UNIQUENESS OF MINIMAL PROJECTIONS ONTO TWO-DIMENSIONAL SUBSPACES

Boris Shekhtman and Lesław Skrzypek†

Abstract. In this paper we prove that the minimal projections from L_p ($1 < p < \infty$) onto any two-dimensional subspace is unique. This result complements the theorems of W. Odyniec ([OL, Theorem I.1.3], [O3]) We also investigate the minimal number of norming points for such projections.

0. Introduction

W. Odyniec ([OL, Theorem I.1.3], [O3]) proved that minimal projections of norm grater than one from a three-dimensional Banach space onto any of its two-dimensional subspace are unique. This result cannot be generalized neither to the subspaces of codimension one nor to the subspaces of dimension two, unless additional assumptions on the space are considered.

However, as proved by Odyniec ([OL, Theorem I.2.22], [O1, O2]) every subspace of codimension one in L_p ($1 < p < \infty$) has unique minimal projection.

In this paper we complete the picture by showing that every two-dimensional subspace of L_p ($1 < p < \infty$) has a unique minimal projection. Specifically we prove the following theorem:

Theorem 0.1. *Let V be a two-dimensional subspace of a $L_p(\mu)$ ($1 < p < \infty$). Then the minimal projection from X onto V is unique.*

We prove this theorem in Section 1. The proof of the above theorem depends on the number of norming points (and functionals) for minimal projections. In Section 2 we investigate one particular minimal projection and its norming pairs.

We use the rest of this section for general remarks and necessary definitions.

It is well known (see [IS] and [CMO]) that for every finite dimensional subspace V of a Banach space X there exists a minimal projection.

The problem of finding a minimal projection and related problems received the attention of many mathematicians [see papers [BP, CF, CL, CM1, CM2, CHM, CMO, CP, F, KTJ, R]] and it turned out to be easier

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in L_1 spaces then in L_p spaces (mostly due to Theorem 1 in [CM2] which can be effectively applied in L_1).

The problem of uniqueness of minimal projection, however, is not well understood yet. It is clear that subspaces of L_1 usually lack uniqueness (see [CM1]) though the classical Fourier projection onto trigonometric polynomials is unique in L_1 as well as in space of continuous functions (compare [CHM, FMW]). For the necessary and sufficient conditions for the uniqueness of minimal projection onto two-dimensional subspaces of ℓ^n_∞ see [L3]

As far as we know, for $1 < p < \infty$ there is no example of subspaces of L_p (finite dimensional or finite codimensional) for which the minimal projection is not unique. Even the uniqueness of minimal projections onto trygonometric polynomials is not known.

To the best of our knowledge the exhaustive list of results consists of previously mentioned theorem of Odyniec and theorem of H.B. Cohen and F.E. Sullivan which states that if the minimal projection in L_p $(1 < p < \infty)$ has norm one then it is unique (see [CS]). In particular all one-dimensional subspaces of L_p $(1 < p < \infty)$ have unique minimal projection. We hope that Theorem 0.1 is a modest contribution to this list.

It is worth mentioning that the result of W. Odyniec has been recently improved by G. Lewicki ([L3] Theorem 2.6.11) by showing that a minimal projection of norm greater than one from a three-dimensional real Banach space onto any two-dimensional subspace is in fact strongly unique.

Let us introduce some basic notions, definitions and facts used in this paper. Let $S(X)$ and $B(X)$ denotes the unit sphere and unit ball of a Banach space X.

A projection P from X onto V is called minimal if it has the smallest possible norm, i.e.,

$$\|P\| = \lambda(V, X) = \inf\{\|Q\| : Q \text{ is a projection from } X \text{ onto } V\}. \quad (0.1)$$

The constant $\lambda(V, X)$ is called the **relative projection constant**.

Definition 0.2. A functional $f \in S(X^*)$ is a **norming functional** for a projection $P : X \to V$ iff $\|f \circ P\| = \|P\|$. It is well known that if V is finite dimensional then P has norming functionals (see [OL, Lemma III.2.1]).

