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Hahn–Banach Operators: A Review

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In this paper we review ongoing work on operators having norm-preserving extensions to every overspace. We call them *Hahn–Banach operators.*

KEY WORDS: Hahn–Banach operators.

1. INTRODUCTION

Let V and W be a pair of Banach spaces. Let $L(V, W)$ be the space of all linear bounded operators from V into W.

Definition 1.1. An operator $T \in L(V, W)$ is called a *Hahn–Banach operator* if for every Banach space X containing V as a subspace there exists an operator $\tilde{T} \in L(X, W)$ such that

$$(1)\ \|\tilde{T}\| = \|T\|$$

$$(2)\ \tilde{T}v = Tv, \qquad \forall v \in V.$$

We use $HB(V, W)$ to denote the set of Hahn–Banach operators.

The classical Hahn–Banach theorem states that every rank-1 operator from V into W is a Hahn–Banach operator. It can also be restated in the following way: if $\dim W = 1$, then $HB(V, W) = L(V, W)$. Observe that the two statements above are slightly different. In the first case we describe a property of an operator T (being of rank 1) and conclude that, for *every pair V and W*, such an operator is a Hahn–Banach operator. The second statement starts with the description of the Banach space W (being one-dimensional) and concludes that for *every Banach space V and every operator* $T \in L(V, W)$, T is a Hahn–Banach operator.

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Hence a generalization of the Hahn–Banach theorem involves these players:

(1) an operator T (described in some way without referencing V and W);
(2) the Banach space V;
(3) the Banach space W.

Keeping some of these variables fixed, we ask to determine the "values" of the remaining variables that would guarantee the Hahn–Banach property. We will now describe some of these questions.

Questions:

1. For what spaces W (V) is it true that $HB(V, W) = L(V, W)$ for all V (W)?
2. Given V (W), what are the spaces W (V) such that $HB(V, W) = L(V, W)$?
3. What are the pairs of spaces V and W such that $HB(V, W) = L(V, W)$?
4. Given an $n \times n$ matrix A, what is a space V such that there exists an (such that every) operator $T \in HB(V, V)$ having (has) A as matrix representation?
5. Given a pair of spaces V and W, describe the class $HB(V, W)$. In particular let $L_k(V, W)$ refer to the set of all operators of rank k from V into W.
6. For what spaces V and W and for what integers k is it true that $L_k(V, W) \cap HB(V, W) \neq \emptyset$?, $L_k(V, W) \subset HB(V, W)$?
7. Let V and W be given. Let $T \in L_k(V, W) \cap HB(V, W)$ and $\|T\| = 1$. Let $B(V)$ and $B(W)$ be the unit balls of V and W respectively. Then $T(B(V)) \subset B(W)$. How large can a k-dimensional volume of $T(B(V))$ be in terms of the properties of V and W?

Question 1 seems to be the only question among those listed that has been studied in detail. There exists a vast literature on it. We refer the reader to the surveys of G. Buskes [1] and L. Nachbin [14]. The survey [1] contains an extensive bibliography that we have not reproduced here.

In particular, the results on the generalizations of the Hahn–Banach theorem obtained in the period 1940–1950 (G. P. Akilov, D. B. Goodner, L. Nachbin and R. Phillips, see references in [1]) lead to understanding that the following properties of a Banach space V are equivalent:

(a) The identity on V is a Hahn–Banach operator.
(b) $L(V, W) = HB(V, W)$ for every Banach space W.

(c) $L(W, V) = HB(W, V)$ for every Banach space W.

This property of the Banach space V is called the *extension property*.

The following result of L. Nachbin [13] gives the complete answer to Question 1 in the finite dimensional case.

Theorem 1.1. (L. Nachbin [13].) An n-dimensional Banach space has the extension property if and only if it is isometric to $l_\infty^{(n)}$.

The complete answer to Question 1 in general case was obtained by J. L. Kelley [9].

In this paper we limit ourselves to the finite-dimensional spaces V and W. We will review the partial results concerning Questions 2–7. Wherever possible we try to illustrate the situation with simple, two-dimensional examples, which provide (in our experience) an invaluable geometric insight into the problem.

