PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 3, Pages 719–724
S 0002-9939(00)05944-X
Article electronically published on November 8, 2000

A TWO-DIMENSIONAL HAHN-BANACH THEOREM

B. L. CHALMERS AND B. SHEKHTMAN

(Communicated by Dale Alspach)

Abstract. Let $\tilde{T} = \sum_{i=1}^{n} \tilde{u}_i \otimes v_i : V \to V = [v_1, ..., v_n] \subset X$, where $\tilde{u}_i \in V^*$ and X is a Banach space. Let $T = \sum_{i=1}^{n} u_i \otimes v_i : X \to V$ be an extension of $\tilde{T}$ to all of X (i.e., $u_i \in X^*$) such that T has minimal (operator) norm. In this paper we show in particular that, in the case $n = 2$ and the field is $\mathbf{R}$, there exists a rank-n $\tilde{T}$ such that $\|T\| = \|\tilde{T}\|$ for all X if and only if the unit ball of V is either not smooth or not strictly convex. In this case we show, furthermore, that, for some $\|T\| = \|\tilde{T}\|$, there exists a choice of basis $v = v_1$, v_2 such that $\|u_i\| = \|\tilde{u}_i\|$, $i = 1, 2$; i.e., each u_i is a Hahn-Banach extension of $\tilde{u}_i$.

1. Introduction

Let X be a Banach space and V be a subspace of X. If T is an operator from X into V, then $T_{|V}$ is the restriction of T onto V. Let $\tilde{T} : V \to V$.

Define

$$e(\tilde{T}; X) = \inf_{T} \{\|T\| : T_{|V} = \tilde{T}\}$$

and

$$e(\tilde{T}) = \sup_{X} \{e(\tilde{T} : X) : X \supset V\}.$$

The classical Hahn-Banach Theorem states that, for every rank-one operator $\tilde{T}$,

$$e(\tilde{T}) = \|\tilde{T}\|.$$

The Nachbin Theorem shows that, if V is n-dimensional and, for every $\tilde{T} : V \to V$,

$$e(\tilde{T}) = \|\tilde{T}\|,$$

then $V \cong \ell_\infty^{(n)}$.

Definition 1.1. An operator $\tilde{T} : V \to V$ is called a Hahn-Banach operator if $e(\tilde{T}) = \|\tilde{T}\|$.

In this paper we are concerned with the existence of non-trivial Hahn-Banach operators on a Banach space V. In particular we show (Theorem 2.2) that the two-dimensional real Banach space V possesses a rank-two Hahn-Banach operator if and only if V is either non-smooth or not strictly convex.

For the proof of the theorem we will need a number of definitions.

Received by the editors January 5, 1999.
2000 *Mathematics Subject Classification.* Primary 46B20; Secondary 47A20.

Let X be a Banach space and V be a finite-dimensional subspace. Let T be a bounded linear operator from X into V and let $\tilde{T}$ be the restriction of T onto V. ($\tilde{T} = T_{|_V}$.)

Definition 1.2. T is a minimal norm extension of $\tilde{T}$ means

$$\|T\| = \inf\{\|S\| : S : X \to V : S|_V = \tilde{T}\}.$$

Definition 1.3. The extremal set of $T : X \to V$ is

$$\mathcal{E}(T) = \{(x^*, x^{**}) \in B(X^*) \times B(X^{**}) : x^*(T^{**}x^{**}) = \|T\|\},$$

where B denotes the unit ball and T^{**} denotes the second adjoint extension of T ($T^{**}x^{**} = \sum_{i=1}^n \langle u_i, x^{**}\rangle v_i$ and $\|T^{**}\| = \|T\|$).

Note 1.1. In the following we will designate by $x^* \otimes x^{**} : X \to X^{**}$ the usual dyad operator given by $x^* \otimes x^{**}(x) = \langle x, x^*\rangle x^{**}$.

Theorem 1.1 ([2]). *T is a minimal norm extension of $\tilde{T}$ if and only if there exists a probability measure μ on $\mathcal{E}(T)$ such that the operator*

$$E_T = \int x^* \otimes x^{**}\, d\mu$$

maps V into V.

Definition 1.4. An n-dimensional subspace $V \subset L_1[\mathcal{T}, \nu]$ is said to be smooth if no non-zero element v of V vanishes on a set of positive ν-measure.