Definition 0.3. A point $x \in S(X)$ is a **norming point** for a projection $P : X \to V$ iff $\|P(x)\| = \|P\|$. If X is a reflexive space and V is finite dimensional then P has a norming functional f and since the functional $f \circ P$ attains its norm, P has a norming point (this is not so in general Banach spaces as Fourier projection does not have a norming point in the space of continuous functions, see [OL, Lemma I.2.7]).

Definition 0.4. A pair (f, x) is called norming pair for a projection P iff $f(Px) = \|P\|$. A set of all norming pairs for a projection P is denoted by $\mathcal{E}(P)$.

As usual, for $g \in X^*$ and $y \in X$, the symbol $g \otimes y$ denotes the one-dimensional operator from X to X given by $g \otimes y(x) = g(x)y$.

For the sake of completeness we will state Rudin Theorem which will be used for proving minimality of a projection given in Section 2.

Definition 0.5. Suppose that a Banach space X and a topological group G are related in the following manner: to every $s \in G$ corresponds a continuous linear operator $T_s : X \to X$ such that

$$T_e = I, \qquad T_{st} = T_s T_t \qquad (s \in G, t \in G).$$

Under these conditions, G is said to act as a group of linear operators on X.

Definition 0.6. A map $L : X \to X$ commutes with G if $T_g L T_{g^{-1}} = L$ for every $g \in G$.

Theorem 0.7 (Rudin) [W, III.B.13]**.** *Let X be a Banach space and V a complemented subspace, i.e., $\mathcal{P}(X, V) \neq \emptyset$. Let G be a compact group which acts as a group of linear operators on X such that*

(1) *$T_g(x)$ is a continuous function of g, for every $x \in X$,*
(2) *$T_g(V) \subset V$, for all $g \in G$.*
(3) *T_g are isometries, for all $g \in G$.*

Furthermore, assume that there exists only one projection $P : X \to V$ which commutes with G. Then this projection is minimal.

Once we know that there is only one projection P commuting with G it can be easily found: fix any projection Q from X onto V, then this projection P equals

$$P(x) = \int_G T_g Q T_{g^{-1}}(x) dg, \qquad \textit{for } x \in X.$$

This theorem, however, does not imply that this projection is the unique minimal projection as there could be projections which do not commute with G but still have a minimal norm (see [S], [L1]).

1. Proof of Theorem 0.1

Lemma 1.1. *Let V be a two dimensional subspace of a Banach space X. Let $x \in S(X) \setminus V$. Then for any arbitrary $\alpha > 0$ there exists a projection Q from X onto V such that $\|Q(x)\| = \alpha$.*

Proof. We can assume that $x \notin V$. Let $v_1, v_2 \in S(V)$ be a basis for V. Since x, v_1, v_2 are linearly independent, using Hahn-Banach theorem we can choose

$$f_1 \in X^* \text{ such that } f_1(v_1) = 1 \text{ and } f_1/span\{x, v_2\} = 0$$

and

$$f_2 \in X^* \text{ such that } f_2(v_2) = 1 \text{ and } f_2/span\{\tfrac{1}{\alpha}x - v_2, v_1\} = 0.$$

We have chosen f_1 and f_2 such that

$$\begin{array}{lll} f_1(x) = 0 & f_1(v_1) = 1 & f_1(v_2) = 0 \\ f_2(x) = \alpha & f_1(v_1) = 0 & f_2(v_2) = 1 \end{array} \tag{1.1}$$

Now take

$$Q = f_1 \otimes v_1 + f_2 \otimes v_2 : X \to V.$$

From (1.1) $Q(v_1) = v_1$ and $Q(v_2) = v_2$ so Q is a projection and from (1.1)

$$Q(x) = f_1(x)v_1 + f_2(x)v_2 = \alpha v_2,$$

hence $\|Q(x)\| = \alpha$. $\square$

Theorem 1.2. *Let V be a two dimensional subspace of a uniformly convex Banach space X. Let P be a minimal projection from X onto V. Then there exists at least two linearly independent norming points for P.*