2. CHARACTERIZATIONS OF THE HAHN–BANACH OPERATORS AND QUESTION 1

The following characterization is a version of Theorem 2 of [3], see also [10] (Section 4(b)).

Theorem 2.1. Let V be a finite-dimensional subspace of a Banach space X. Let $T \in L(V, W)$ and let $\tilde{T}$ be an extension of T to X. $\tilde{T}$ has minimal norm among all extensions if and only if there exists an operator $E_{\tilde{T}}$: $W \to X$ such that

$$\text{(a)}\ \operatorname{tr}(E_{\tilde{T}}\tilde{T}) = \|\tilde{T}\|\nu(E_{\tilde{T}})$$

$$\text{(b)}\ \operatorname{Range} E_{\tilde{T}} \subset V,$$

where ν denotes the nuclear norm on $L(W, X)$.

Proof. Let $\mathcal{A} = \{A \in L(X, W): Av = 0, \forall v \in V\}$. Then $\tilde{T}+A \in L(X, W)$ is an extension of T for all $A \in \mathcal{A}$. Hence $\tilde{T}$ has minimal norm among all extensions if and only if 0 is the best approximation to $\tilde{T}$ from $\mathcal{A}$. Observe that the dual of $L(X, W)$ under trace duality is $L(W, X^{**})$ with the nuclear norm. (This statement can be proved using a minor modification of similar results in [8]; see (1.8) and (1.11).) Hence there exists an operator

$E_{\tilde{T}} \in L(W, X^{**})$ such that

$$\operatorname{tr}(E_{\tilde{T}}\tilde{T}) = \|\tilde{T}\|\nu(E_{\tilde{T}})$$

$$\operatorname{tr}(E_{\tilde{T}}A) = 0, \qquad \forall A \in \mathcal{A}.$$

Let $w \in W$. Then

$$A := x^* \otimes w \in \mathcal{A} \qquad \text{for each } x^* \in V^{\perp}.$$

Hence

$$0 = \operatorname{tr}(E_{\tilde{T}}A) = x^*(E_{\tilde{T}}w)$$

for all $x^* \in V^{\perp}$. Thus $E_{\tilde{T}}w \in V$ for all $w \in W$ and $E_{\tilde{T}} \in L(W, X)$. It remains to observe that by local reflexivity (see [8], (17.2)) the nuclear norm of $E_{\tilde{T}}$ considered as an operator from $L(W, X)$ is the same as the nuclear norm of $E_{\tilde{T}}$ considered as an operator from $L(W, X^{**})$. □

Theorem 2.2. (*cf*. [5].) An operator $T \in L(V, W)$ is a Hahn–Banach operator if and only if there exists a pair $(w^*, v) \in W^* \times V$ such that the operator defined by

$$E = w^* \otimes v$$

satisfies the conclusion of Theorem 2.1.

Proof. Let $w^* \in W^*$ and $v \in V$ be such that

$$1 = \|w^*\| = \|v\| \quad \text{and} \quad w^*(Tv) = \|T\|.$$

Then clearly $E = w^* \otimes v$ satisfies (a) and (b) from Theorem 2.1.

Conversely, if $E = w^* \otimes v$ satisfies Theorem 2.1, then

$$||T||\,||v||\,||w^*|| \geq w^*(Tv) = w^*(\tilde{T}v) = \operatorname{tr}(E\tilde{T}) = ||\tilde{T}||\nu(E) = ||\tilde{T}||\,||v||\,||w^*||,$$

and hence $\|\tilde{T}\| = \|T\|$. □

In addition to this dual characterization of the Hahn–Banach operators, let us mention a characterization of $HB(V, W)$ in terms of factorizations.

Definition 2.1. For every $T \in L(V, W)$ define

$$\gamma_\infty(T) = \inf \|A\|\|B\|,$$

where the infimum is over all $A \in L(V, l_\infty^{(N)})$, $B \in L(l_\infty^{(N)}, W)$ such that

$$T = BA.$$

It is well known (see e.g. [19]) that

Theorem 2.3. $T \in L(V, W)$ is a Hahn–Banach operator if and only if

$$\|T\| = \gamma_\infty(T).$$

In the study of Hahn–Banach operators that are linear isomorphisms it turns out to be useful to exploit the following geometric language introduced in [15] and [16].

