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CONSTRUCTIVE APPROXIMATION

On the Divergence of Polynomial Interpolation in the Complex Plane

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Abstract. We extend the results in [1] and [2] from the divergence of Hermite–Fejér interpolation in the complex plane to the divergence of arbitrary polynomial interpolation in the complex plane. In particular, we prove the following theorem: Let $\Delta_n = -1 \le t_1^{(n)} < \cdots < t_n^{(n)} < 1$. Let $\varphi_k^{(n)}$ be polynomials of arbitrary degree such that $\varphi_k^{(n)}(t_j^{(n)}) = \delta_{kj}$. Then the Lebesgue function $\Lambda_n(x) = \sum_{j=1}^n |\varphi_j^{(n)}(x)|$ tends to infinity at every *complex* neighborhood of some point in $[-1, 1]$.

1. Introduction

Let $\Delta_n = -1 \le t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)} \le 1$ and let $\varphi_k^{(n)}$ be polynomials of degree $n-1$ such that $\varphi_k^{(n)}(t_j^{(n)}) = \delta_{jk}$. Define a linear interpolation operator $L(\Delta_n) = L(\Delta_n, \varphi)$ by

$$L(\Delta_n)f = \sum f(t_j^{(n)})\varphi_j^{(n)} : C([-1,1]) \to C([-1,1]),$$

and the corresponding Lebesgue function

$$\Lambda_n(x) = \sum_{j=1}^{n} |\varphi_j^{(n)}(x)|.$$

Then it is well known (Faber theorem) that

$$\|L(\Delta_n)\| = \sup\{\Lambda_n(x) : x \in [-1,1]\} \to \infty \qquad \text{as} \quad n \to \infty. \tag{1.1}$$

The following strengthening of this theorem is due to P. Erdős and P. Vértesi (see [7]).

Theorem 1.1. *Let* $x \in [-1, 1]$ *and* U *be a neighborhood of* x *in* $\mathbf{R}$. *Then*

$$\sup\{\Lambda_n(x) : x \in U\} \to \infty \qquad \textit{as} \quad n \to \infty. \tag{1.2}$$

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The Faber theorem is no longer valid if we do not restrict the degree of polynomials $\varphi_j^{(n)}$. In fact, the clasical Hermite–Fejér interpolation (see [7]) has the property

$$\|f - L(\Delta_n)f\| \to 0 \qquad \text{for all} \quad f \in C([-1,1]).$$

It was recently observed by L. Brutman and I. Gopengauz [2] (see also [1]) that (1.2) still holds for the Hermite–Fejér interpolants if U is a *complex* neighborhood of x.

Theorem 1.2. *Let $\varphi_j^{(n)}$ be polynomials of degree at most $2n-1$ such that $\varphi_j^{(n)}(t_k) = \delta_{jk}$ and $(\varphi_j^{(n)})'(t_k) = 0$ for all j and $k = 1, \dots, n$. Then for every $x \in [-1,1]$ and for every complex neighborhood*

$$U = \{z \in \mathbf{C} : |x - z| < r\}$$

we have

$$\sup \left\{ \sum_{k=1}^{n} |\varphi_k^{(n)}(z)| : z \in U \right\} \to \infty. \tag{1.3}$$

Moreover, for every nonconstant polynomial p:

$$\sup \left\{ \left| \sum_{j=1}^{n} p(t_j^{(n)}) \varphi_j^{(n)}(z) \right| : z \in U \right\} \to \infty.$$

The main purpose of this paper is to show that (1.2) is valid for any choice of polynomials $\varphi_j^{(n)}$ of arbitrary degree satisfying the biorthogonality condition

$$\varphi_j^{(n)}(t_k^{(n)}) = \delta_{jk}.$$

This result is proved in the next section (Theorem 2.3).