Note 1.2. It is well known that the n-dimensional real space V is smooth if and only if the unit sphere $S(V)$ is differentiable (has a unique $(n-1)$-dimensional tangent plane) at every point. This fact follows immediately from $\|\vec{a}\| := \|\vec{a} \cdot \vec{v}\|$ and the formula (see e.g. [1])

$$\frac{\partial}{\partial a_i}\|\vec{a}\| = \int_T v_i(s)\,\mathrm{sgn}[\vec{a} \cdot \vec{v}(s)]\, d\nu(s) \quad \text{for } \vec{a} \neq 0$$

and the Implicit Function Theorem. A point where $S(V)$ is non-differentiable is called a point of non-smoothness.

Theorem 1.2 ([1]). *Let $V = [\vec{v}]$ be a smooth real subspace of $L_1[\mathcal{T}, \nu]$. Then $T = \sum_{i=1}^n u_i \otimes v_i$ is a minimal extension of $\tilde{T}$ (to $L_1[\mathcal{T}, \nu]$) implies that, for almost all t,*

$$\vec{u}(t) = \|T\|\, \vec{z}(t)$$

with $\vec{z}(t)$ being a point of intersection of $S(V)$ and its tangent plane perpendicular to $M\vec{v}(t)$, where M is the $n \times n$ matrix M of $(E_T)_{|_V}$ with respect to the basis $\vec{v}$. (See Figure 1.)

Corollary 1.1. *Let $V = [\vec{v}]$ be a smooth real subspace of $L_1[\mathcal{T}, \nu]$. Then without loss $\mathcal{T}$ can be enlarged so that there exists $T = \sum_{i=1}^n u_i \otimes v_i$ which is a minimal extension of $\tilde{T}$ and the operator T^* is an extension of $\tilde{T}^* : V^* \to V^*$.*

Proof. Let $T = \sum_{i=1}^n u_i \otimes v_i$ be a minimal extension of $\tilde{T}$. Theorem 1.2 provides that $\vec{u}/\|T\|$ lies on $S(V)$ a.e.$[\nu]$. Next replace $\mathcal{T}$ by $\mathcal{T} \cup \Delta\mathcal{T}$, where $\Delta\mathcal{T} = S(V) -$range $(\vec{u}/\|T\|)$, enlarge ν and $\vec{v}$ to be zero on $\Delta\mathcal{T}$, and enlarge $\vec{u}/\|T\|$ to be the coordinate functions on $\Delta\mathcal{T}$. Then let T and $\tilde{T}$ be replaced by their corresponding enlargements, recall that V^* is isometric to the space of coordinate functions of $S(V)$ regarded as a subspace of $L_\infty(S(V))$, and note that $T^*_{|_{V^*}} = \tilde{T}^*$. $\square$

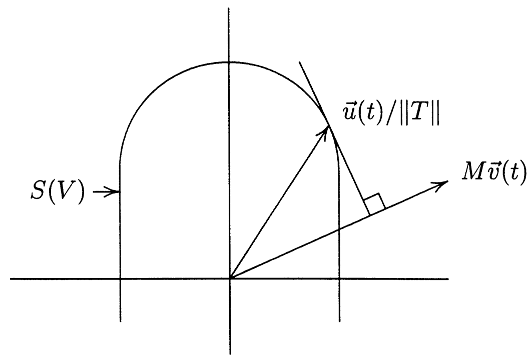

FIGURE 1.

2. Main theorem

Lemma 2.1. *Let T be an extension of $\tilde{T}$. Then $\|T\| = \|\tilde{T}\|$ if and only if there exists a pair*

$$(x^*,\, v) \in \mathcal{E}(T),\ v \in V.$$

Hence the dyad operator $E_T = x^* \otimes v$ satisfies the conclusion of Theorem 1.1.