Let V be a finite dimensional normed space.

Definition 2.2. A symmetric with respect to 0, bounded, closed convex body $A \subset V$ is called a *sufficient enlargement* of (the unit ball) $B(V)$ (or for V) if for arbitrary isometric embedding $V \subset X$ there exists a projection $P: X \to V$ such that $P(B(X)) \subset A$.

With this definition we have the following obvious statement: a linear isomorphism T between finite dimensional normed spaces V and W is a Hahn–Banach operator if and only if $\|T\|\,T^{-1}(B(W))$ is a sufficient enlargement of $B(V)$.

3. QUESTIONS 2 AND 3

Questions 2 and 3 are essentially the same question. Based on purely aesthetic considerations it would be nice if the following conjecture was true.

Conjecture 3.1. A pair of finite-dimensional Banach spaces (V, W) satisfies

$$HB(V, W) = L(V, W)$$

if and only if one of the spaces V or W is isometric to an $l_\infty^{(n)}$.

Only the following result is known to us:

Theorem 3.1. Let V be a strictly convex or a smooth n-dimensional normed space and let W be n-dimensional. Then

$$HB(V, W) = L(V, W)$$

if and only if W is isometric to $l_\infty^{(n)}$.

Proof. Let us consider an operator $T\colon V \to W$ such that $||T|| = 1$ and such that $T(B(V))$ has the maximal possible volume among all operators of norm 1. From now on we identify V and W using T and consider the obtained space with two norms $||\cdot||_V$ and $||\cdot||_W$. If T is a Hahn–Banach operator, then $B(W)$ is a sufficient enlargement of $B(V)$.

Using a result due to D. R. Lewis and V. D. Milman (see Theorem 1.3 in [11] and Theorem 14.5 in [19]) and the Binet–Cauchy theorem, we can find n linearly independent points $x_1, \dots, x_n$ and n linearly independent functionals $f_1, \dots, f_n$ such that

$$||f_i||_V = ||f_i||_W = ||x_i||_V = ||x_i||_W = f_i(x_i) = 1.$$

It implies that $B(W)$ is contained in the parallelepiped

$$P = \{x\colon\ |f_i(x)| \le 1 \text{ for every } i\}.$$

In order to finish the proof it is enough to show that $P = B(W)$. By a result of [16] (see Corollary on p. 317) it suffices to show that there is no $f \in B(V^*)$ such that $|f(x_i)| = |f(x_j)| = 1$ for $i \neq j$.

In the strictly convex case it immediately follows from the definition.

In the smooth case it follows from the fact that f_i is the only functional in $B(V^*)$ such that $f_i(x_i) = 1$ and the condition that $\{f_i\}_{i=1}^n$ are linearly independent, and, hence, are different. □

4. QUESTION 4

This question was addressed in the papers [6], [7], and [17].

Theorem 4.1. (*cf.* [6], [17].) Let A be an $n \times n$ matrix different from a scalar multiple of the identity. Then there exists a Banach space V and a Hahn–Banach operator $T \in L(V, V)$ such that A is a matrix representation of the operator T with respect to some basis in V, and V is not isometric to $l_\infty^{(n)}$.

Instead of giving the proof for this theorem we will illustrate it by two examples.

Example 4.1.

$$\text{Let } A = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \quad \text{where } |d| < 1.$$

