If $c_{0,2} = 0$ *then* $Z(P) = \{(d_{1,1}, d_{0,2} - c_{1,1}), (x_1, c_{1,1}), (x_2, c_{1,1})\}$ *where* x_1 *and* x_2 *are the roots of quadratic equation*

$$x^2 - c_{2,0}x - 2c_{1,1}d_{2,0} - d_{1,1}^2 + c_{2,0}d_{1,1} + d_{2,0}d_{0,2} = 0.$$

Proof. The description of $Z(P)$ follows from solving equations (4.1) directly. ■

§5. Application to deBoor's Conjecture.

In this section we will use description of ideal projections, to verify Conjecture 2 for ideal projections with three dimensional range.

Theorem 7. *Let* P *be an ideal projection in* $\mathbb{C}[x, y]$ *with* $dim(ImP) = 3$*. Then* P *is a lim-interpolating projection.*

Proof. An ideal projection P is an interpolating projection if (and only if) the corresponding ideal variety $Z(P)$ consists of precisely three points. Next observe that for ideal projections described in the previous section, the variety $Z(P)$ consists of three distinct points for a dense set of "free parameters" c-s and d-s. Hence if P is an ideal projection onto $\mathbb{C}_1[x, y]$, there exists a sequence of interpolating projections P_n such that $P_n f \to Pf$ for every $f \in \mathbb{C}[x, y]$. That means that the ideal $J := \ker P$ can be approximated by radical ideals $J_n = \ker P_n$. Thus if Q is a projection onto a different three dimensional subspace with $\ker Q = \ker P$, then Q is LIP. In other words if Q is any three-dimensional ideal projection with

$$\ker Q \cap \mathbb{C}_1[x, y] = \{0\} \tag{5.1}$$

then Q is a limit interpolating projection.

Assume now that $0 \in Z(\ker Q)$ and hence no polynomial with a constant coefficient belongs to $\ker Q$. If (5.1) does not hold, then $\ker Q$ contains one (and up to a constant multiple, only one) linear function. Indeed, if it contains two linearly independent linear functions, then $codim(\ker Q) =$

1. So without loss of generality, let $f(x,y) = y \in \ker Q$. Then $\ker Q$ contains all polynomials that depend on y, and $x \notin \ker Q$. Since $codim(\ker Q) = 3$, it must not contain at least one other polynomial that does not depend on the indeterminate y. Let $E = span\{1, x, q(x)\}$ be a three dimensional space with $E \oplus \ker Q = \mathbb{C}[x,y]$ and let R be a projection from $\mathbb{C}[x,y]$ onto E with $\ker R = \ker Q$. Then R is an ideal projection and it suffices to prove that R is a lim-interpolating projection.

Let $\tilde{R}$ be the restriction of R onto $\mathbb{C}[x]$. Then $\tilde{R}$ is an ideal projection and by Theorem1, there exist distinct points $\Delta_n := \{x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\} \subset \mathbb{C}$ such that the interpolating projections $\tilde{R}_n := \tilde{R}_{\Delta_n} \to \tilde{R}$. That is

$$\tilde{R}_n f \to \tilde{R} f \text{ for all } f \in \mathbb{C}[x]. \tag{5.2}$$

For $j = 1, 2, 3$ define

$$y_j^{(n)} = R(y)(x_j^{(n)}); \mathbf{u}_j^{(n)} = (x_j^{(n)}, y_j^{(n)}) \in \mathbb{C}^2 \text{ and } \tau_n := \{\mathbf{u}_j^{(n)}\}. \tag{5.3}$$

Let R_n be projections onto E that interpolate at τ_n. We want to prove that

$$R_n(x^k y^m) \to R(x^k y^m) \tag{5.4}$$

for all k and m.

Observe that $(R(y) - y)(\mathbf{u}_j^n) = R(y)(x_j^{(n)}) - y_j^{(n)} = 0$, by (5.3). Hence

$$R(y) = R_n(y). \tag{5.5}$$

The rest of the proof proceeds by induction on m. If $m = 0$ then (5.4) is the same as (5.2). Assume that (5.4) holds for m. Then

$$R_n(x^k y^{m+1}) = R_n(x^k y^m R_n(y)) = R_n(x^k y^m R(y)).$$

Since $R(y)$ is a polynomial in x only, we use the inductive assumption to conclude that $R_n(x^k y^{m+1}) \to R(x^k y^{m+1})$ ■

Remark 8. *Using "Maple" we can demonstrate that every 6-dimensional ideal projection P is LIP. We use the same procedure as the one described in the last two sections. First we describe all ideal projections onto $\mathbb{C}_2[x,y]$. Depending on the coefficients of polynomials $h_{j,k}$ there are19 different descriptions of the varieties $Z(P)$ (in Theorem 8, there are only two). However in each case the variety $Z(P)$ depends on the roots of an equation of degree equal to* $\dim(ImP)$*, which "generically" has all distinct solutions. Next we use a procedure similar to the one involved in the proof of the last theorem to verify the claim for all 6-dimensional projections.*

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