Thus, we proved a (surprising to us) "complex extension" of the Erdős–Vértesi theorem for interpolation by polynomials of arbitrary degree. The result has a flavor of Somorjai's theorem [6] (see also [5]). The method of proof is reminiscent of the ones in [3]. Observe that the quantity

$$\sup \left\{ \sum_{k=1}^{n} |\varphi_k^{(n)}(z)| : z \in U \right\}$$

is the norm of the interpolation operator

$$L(\Delta_n) f = \sum_{j=1}^{n} f(t_j^{(n)}) \varphi_j^{(n)}$$

as an operator from $C([-1,1])$ into $C(\overline{U})$.

In Section 3 we show that the "moreover" part of Theorem 1.2 is not valid for the arbitrary choice $\varphi_j^{(n)}$. That is, we show that there exist $\varphi_j^{(n)}$ such that

$$p(z) - \sum p(t_j^{(n)}) \varphi_n^{(n)}(z) \to 0$$

uniformly in $\mathbf{C}$ for every polynomial $p(z)$. We also investigate the behavior of $\Lambda_n(z)$ in the arbitrary neighborhoods of $\mathbf{C}$.

2. The Main Theorems

Most of the proofs in this paper will rely on the following simple perturbation argument.

Lemma 2.1. *Let X be a Banach space and let U, V be subspaces of X. Let P be a projection from X onto U and let J be an isomorphism from V onto U such that*

$$\|v - Jv\| < \varepsilon\|v\| \qquad \textit{for all} \quad v \in V. \tag{2.1}$$

Then:

(1) *if $\varepsilon\|P\| < 1$, then there is a projection Q from X onto V such that $\|Q\| \leq \|P\|/(1 - \varepsilon\|P\|)$;*
(2) *if U and V are n-dimensional and*

$$Px = \sum_{j=1}^{n} f_j(x)u_j \qquad \textit{for some} \quad f_j \in X^*, \quad \|f_j\| = 1,$$

then the projection Q can be chosen as

$$Qx = \sum_{j=1}^{n} f_j(x)v_j.$$

Proof. Let $R = P|V$ be a restriction of the operator P onto V. We claim that R is an isomorphism from V onto U. Indeed, for every $v \in V$ we have

$$Rv = Pv = P(v - Jv) + Jv. \tag{2.2}$$

From this we obtain

$$\|Rv\| \geq \|V\| - \varepsilon\|P\|\,\|V\| \geq (1 - \varepsilon\|P\|)\|V\| \tag{2.3}$$

and

$$\|R - J\| \leq \varepsilon\|P\| < 1. \tag{2.4}$$

Now (2.4) implies that R is an isomorphism and (2.3) gives

$$\left\|R^{-1}\right\| \leq 1/(1 - \varepsilon\|P\|).$$

Hence the operator

$$Q = R^{-1}P : X \to V$$

defines a projection that satisfies the conclusion (1) of the lemma.

To prove the second part of the lemma it is enough to prove that

$$f_j(x) = 0 \qquad \text{for all} \quad j = 1, \dots, n$$

implies

$$Qx = 0.$$

Indeed

$$Qx = R^{-1}\left(\sum_{j=1}^{n} f_j(x)v_j\right) = 0$$

and we are done. ■

Let Δ_n be finite subsets of $\mathbf{C}$.

Definition 2.1. *A point $z_0 \in \mathbf{C}$ is called an accumulation point of $\{\Delta_n\}_{n=1}^{\infty}$ if for every $\varepsilon > 0$ and every integer $k \in \mathbf{N}$ there exists an integer N such that if $n \geq N$ then*

$$|\{z \in \mathbf{C} : |z - z_0| < \varepsilon\} \cap \Delta_n| \geq k. \tag{2.5}$$

Theorem 2.1. *Let $\Delta_n \subset \mathbf{C}$; $\Delta_n = \{z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)}\}$. Let $\varphi_1^{(n)}, \ldots, \varphi_n^{(n)}$ be polynomials (of arbitrary degree) such that $\varphi_j^{(n)}(z_k^{(n)}) = \delta_{jk}$. Let $z_0 \in \mathbf{C}$ be an accumulation point of $\{\Delta_n\}$. Then for any neighborhood $U \ni z_0$; $U \subset \mathbf{C}$:*

$$\sup_{z \in U} \Lambda_n(z) \to \infty \qquad \text{as} \quad n \to \infty. \tag{2.6}$$

Here $\Lambda_n(z) = \sum_{k=1}^{n} |\varphi_k^{(n)}(z)|$.