We show there exists a two-dimensional Banach space $V = V(d)$ such that $V(d)$ is not isometric to $l_\infty^{(2)}$ and $\lambda(V(d)) = 1$. Indeed let $V(d)$ be the space with unit ball $B(V(d))$ given in $\mathbf{R}^2$ by

$$\|(a_1, a_2)\| := \max(|a_1|, |(1-d)a_1 + a_2|, |(1-d)a_1 - a_2|) \le 1.$$

I.e., $B(V(d))$ is the (convex hull of the) hexagon with vertices $\pm(0, 1)$, $\pm(1, d)$, $\pm(1, -d)$. The coordinate functionals $\phi_i \in V(d)^*$ defined by

$$\phi_i(a_1, a_2) = a_i, \qquad i = 1, 2, \tag{1}$$

have norms equal to 1 (since $B(V(d))$ is inside the unit square). It follows from (1) that, for the operator T defined on $V(d)$ by

$$T(a_1, a_2) = (a_1, da_2) \quad \text{or} \quad T = \phi_1 \otimes e_1 + d\phi_2 \otimes e_2,$$

we have

$$\|T\| = \|T(1, 0)\| = 1.$$

Let $X \supset V(d)$. Let $\tilde{\phi}_i$ be the Hahn–Banach extension of the functionals ϕ_i in (2.2). Define an operator $\tilde{T}\colon X \to V(d)$ by

$$\tilde{T} = \tilde{\phi}_1 \otimes e_1 + d\tilde{\phi}_2 \otimes e_2, \qquad \text{i.e., } \tilde{T}x = (\tilde{\phi}_1(x), d\tilde{\phi}_2(x)).$$

If $\|x\| = 1$ then $|\tilde{\phi}_1(x)| \le 1$; $|\tilde{\phi}_2(x)| \le 1$ and since $(1, d) \in B(V(d))$ we have

$$\tilde{T}x \in B(V(d)).$$

Hence $\|\tilde{T}\| \le 1$.

Finally, observe that, since $B(V(d))$ is a hexagon, $V(d)$ is not isometric to $l_\infty^{(2)}$ (whose unit ball is the square).

The next example deals specifically with the one remaining diagonal case $d = -1$.

Example 4.2.

$$\text{Let } A = \begin{bmatrix} 1 & 0 \\ 0 & {}^-1 \end{bmatrix}.$$

Let V be the space with unit ball $B(V)$ given in $\mathbf{R}^2$ by

$$\|(a_1, a_2)\| := \max(|a_1|, |a_2|, |a_1 - a_2|) \le 1.$$

I.e., $B(V)$ is the (convex hull of the) hexagon with vertices $\pm(1, 1)$, $\pm(1, 0)$, $\pm(0, 1)$.

Again, as in Example 4.1, the coordinate functionals ϕ_i, $i = 1, 2$, in V^* have norms equal to 1 (since $B(V)$ is inside the unit square). Consider the operator T defined on V by

$$T(a_1, a_2) = (a_1, -a_2) \quad \text{or} \quad T = \phi_1 \otimes e_1 - \phi_2 \otimes e_2.$$

Let $X \supset V$ and define an operator $\tilde{T}\colon X \to V$ by

$$\tilde{T} = \tilde{\phi}_1 \otimes e_1 - \tilde{\phi}_2 \otimes e_2, \qquad \text{i.e., } \tilde{T}x = (\tilde{\phi}_1(x), -\tilde{\phi}_2(x)),$$

where, as in Example 4.1, $\tilde{\phi}_i$ is a Hahn–Banach extension of the functional ϕ_i, $i = 1, 2$. Since $\|(1, 1)\| = 1$ and $\|T(1, 1)\| = 2$, we have $\|T\| \geq 2$. On the other hand, clearly $\|T\| \leq \|\phi_1\| + \|\phi_2\| = 2$ and, similarly, $\|\tilde{T}\| \leq 2$.

As in Example 4.1, the unit ball of V is an hexagon and thus V is not isometric to $l_\infty^{(2)}$.

In general the following is shown in [6] and [17]:

Theorem 4.2. For every 2×2 matrix A, different from a scalar multiple of the identity, there exists a two-dimensional Banach space V with an hexagonal unit ball and a Hahn–Banach operator $T \in L(V, V)$ that corresponds to the matrix A.

The case of a multiple of the identity is excluded for a reason. Indeed, if $\alpha I \in L(V, V)$ is a Hahn–Banach operator, then I is also a Hahn–Banach operator and by Theorem 1.1 V is isometric to $l_\infty^{(n)}$.

Finally, observe that the operator of Example 4.2 has norm 2 and spectral radius 1. That these quantities are not equal, turns out to be a necessity, as the following theorem shows.