Proof. Take P to be a minimal projection from X onto V. Since uniformly convex spaces are reflexive and P is a compact operator and every compact operator attains its norm in reflexive Banach space P has at least one norming point. If $\|P\| = 1$ then the statement is obvious. Now, suppose that $\pm x_0 \in S(X)$ are the only norming points for P. From Lemma 1.1 there is a projection Q from X onto V such that

$$\|Q(\pm x_0)\| \leq \frac{1}{2}$$

and by continuity we can find $\epsilon > 0$ such that

$$\|Q(x)\| < 1, \quad \text{for any } x \in B(x_0, \epsilon) \cup B(-x_0, \epsilon). \tag{1.3}$$

We now claim that there exists $\eta > 0$ such that

$$\|P(x)\| < \|P\| - \eta, \quad \text{for any } x \notin B(x_0, \epsilon) \cup B(-x_0, \epsilon) \text{ and } x \in S(X). \tag{1.4}$$

Indeed if it is not so then for any $1/n$ we can find $x_n \in S(X)$ such that $x_n \notin B(x_0, \epsilon) \cup B(-x_0, \epsilon)$ and $\|P(x_n)\| \to \|P\|$. A sequence $\{P(x_n)\}$ is contained in two-dimensional space V therefore, choosing a subsequence if necessary, we can assume that $P(x_n) \to y_0$, since $\|P(x_n)\| \to \|P\|$ then $\|y_0\| = \|P_0\|$. Since uniformly convex spaces have Banach-Saks property (see Theorem III.7.1 [D]) and a sequence $\{x_n\}$ is bounded in norm we can choose a subsequence $\{x_{n_k}\}$ which arithmetic means converges in norm, i.e.,

$$y_k := \frac{x_{n_1} + ... + x_{n_k}}{k} \to y.$$

We will show that y is a norming point for P (of course $y \neq x_0$ and $y \neq -x_0$, hence contrary). First observe that since $\|x_n\| \leq 1$ then $\|y_k\| \leq 1$ which implies $\|y\| \leq 1$. Now

$$P(y_k) := \frac{P(x_{n_1}) + ... + P(x_{n_k})}{k} \to P(y),$$

but the sequence $P(x_{n_k}) \to y_0$, hence also its arithmetic means $P(y_k) \to y_0$. Therefore $\|y_0\| = \|P\|$ implies $\|P(y)\| = \|P\|$ hence y is a norming point for P different from $\pm x_0$, contrary to the assumption that $\pm x_0$ are the only norming points for P.

Now for every $t \in (0,1)$ consider a projection

$$R_t = tQ + (1-t)P : X \to V.$$

If $x \in B(x_0, \epsilon) \cup B(-x_0, \epsilon)$ and $x \in S(X)$ then by (1.3)

$$\begin{aligned} \|R_t(x)\| &= \|tQ(x) + (1-t)P(x)\| \\ &\le t\|Q(x)\| + (1-t)\|P(x)\| \\ &< t\|P\| + (1-t)\|P\| = \|P\|. \end{aligned} \tag{1.5}$$

If $x \notin B(x_0, \epsilon) \cup B(-x_0, \epsilon)$ and $x \in S(X)$ then by (1.4)

$$\begin{aligned} \|R_t(x)\| &= \|tQ(x) + (1-t)P(x)\| \le t\|Q(x)\| + (1-t)\|P(x)\| \\ &< t\|Q\| + (1-t)(\|P\| - \eta) = t(\|Q - \|P\| + \eta) + (\|P\| - \eta), \end{aligned} \tag{1.6}$$

the last term tends to $\|P\| - \eta$ if t tends to zero. Therefore for t_0 sufficiently small using (1.5) and (1.6)

$$\|R_{t_0}\| < \|P\|,$$

which contradicts minimality of P. □

Theorem 1.3. *Let V be a two-dimensional subspace of a smooth and uniformly convex space X. Then the minimal projection from X onto V is unique.*