Proof. If $(x^*,\, v) \in \mathcal{E}(T)$, then

$$\|T\| = x^*(Tv) = \|Tv\|.$$

Thus T attains its norm on V and

$$\|T\| = \|Tv\| = \|\tilde{T}v\| \le \|\tilde{T}\|.$$

Conversely, if $\|T\| = \|\tilde{T}\|$, let $v \in V$ be such that $\|\tilde{T}v\| = \|Tv\| = \|\tilde{T}\| = \|T\|$. Let $x^* \in X^*$ be such that $x^*(Tv) = \|Tv\|$. Then $(x^*,\, v) \in \mathcal{E}(T)$. □

The following theorem shows that, if $n > 1$ and the field is $\mathbf{R}$, a smooth, strictly convex, n-dimensional L_1-subspace V possesses no rank-n Hahn-Banach operator. In fact it shows more, namely, that V possesses no rank-k operator which extends to L_1 of the same norm, $k > 1$.

Theorem 2.1. *Let V be a smooth, strictly convex, n-dimensional real subspace of $L^1[-1,1]$. Let $\tilde{T} : V \to V$ be such that there exists an extension $T : L^1[-1,1] \to V$ with $\|T\| = \|\tilde{T}\|$. Then $\tilde{T}$ is a rank-one operator.*

Proof. By Lemma 2.1, corresponding to $T = \sum_{i=1}^{n} u_i \otimes v_i$, there exists a rank-one operator E_T satisfying the conclusion of the lemma. Hence the matrix M defined as the matrix of $(E_T)_{|V}$, with respect to the basis $\vec{v}$, has rank one.

Since V is smooth there exists a minimal extension T of $\tilde{T}$ satisfying the conditions of Theorem 1.2. But M has rank one, whence the vectors $M\vec{v}(t)$ lie on the same one-dimensional subspace of $\mathbf{R}^n$ for almost all $t \in [-1,1]$.

Since V is strictly convex, the vectors $\vec{u}(t)$ also lie in the same one-dimensional subspace of $\mathbf{R}^n$ for almost all $t \in [-1,1]$. Hence the operator T given by Theorem 1.2 has rank one and thus so does $\tilde{T}$. □

Corollary 2.1. *n-dimensional Hilbert space $(\ell_2^{(n)})$ possesses no Hahn-Banach operator with rank > 1.*

Proof. It is well known (see [5]) that $\ell_2^{(n)}$ can be realized as a subspace of L_1. ☐

Theorem 2.2. *Let V be a two-dimensional real Banach space. Then the following are equivalent:*
(1) *V is smooth and strictly convex.*
(2) *For every rank-two operator*

$$\tilde{T} : V \to V, \quad e(\tilde{T}) > \|\tilde{T}\|.$$

Proof. (1) implies (2) by Theorem 2.1 since every two-dimensional space V can be isometrically imbedded into $L_1[-1,1]$ (see e.g. [6]).

We now show (2) implies (1).

Suppose that V with unit sphere $S(V) \subset \mathbf{R}^2$ is not strictly convex. Then (via a linear transformation) we can assume that $S(V)$ lies inside the unit square C, touches C on all four sides, and has a flat spot running from $(1,d)$ to $(1,-d)$ (and of course a corresponding flat spot running from $(-1,-d)$ to $(-1,d)$), where "flat spot" is defined in the usual obvious way. Let P be the rectangle with corners $(\pm 1, \pm d)$, whence P lies inside $S(V)$. (See Figure A.)

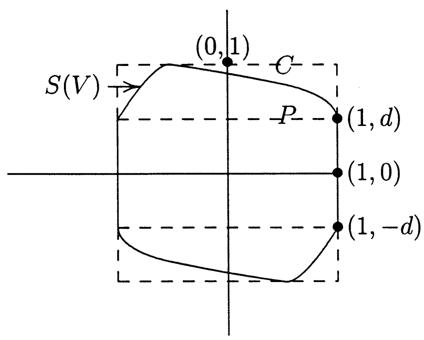

Figure A.

Now consider the linear transformation $\tilde{T}(a_1, a_2) := (a_1, da_2)$ taking C into P. Thus $\tilde{T}(S(V))$ lies inside P, which lies inside $S(V)$, and, since $\tilde{T}(1,0) = (1,0) \in S(V) \cap \tilde{T}(S(V))$, we see that $\tilde{T} : V \to V$ has norm one. Note that we can also write $\tilde{T}$ in the form

$$\tilde{T} = \tilde{\phi}_1 \otimes (1,0) \,+\, d\tilde{\phi}_2 \otimes (0,1),$$