Theorem 4.3. (*cf.* [7].) Let A be an $n \times n$ matrix with the spectrum in the unit circle. Suppose that there exists a Hahn–Banach operator $T \in L(V, V)$ such that $\|T\| = 1$ and T is represented by A. Then V is isometric to $l_\infty^{(n)}$.

5. QUESTIONS 5 AND 6

In this section we will concern ourselves primarily with the existence of an operator $T \in L_k(V, W) \cap HB(V, W)$. In other words we focus on the description of finite-dimensional spaces V and W for which there exists a Hahn–Banach operator of rank k. These questions have been dealt with in [5] and [18]. We will need a few definitions. Let X be a Banach space, let $B(X)$ be its unit ball and $S(X)$ be its unit sphere. Let $x \in S(X)$ and let H be a supporting hyperplane to $B(X)$ containing x. Let $F_{x,H} = S(X) \cap H$. The set $F_{x,H}$ we call a "flat spot." The space X is strictly convex if $\dim F_{x,H} = 0$ for every x and H. The space X is called smooth if for each $x \in S(X)$ there exists only one supporting hyperplane of $B(X)$ containing x. In finite dimensional case X is smooth if and only if X^* is strictly convex.

It turns out that the flat spots on the spheres of V^* and W play the fundamental role in constructing the Hahn–Banach operators. We need the following parameter.

$\phi(X) := \max\{\dim F_{x,H} : x \in S(X),\ H$ is a supporting hyperplane to $B(X)$ containing x$\}$.

We will now indicate how we can construct Hahn–Banach operators using flatness. Once again for simplicity we will use two-dimensional examples.

Suppose that V is a two-dimensional space such that $\phi(V^*) = 1$. That means that there exist two functionals v_1^* and $v_2^* \in S(V^*)$ and an element $v_0 \in S(V)$ such that

$$v_1^*(v_0) = v_2^*(v_0) = 1.$$

We will now choose vectors $w_1, w_2 \in W$ so that the operator

$$T = v_1^* \otimes w_1 + v_2^* \otimes w_2$$

is a Hahn–Banach operator. In order to do so, let w_1 and w_2 have the property that

$$1 = \|w_1 + w_2\| \geq \|\alpha_1 w_1 + \alpha_2 w_2\|, \qquad \forall |\alpha_j| \leq 1. \tag{2}$$

We will establish the existence of such a basis below. For now let $X \supset V$. Let x_1^* and x_2^* be the Hahn–Banach extensions of v_1^* and v_2^*. Then

$$\tilde{T} = x_1^* \otimes w_1 + x_2^* \otimes w_2.$$

For every $x \in S(X)$

$$\|\tilde{T}(x)\| = \|x_1^*(x)w_1 + x_2^*(x)w_2\| \leq \|w_1 + w_2\| = \|T(v_0)\| \leq \|T\|.$$

Hence $\|\tilde{T}\| \leq \|T\|$ and $T \in HB(V, W)$. To establish the existence of w_1 and w_2 with property (2), let $w_0 \in S(W) \subset \mathbf{R}^2$ be a point such that its Euclidean distance to the origin is minimal among all the points of $S(W)$. Let $\tilde{w}_1$ and $\tilde{w}_2$ be an orthonormal basis in $\mathbf{R}^2$ with $\tilde{w}_i \neq w_0$. By the symmetry of $S(W)$ we may assume $\tilde{w}_1 = (1, 0)$ and $\tilde{w}_2 = (0, 1)$. Then $w_0 = \alpha\tilde{w}_1 + \beta\tilde{w}_2$. Denoting $w_1 = \alpha\tilde{w}_1$ and $w_2 = \beta\tilde{w}_2$, we have

$$1 = \|w_0\| = \|w_1 + w_2\|.$$

Then, by the definitions of w_1 and w_2, we have $\|\alpha_1 w_1 + \alpha_2 w_2\|_2 \leq \|w_1 + w_2\|_2$ for $|\alpha_j| \leq 1$, where $\|\cdot\|_2$ denotes the Euclidean norm. But, by definition of $w_0 = w_1 + w_2$, the Euclidean ball of radius $\|w_1 + w_2\|_2$ is contained in the W-ball of radius 1. Therefore

$$1 = \|w_1 + w_2\| \geq \|\alpha_1 w_1 + \alpha_2 w_2\|, \qquad \forall |\alpha_j| \leq 1$$

and we are done.