Proof. Assume that there are two different minimal projections, say P_1 and P_2. Then $Q = (P_1 + P_2)/2$ is also a minimal projection (since $\|Q\| \le \|(P_1 + P_2)/2\| \le (\|P_1\| + \|P_2\|)/2 \le \lambda(V, X)$). Now take any $(f, x) \in \mathcal{E}(Q)$ (see Definition 0.3) and compute

$$\lambda(V,X) = f(Qx) = \tfrac{1}{2}f(P_1x) + \tfrac{1}{2}f(P_2x) \le \tfrac{1}{2}\lambda(V,X) + \tfrac{1}{2}\lambda(V,X) = \lambda(V,X),$$

therefore, since $f(P_ix) \le \|P_i\| = \lambda(V, X)$,

$$f(P_1x) = \lambda(V,X) = \|P_i\| \quad \text{and f}(P_2x) = \lambda(V,X) = \|P_i\|.$$

As a consequence we have

$$\begin{gathered} \mathcal{E}(Q) \subset \mathcal{E}(P_1) \quad \text{and} \quad \mathcal{E}(Q) \subset \mathcal{E}(P_2), \\ \text{i.e., any norming pair for } Q \text{ is also a norming pair for } P_1 \text{ and } P_2. \end{gathered} \tag{1.7}$$

Since Q is a minimal projection, by Theorem 1.2, there are x_1 and x_2 two linearly independent norming points for Q. Let (f_1, x_1) and (f_2, x_2) be corresponding norming pairs for Q. Observe that

$$f_1/_{V^*}, f_2/_{V^*} \quad \text{are linearly independent.} \tag{1.8}$$

Indeed, if not then $f_1 = \pm f_2$ and $f_1(Qx_1) = f_1(Q(\pm x_2)) = \|Q\|$. Hence

$$f_1(Q(\tfrac{x_1 + (\pm x_2)}{2}) = \|Q\|$$

so $\|\frac{x_1+(\pm x_2)}{2}\| = 1$ which is not possible if X is strictly convex.

From (1.7)

$$f_i(P_1 x_i) = \|P_1\| \quad \text{and} \quad f_i(P_2 x_i) = \|P_2\|.$$

Therefore

$$(P_1^* f_i)(x_i) = \|P_1\| = \lambda(V, X) \quad \text{and } (P_2^* f_i)(x_i) = \|P_2\| = \lambda(V, X).$$

It follows now that $(P_1^* f_i)/\|P_1\|$ and $(P_2^* f_i)/\|P_2\|$ are two norming functionals for x_i. Since X is smooth they have to be equal. Hence

$$P_1^* f_i = P_2^* f_i,$$

and since $f_i/_{V^*}$ span V^* ((1.8)) we have

$$P_1^* = P_2^*.$$

Hence $P_1 = P_2$. □

Corollary 1.4. *Let V be a two-dimensional subspace of $L_p(\mu)$ with $1 < p < \infty$. Then the minimal projection from $L_p(\mu)$ onto V is unique (this covers both classical cases $L_p[0,1]$ and ℓ_p).*

2. Norming pairs

It was seen in the previous section that there are at least two linearly independent norming points for a minimal projection onto two-dimensional subspace. In this section we show that there are at least six norming points, all together, for such a projection. We show, by means of the example, that number six cannot be increased.