where $\tilde{\phi}_i$ are the norm-1 coordinate functionals on V defined by

$$\tilde{\phi}_i(a_1, a_2) = a_i, \ i = 1, 2.$$

Next let ϕ_i be a Hahn-Banach extension to X of the functional $\tilde{\phi}_i$ on V, $i = 1, 2$. Then the extension operator $T : X \to V$ has the form

$$T = \phi_1 \otimes (1,0) \,+\, d\phi_2 \otimes (0,1).$$

Finally, since $T(S(X)) \subset \tilde{T}(\overline{co}C) = \overline{co}P \subset \overline{co}S(V) = B(V)$, we see that $\|T\| \leq 1$ and hence, since $\|T\| \geq \|\tilde{T}\| = 1$, it follows that $\|T\| = \|\tilde{T}\|$.

Furthermore, suppose that V is strictly convex but not smooth. Then, as is well known, V^* is smooth but not strictly convex, and the above argument shows that V^* possesses a rank-two Hahn-Banach operator. But then, by Corollary 1.1

($1 = \|T^*\| = \|\tilde{T}^*\| = \|\tilde{T}\| = \|T\|$ and $e(\tilde{T}) = e(\tilde{T}; L_\infty)$), V also possesses a rank-two Hahn-Banach operator. □

Corollary 2.2. *If the unit ball of the 2-dimensional real space V is either not smooth or not strictly convex, then V possesses a basis v_1, v_2 and a rank-2 Hahn-Banach operator $\tilde{T} = \sum_{i=1}^2 \tilde{u}_i \otimes v_i$ such that, for the extension $T = \sum_{i=1}^2 u_i \otimes v_i$, each $\|u_i\| = \|\tilde{u}_i\|$, $i = 1, 2$, i.e., each u_i is a Hahn-Banach extension of $\tilde{u}_i$. A reasonable conjecture is that this phenomenon holds for every rank-2 Hahn-Banach operator on V.*

Note 2.1. In [3] it is shown in particular that, for each real non-singular 2×2 real matrix A not equal to a scalar multiple of the identity I, there exists a real space $V = [\vec{v}] \ncong \ell_\infty^{(2)}$ which possesses a Hahn-Banach operator $\tilde{T}$ such that $\tilde{T}\vec{v} = A\vec{v}$. Theorem 2.2 explains why any such space must be either non-smooth or non-strictly convex. Indeed all the spaces provided in [3] are hexagonal spaces.

Note, furthermore, that the rectangle P in the proof of Theorem 2.2 can be replaced by any parallelogram (symmetric with respect to the origin) contained inside $S(V)$ and intersecting the flat sides of $S(V)$. In this way we see that V possesses an entire family of two-dimensional Hahn-Banach operators of norm-1 ($\tilde{T}p = p$ for some $p \in S(V) \cap \tilde{T}(S(V))$ of the form

$$\tilde{T} = (\alpha\tilde{\phi}_1 + \beta\tilde{\phi}_2) \otimes (1, 0) \,+\, (\gamma\tilde{\phi}_1 + \delta\tilde{\phi}_2) \otimes (0, 1),$$

all of which have two real eigenvalues (1 and e, where $|e| \leq 1$). In [4] we show that, if the spectrum of $\tilde{T} \neq I$ lies on the unit circle, then the Hahn-Banach operator $\tilde{T}$ has norm > 1.

Corollary 2.3. *If the two-dimensional real Banach space V possesses a rank-two Hahn-Banach operator $\tilde{T} \neq I$ whose spectrum lies on the unit circle, then V also possesses a rank-two Hahn-Banach operator with eigenvalues 1, d, $|d| < 1$.*

Corollary 2.4. *The two-dimensional real Banach space V possesses a rank-two Hahn-Banach operator if and only if its dual space V^* possesses a rank-two Hahn-Banach operator.*

Corollary 2.5. *For every two-dimensional subspace V of ℓ_1 or ℓ_∞ there exists a rank-two operator $\tilde{T} : V \to V$ such that $e(\tilde{T}) = \|\tilde{T}\|$.*

For every two-dimensional subspace V of ℓ_p ($p \in (1, \infty)$) and for every rank-two operator $\tilde{T} : V \to V$ we have $e(\tilde{T}) > \|\tilde{T}\|$.