We will now reproduce the two-dimensional construction in [5] of a Hahn–Banach operator if $\phi(W) = 1$. (Note that in this construction

$W = V$.) Suppose that V with unit sphere $S(V) \subset \mathbf{R}^2$ is not strictly convex. Then (via a linear transformation) we can assume that $S(V)$ lies inside the unit square C, touches C on all four sides, and has a flat spot running from $(1, d)$ to $(1, -d)$ (and of course a corresponding flat spot running from $(-1, -d)$ to $(-1, d)$), where "flat spot" is defined in the usual obvious way. Let P be the rectangle with corners $(\pm 1, \pm d)$, whence P lies inside $S(V)$. (See Figure 1 below.)

Now consider the linear transformation $T(a_1, a_2) := (a_1, da_2)$ taking C into P. Thus $T(S(V))$ lies inside P, which lies inside $S(V)$, and, since $T(1, 0) = (1, 0) \in S(V) \cap T(S(V))$, we see that T: $V \to V$ has norm one. Note that we can also write T in the form

$$= \phi_1 \otimes (1, 0) + d\phi_2 \otimes (0, 1),$$

where ϕ_i are the norm-1 coordinate functionals on V defined by

$$\phi_i(a_1, a_2) = a_i, \qquad i = 1, 2.$$

Next let $\tilde{\phi}_i$ be a Hahn–Banach extension to X of the functional ϕ_i on V, $i = 1, 2$. Then the extension operator $\tilde{T}$: $X \to V$ has the form

$$\tilde{T} = \tilde{\phi}_1 \otimes (1, 0) + d\tilde{\phi}_2 \otimes (0, 1).$$

Finally, since $\tilde{T}(S(X)) \subset T(\overline{co}C) = \overline{co}P \subset \overline{co}S(V) = B(V)$, we see that $\|\tilde{T}\| \leq 1$ and hence, since $\|\tilde{T}\| \geq \|T\| = 1$, it follows that $\|\tilde{T}\| = \|T\|$.

Observe that this construction is essentially the same as in Example 4.1.

Of course it is easy to generalize the second example above to $W \neq V$ and both examples to a k-dimensional situation (see [18] for details). What is amazing is that these are (in a sense) the only possibilities (up to the choice of basis).

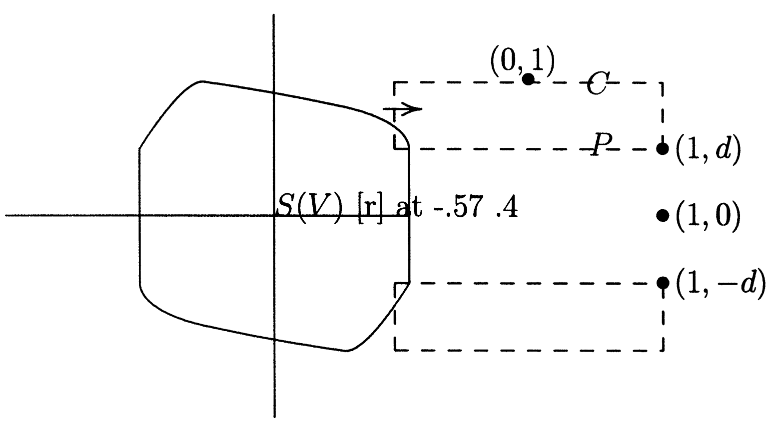

Fig. 1

Theorem 5.1. (*cf.* [5], [18].) Let V and W be a pair of finite-dimensional Banach spaces. Let k be an integer such that $\min\{\dim V, \dim W\} \geq k$. Then

$$L_k(V, W) \cap HB(V, W) \neq \emptyset$$

if and only if

$$\phi(V^*) + \phi(W) \geq k - 1.$$

For the following Example 5.1 we need the following two theorems and will need to observe the proof of the second:

Theorem 5.2. ([2].) Let $V = [\vec{v}\,]$ be a smooth real subspace of $L_1[\mathcal{T}, \nu]$. Then $\tilde{T} = \sum_{i=1}^{n} u_i \otimes v_i$ is a minimal extension of T (to $L_1[\mathcal{T}, \nu]$) implies that, for almost all t,

$$\vec{u}(t) = \|\tilde{T}\|\, \vec{z}(t)$$

with $\vec{z}(t)$ being a point of intersection of $S(V)$ and its tangent plane perpendicular to $M\vec{v}(t)$, where M is the $n \times n$ matrix M of $(E_{\tilde{T}})_{|V}$ (from Theorem 2.1) with respect to the basis $\vec{v}$. (See Fig. 2 below.)

Theorem 5.3. Let V be a smooth, strictly convex, n-dimensional real subspace of $L^1[-1, 1]$. Let $T\colon V \to V$ be such that there exists an extension $\tilde{T}\colon L^1[-1, 1] \to V$ with $\|\tilde{T}\| = \|T\|$. Then T is a rank-one operator.

Proof. By Theorem 2.2, corresponding to $\tilde{T} = \sum_{i=1}^{n} u_i \otimes v_i$, there exists a rank-one operator E satisfying the conclusion of the theorem. Hence the matrix M defined as the matrix of E, with respect to the basis $\vec{v}$, has rank one.

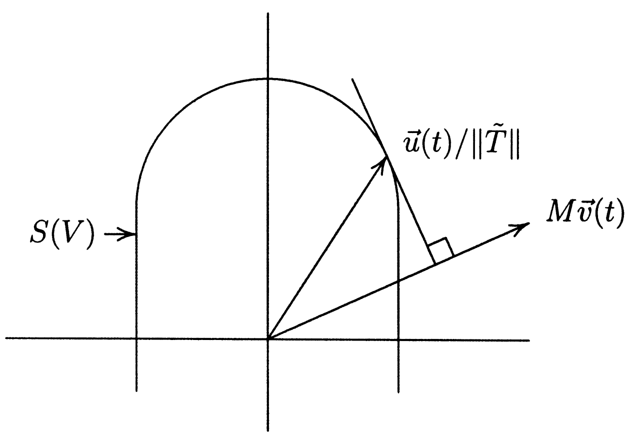

Fig. 2

Since V is smooth there exists a minimal extension $\tilde{T}$ of T satisfying the conditions of Theorem 5.2. But M has rank one, whence the vectors $M\vec{v}(t)$ lie on the same one-dimensional subspace of $\mathbf{R}^n$ for almost all $t \in [-1, 1]$.

Since V is strictly convex, the vectors $\vec{u}(t)$ also lie in the same one-dimensional subspace of $\mathbf{R}^n$ for almost all $t \in [-1, 1]$. Hence the operator $\tilde{T}$ given by Theorem 5.2 has rank one and thus so does T. □

Example 5.1. We now turn to the description of $HB(V, W)$ in the specific case where

$$B(V) = B(W) = \left\{(a, b) \in \mathbf{R}^2 : \int_{-1}^{1} \left|\frac{a}{2} + bt\right| dt \le 1\right\}.$$

In this case $S(V)$ is comprised of the line segments $(\pm 1, b)$, $-\frac{1}{2} \le b \le \frac{1}{2}$, and the half ellipses given by $(a^2 + 4b^2)/4b = 1$, $\frac{1}{2} \le |b| \le 1$. It is easy to see that $S(V)$ is smooth (but not strictly convex, having flat spots given by the line segments $(\pm 1, b)$, $-\frac{1}{2} \le b \le \frac{1}{2}$, and thus from the proof of Theorem 5.3 we see that $\tilde{T}$ must have the following form (without loss we can choose the "right-hand" flat spot; this corresponds to normalizing the "action matrix" A below so that $A_{11} > 0$):

$$\tilde{T} = 1 \otimes \tfrac{1}{2} + u_2 \otimes t, \qquad 0 \le |u_2(t)| \le \tfrac{1}{2},$$