Theorem 2.1. *A minimal projection from $L_p(\mu)$ (with $1 < p < \infty$) onto two-dimensional subspace has at least six different norming functionals $\pm f_1, \pm f_2, \pm f_3$. Moreover a set of restrictions to V^* of these functionals $\pm f_1/_{V^*}, \pm f_2/_{V^*}, \pm f_3/_{V^*}$ contains six different elements.*

Proof. By Theorem III.2.8 and Remark III.2.9 [OL], every set C such that

$$C \cup -C = \{ \text{ the set of norming functionals of P restricted to } V^*\}$$

and

$$C \cap -C = \varnothing$$

is linearly dependent over V^*. But by Theorem 1.2 and reasoning in proof of Theorem 1.3 (see (1.8)) we have at least two norming functionals for P which are linearly independent over V^*. Hence C has to contain at least three elements. □

Theorem 2.2. *A minimal projection from $L_p(\mu)$ (with $1 < p < \infty$) onto two-dimensional subspace has at least six different norming points $\pm x_1, \pm x_2, \pm x_3$.*

Proof. By previous theorem there are three norming functionals for P such that

$$f_1/_{V^*}, f_2/_{V^*}, f_3/_{V^*} \qquad \text{are three different functionals.} \tag{2.1}$$

To these functionals there corresponds three norming points x_1, x_2, x_3. Let

$$g_i = \frac{f_i \circ P}{\|P\|}.$$

By (2.1) g_1, g_2, g_3 are three different functionals on X^* of norm one. Also

$$g_i(x_i) = 1.$$

Now if $x_i = x_j$ (for some $i \neq j \in \{1,2,3\}$) then g_i and g_j are norming functionals for the same point $x = x_i = x_j$. Since the $L_p(\mu)$ is smooth that would imply $g_i = g_j$, contrary. Hence x_1, x_2, x_3 are all diferent. □

Theorem 2.3. *Let P be a minimal projection from ℓ_p^3 onto two-dimensional subspace V. Let $W = \{x \in \ell_p^n : (x_1, x_2, x_3) \in V$ and $x_4 = ... = x_n = 0\}$. Take a projection Q from ℓ_p^n onto W defined by*

$$Q(x_1, x_2, x_3, x_4, ..., x_n) = (P(x_1), P(x_2), P(x_3), 0, ..., 0).$$

Then Q is also a minimal projection having the same number of norming points and norming functionals as projection P.

Proof. By the very construction of Q, if $x = (x_1, ..., x_n)$ is a norming point for Q then $x_4 = ... = x_n = 0$. If $f = (f_1, ..., f_n)$ is a norming functional for Q then by the form of norming functionals (i.e., $f_i = sgn(a_i) \cdot |a_i|^{p/q}$) and the form of Q we get $f_4 = ... = f_n = 0$. Hence $\|Q\| = \|P\|$, moreover

$$x = (x_1, ..., x_n) \text{ is a norming point for } Q$$
$$\Updownarrow$$
$$x = (x_1, x_2, x_3) \text{ is a norming point for } P$$

and

$$f = (f_1, ..., f_n) \text{ is a norming functional for } Q$$
$$\Updownarrow$$
$$f = (f_1, f_2, f_3) \text{ is a norming functional for } P.$$

Since $L : \ell_p^n \to \ell_p^3$ given by $L(x_1, ..., x_n) = (x_1, x_2, x_3, 0, ..., 0)$ is norm one projection then by Proposition I.3.1 [OL] projection Q is also minimal projection. □

Now we will compute the norm, all norming points and all norming functionals for a particular minimal projection

Theorem 2.4. *Let $f = (1,1,1) \in \ell_q^3$ be a representation of a functional. Then $P : \ell_p^3 \to kerf$ given by*

$$P = Id - \tfrac{1}{3}(1,1,1) \otimes (1,1,1) \tag{2.2}$$

is a minimal projection for any $1 \leq p \leq \infty$.

Proof. First we will prove that P given by (2.2) is indeed a minimal projection. We will use Rudin Theorem. Observe that the following operators

$$L_\sigma(x_1, x_2, x_3) := (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}), \tag{2.3}$$

(where σ is any permutation of a set $\{1,2,3\}$) are isometries in ℓ_p^3. Furthermore

$$L_\sigma(ker(1,1,1)) \subset ker(1,1,1). \tag{2.4}$$

Now according to Theorem 0.7 it is enough to prove that P is the only projection which commutes with L_σ.