Example 2.1. Let $V = [\vec{v}] \subset L^1[-1, 1]$, where $(v_1, v_2) = (\frac{1}{2}, t)$ and let $\tilde{T}v_1 = v_1$ and $\tilde{T}v_2 = av_2$, $a \geq \frac{1}{2}$. Then Theorem 1.2 yields that the minimal extension operator $T = \sum_{i=1}^2 u_i \otimes v_i : L^1[-1, 1] \to V$ is given by

$$\vec{u} = \kappa\Big[\frac{(1, 2mt)}{\sqrt{1 + 4m^2t^2}} + (0, \frac{\operatorname{sgn} t}{2})\Big],$$

where $\kappa = \|T\|$, m is a non-negative constant, and $M = (\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix})$. (In fact $a = (\frac{1}{2} + \int_0^1 \frac{4mt^2\, dt}{\sqrt{1+4m^2t^2}}) / \int_0^1 \frac{dt}{\sqrt{1+4m^2t^2}}$.) Since V is smooth, Theorem 2.2 provides that $\|T\| = \|\tilde{T}\| = 1$ precisely when $m = 0$. Note that then $u_1 = 1$ and $u_2 = \frac{1}{2}\operatorname{sgn} t$ shows that $u_i \in L_\infty[-1, 1]$ is the Hahn-Banach extension of $\tilde{u}_i$, $i = 1, 2$. (See Figure 2.)

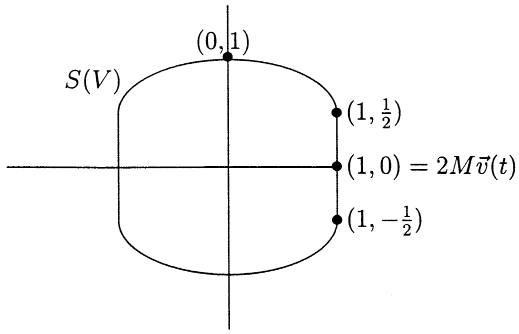

FIGURE 2.

As a further example, following Corollary 1.1, note that the space V is also realized as $V=[\frac{1}{2},t]\subset L^1[\overline{\mathbf{R}},\,\nu]$, where ν is Lebesgue measure on $[-1,1]$ and zero off $[-1,1]$. Thus replacing $[-1,1]$ by $\overline{\mathbf{R}}$ in the above, we see that, for $m>0$, $\vec{u}/\|T\|$ now covers all of $S(V)$ and, for $m=0$, let $\vec{u}(t)/\|T\|=\vec{u}(t)$, $|t|>1$, be any convenient parameterization of the curved portion of $S(V)$. Thus for $m=0$ the adjoint operator $\tilde{T}^*$ (with same-norm extension T^*) provides a Hahn-Banach operator for the non-smooth, strictly convex space $V^*=[t,1-t^2]\subset L_\infty[-1,1]$ with a "corner" at $(1,0)$. ($V^*\cong[\vec{u}/\|T\|]\subset L_\infty[\overline{\mathbf{R}}]$.)

Acknowledgment

The authors would like to thank Professor Mikhail Ostrovskii for his very valuable help with respect to this paper.

References

[1] B. L. Chalmers and F. T. Metcalf, The determination of minimal projections and extensions in L^1, Trans. Amer. Math. Soc., **329**(1992), 289-305. MR **92e:**41017

[2] B. L. Chalmers and F. T. Metcalf, A characterization and equations for minimal projections and extensions, J. Operator Theory, **32**(1994), 31-46. MR **96c:**46014

[3] B. L. Chalmers and B. Shekhtman, Actions that characterize $\ell_\infty^{(n)}$, Lin. Alg. and Appl., **270**(1998), 155-169. MR **98h:**46008

[4] B. L. Chalmers and B. Shekhtman, Spectral properties of operators that characterize $\ell_\infty^{(n)}$, Abst. Appl. Anal. **3** (1998), 237–246. CMP 2000:11

[5] L. E. Dor, Potentials and isometric embeddings in L_1, Israel J. Math. **24**(1976), 260-268. MR **54:**5806

[6] D. Yost, L_1 contains every two-dimensional normed space, Ann. Polonici Math. **49**(1988), 17-19. MR **90b:**46047

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CALIFORNIA 92507
E-mail address: `blc@math.ucr.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33620
E-mail address: `boris@math.usf.edu`