i.e., $(u_1(t), u_2(t))$ must lie on the right-hand flat spot. Next let $\|x\| = \|x\|_{L^1[-1,1]} = 1$. Then it is immediate that $\|\tilde{T}x\| \le 1$, and so $\|\tilde{T}\| \le 1$. On the other hand it is clear that $\tilde{T}(\frac{1}{2}) = 1$, and thus $\|\tilde{T}\| = \|T\| = 1$, where $T = \tilde{T}_{|_V}$. Now, letting $(s, r) = \int_{-1}^{1} (\frac{1}{2}, \tau) u_2(\tau)\, d\tau$, we conclude that

$$HB(V, V) = \{T\colon\ T(a + bt) = \rho(a + (rb + 2sa)t),\ |\rho| > 0\},$$

where u_2 ranges over all functions such that $\|u_2\|_\infty \le \frac{1}{2}$ and $r \ne 0$. Note that each T has the corresponding matrix A (with respect to $(\frac{1}{2}, t)$) given by

$$A = \rho \begin{pmatrix} 1 & s \\ 0 & r \end{pmatrix}.$$

6. QUESTION 7

In this last section of the paper we offer a few remarks regarding the last question: Let $T \in HB(V, W) \cap L_k(V, W)$ be such that $\|T\| = 1$. How large can the volume of $T(B(V))$ be? If $V = W$ we can rephrase this question in terms of the determinant of T.

Since the very existence of a k-dimensional Hahn–Banach operator depends on the dimension of the flat spots of V^* and W, we would expect that the size of the range $T(B(V))$ depends on the size of the flat spots of W^* and V. Here are a few preliminary examples that indicate that this is in fact the case.

In the following discussion we restrict ourselves to the case $k = 2$. First we recall that any real two-dimensional space V can be isometrically imbedded in an L^1-space (see e.g., [20]), which is in turn a maximal over-space for V (i.e., $\sup_X\{\|\tilde{T}\| : X \to V\}$ is achieved for X equal to this L^1-space (see e.g., [12], [14])). We can thus conclude, from the construction preceeding Fig. 1, that, by normalizing the $B(V)$ as in Fig. 2, if $S(V)$ has a flat spot of half-length d, then there exists an Hahn–Banach operator on V taking the unit square (corners $(\pm 1, \pm 1)$) into itself and with determinant d.

Conversely, suppose that V is smooth and has a Hahn–Banach operator T. Then, analogously as in Example 5.1, $S(V)$ must have a flat spot corresponding to T. Suppose furthermore that, after normalizing $B(V)$ as in Fig. 1, the maximum height of $S(V)$ occurs at $(0, 1)$ (e.g., if $S(V)$ is symmetric with respect to the vertical b-axis).

Then

$$V = \left\{(a, b) \in \mathbf{R}^2 : \int_{-1}^{1} \left| \frac{a}{2} + \frac{b}{2d}\, t \right| d\mu(t) \le 1 \right\},$$

for some positive measure μ, where μ is normalized by $\int_{-1}^{1} \frac{1}{2}\, d\mu(t) = 1$ and $d = \int_{-1}^{1} |t|\, d\mu(t)/2$. Thus, generalizing Example 5.1, we see that the size of the flat spot is given by d. We argue analogously as in Example 5.1 to conclude that by letting $(s, r) = \int_{-1}^{1} (\frac{1}{2}, \tau) u_2(\tau)\, d\mu(\tau)$, we have

$$HB(V, V) \supset \{T\colon T(a + bt) = \rho(a + (rb + 2sa)t),\ |\rho| > 0\},$$

where u_2 ranges over all functions such that $\|u_2\|_\infty \le d$ and $r \ne 0$. Note that each T has the corresponding matrix A (with respect to $(\frac{1}{2}, t)$) given by

$$A = \rho \begin{pmatrix} 1 & s \\ 0 & r \end{pmatrix}.$$

As an illustrative subexample of the foregoing, note that the case V being isometric to $l_\infty^{(2)}$ (considered as a limit of spaces with the corners smooth) corresponds to $\mu = \delta_{-1} + \delta_1$, where then $d = 1$.

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