Any projection $Q : \ell_p^3 \to \ker(1,1,1)$ is given by

$$Qx = x - (1,1,1) \otimes (v_1, v_2, v_3), \quad \text{where } v_1 + v_2 + v_3 = 1. \tag{2.5}$$

Assume that Q commutes with L_σ, then

$$((1,1,1) \otimes (v_1, v_2, v_3)) \circ L_\sigma = L_\sigma((1,1,1) \otimes (v_1, v_2, v_3)).$$

Taking a value at $x = (x_1, x_2, x_3)$ at both sides of the above equality results in

$$(\sum_1^3 x_i, \sum_1^3 x_i, \sum_1^3 x_i) \cdot (v_1, v_2, v_3) =$$
$$= (\sum_1^3 x_i, \sum_1^3 x_i, \sum_1^3 x_i) \cdot (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}),$$

for any $\sigma \in S_3$ and for any $x = (x_1, x_2, x_3)$. Therefore $v_1 = v_2 = v_3$ and since $v_1 + v_2 + v_3 = 1$ we have

$$v_1 = v_2 = v_3 = \frac{1}{3}.$$

Hence $Q = P$. On the other hand it is easy to see that P indeed commutes with L_σ. Therefore P is minimal. □

Now we will restrict ourselves to $p = 4$.

Theorem 2.5. *Let $p = 4$. Then the minimal projection from Theorem 2.4 (see (2.2)) has exactly six norming points*

$$\begin{aligned}
x_0 &= \frac{1}{(2+2^{4/3})^{1/4}}(2^{1/3}, -1, -1), \quad x_3 = -x_0,\\
x_1 &= \frac{1}{(2+2^{4/3})^{1/4}}(-1, 2^{1/3}, -1), \quad x_4 = -x_1,\\
x_2 &= \frac{1}{(2+2^{4/3})^{1/4}}(-1, -1, 2^{1/3}), \quad x_5 = -x_2,
\end{aligned} \tag{2.6}$$

and exactly six norming functionals

$$\begin{aligned} f_0 &= \frac{1}{(2+2^4)^{3/4}}(2^3,-1,-1), \quad f_3 = -f_0, \\ f_1 &= \frac{1}{(2+2^4)^{3/4}}(-1,2^3,-1), \quad f_4 = -f_1, \\ f_2 &= \frac{1}{(2+2^4)^{3/4}}(-1,-1,2^3), \quad f_5 = -f_3. \end{aligned} \tag{2.7}$$

Additionally

$$\|P\| = \lambda(ker(1,1,1),\ell_p^3) = \frac{1}{3}(1+2^3)^{1/4}(1+2^{1/3})^{3/4}. \tag{2.8}$$

Proof. Projection P from (2.2) is given by

$$P(x_1,x_2,x_3) = \frac{1}{3}(2x_1 - x_2 - x_3, -x_1 + 2x_2 - x_3, -x_1 - x_2 + 2x_3),$$

therefore the problem of finding its norm and all norming points is equivalent to finding the maximum of the function

$$\begin{aligned} h(x_1,x_2,x_3) &= \left(\tfrac{2x_1-x_2-x_3}{3}\right)^4 + \left(\tfrac{-x_1+2x_2-x_3}{3}\right)^4 + \left(\tfrac{-x_1-x_2+2x_3}{3}\right)^4 \\ &\text{in the set:} \quad (x_1)^4 + (x_2)^4 + (x_3)^4 = 1, \end{aligned} \tag{2.9}$$

and finding all points at which this maximum is attained.

Let

$$\begin{aligned} z_1 &= \tfrac{2x_1-x_2-x_3}{3}, \; z_2 = \tfrac{-x_1+2x_2-x_3}{3}, \\ z_3 &= \tfrac{-x_1-x_2+2x_3}{3}, \; d = \tfrac{x_1+x_2+x_3}{3}. \end{aligned} \tag{2.10}$$

Then (2.9) is equivalent to finding the maximum and all points at which this maximum is attained of the following function

$$\begin{aligned} &f(z_1,z_2,z_3,d) = (z_1)^4 + (z_2)^4 + (z_3)^4 \\ &\text{in the set:} \quad (z_1+d)^4 + (z_2+d)^4 + (z_3+d)^4 = 1 \text{ and } z_1+z_2+z_3 = 0. \end{aligned} \tag{2.11}$$

Using standard Lagrange multipliers argument we construct the function

$$\begin{aligned} \varphi(z_1,z_2,z_3,d) =& (z_1)^4 + (z_2)^4 + (z_3)^4 \\ &- \lambda_1((z_1+d)^4 + (z_2+d)^4 + (z_3+d)^4) \\ &- \lambda_2(z_1+z_2+z_3) \end{aligned}$$

and in particular we obtain that z_1, z_2, z_3 has to fulfill the equations

$$g(z_1) = g(z_2) = g(z_3) = 0, \quad \text{where} \quad g(x) = 4x^3 - 4\lambda_1(x+d)^3 - \lambda_2. \tag{2.12}$$

Now assume that z_1, z_2, z_3 are distinct. Then by (2.12) z_1, z_2, z_3 will be three distinct zeros of g. That implies $\lambda_1 \neq 0$ (in that case g has only one zero), $\lambda_1 \neq 1$ (in that case g is polynomial of degree 2 hence has at most two zeros) and

$$g(x) = (4 - 4\lambda_1)(x - z_1)(x - z_2)(x - z_3). \tag{2.13}$$

Now by comparing the coefficients of g in (2.12) and (2.13) gives

$$z_1 + z_2 + z_3 = \frac{3\lambda_1 d}{1 - \lambda_1}.$$

On the other hand $z_1 + z_2 + z_3 = 0$, hence $d = 0$. But clearly a four tuple $(z_1, z_2, z_3, 0)$ is not a maximum of function f (2.11) since $f(z_1, z_2, z_3, 0) = 1$. Therefore we proved that

$$z_1 = z_2 \quad \text{or} \quad z_2 = z_3 \quad \text{or} \quad z_3 = z_1$$

which is equivalent to

$$x_1 = x_2 \quad \text{or} \quad x_2 = x_3 \quad \text{or} \quad x_3 = x_1. \tag{2.14}$$

By symmetry it is enough to let $x_2 = x_3$. Letting $x_2 = x_3$ from (2.9) we have to find the maximum (and all points at which this maximum is attained) of the function

$$h(x_1, x_2) = \frac{2 + 2^4}{3^4}(x_1 - x_2)^4 \quad \text{in the set:} \quad x_1^4 + 2x_2^4 = 1.$$

This can be easily solved using Lagrange multipliers and with (2.14) it leads to (2.6). Note that (2.7) follows immediately from (2.6). □

Using Theorem 2.3 and 2.5 we may observe

Corollary 2.6. *For any ℓ_4^n we can construct a two-dimensional subspace V of ℓ_4^n such that minimal projection P from ℓ_4^n onto V has only six norming functionals and six norming points.*

Remark 2.7. *Theorem 2.2 is not true if the word minimal is dropped - as we can easily find a projection (not minimal of course) which has only 2 different norming points. For instance,*

$$Q = Id - (1,1,1) \otimes (0,0,1)$$

has only 2 norming points $\pm(\frac{1}{2^{1/p}}, \frac{1}{2^{1/p}}, 0)$.

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Deptartment of Mathematics,
University of South Florida,
4202 E. Fowler Ave., PHY 114
Tampa, FL 33620-5700
USA

E-mail address: boris@math.usf.edu, skrzypek@chuma.cas.usf.edu.