Proof. Without loss of generality we may assume that zero is an accumulation point of $\{\Delta_n\}$ and that the neighborhood of zero is the unit disk D. For a finite set of points $\Delta \subset \mathbf{C}$, let b_Δ denote the Blaschke product with zeros at Δ.

We proceed by contradiction. Let $M > 0$ be such that

$$\sup_{z \in D} \Lambda_n(z) \leq M \qquad \text{for all } n. \tag{2.7}$$

Choose $\eta > 0$ so that

$$\eta \leq \frac{1}{2M}. \tag{2.8}$$

Next, for an arbitrary integer n we choose $\varepsilon > 0$ so that if

$$|\Delta| = k \qquad \text{and} \qquad \Delta \subset \{z \in \mathbf{C} : |z| < \varepsilon\} \tag{2.9}$$

then

$$\|z^k - b_\Delta(z)\| < \eta. \tag{2.10}$$

Choose $N > 0$ so that if $n \geq N$, then there exists

$$\Delta = \Delta_n' \subset \Delta_n$$

satisfying (2.9).

Consider the projection $P_n : A(D) \to A(D)$ defined by

$$P_n f = \sum_{t_j^{(n)} \in D_n'} f(t_j^{(n)})\varphi_j^{(n)}.$$

where $A(D)$ stands for the usual disk algebra. In view of (2.7), we have

$$\|P_n\| \le M \qquad \text{for all } n. \tag{2.11}$$

Now observe that the projection $P_n' = I - P_n : A(D) \to A(D)$ has the ideal $b_\Delta \cdot A(D)$ as its range. We now intend to prove that there exists a projection

$$Q_n : A(D) \to A(D), \qquad \text{Range } Q_n = z^k \cdot A(D),$$

and

$$\|Q_n\| \le \frac{\|P_n\|}{1 - \eta\|P_n\|}. \tag{2.12}$$

Consider the mapping $J_n : z^k \cdot A(D) \to b_\Delta(z) \cdot A(D)$ defined by

$$J_n(z^k \cdot f) = b_\Delta(z) \cdot f.$$

Then, by the maximum modulus principle, J_n is an isomorphism from $V = z^k \cdot A(D)$ onto $U = b_\Delta \cdot A(D)$, and by Lemma 2.1 we obtain the projection Q satisfying (2.12).

For every $s \in \mathbf{T}$, define $T_s : A(D) \to A(D)$ by

$$(T_s f)(z) = f(sz).$$

Let

$$F_n f = \int T_{\bar{s}} Q_n T_s f \, d\mu(x), \tag{2.13}$$

where μ is a normalized Haar measure on $\mathbf{T}$.

It is easy to check that F_n is a projection from $A(D)$ onto $z^k A(D)$. Moreover,

$$F_n(z^m) = \begin{cases} z^m & \text{if} \quad m \ge k, \\ 0 & \text{if} \quad m < k. \end{cases}$$

Hence, F_n maps a function into the remainder of its Taylor series. Thus, there exists a universal constant $c > 0$ such that

$$\|F_n\| \ge c \log k. \tag{2.14}$$

On the other hand, it follows from (2.13) that

$$c \log k \le \|F_n\| \le \|Q_n\|. \tag{2.15}$$

Combining the last inequality with (2.12) and (2.8), we get

$$\|P_n\| \ge \frac{c}{2} \log k$$

which tends to infinity with k and hence contradicts (2.7). ■

As an immediate corollary, we obtain the following analog of Faber's theorem for interpolation by polynomials of arbitrary degree.

Theorem 2.2. *Let* $\Delta_n \subset [-1, 1]$ *and let* $L(\Delta_n)$ *be an arbitrary interpolation projection at the nodes* Δ_n *with polynomial range. Let* $\mathcal{E}$ *be an open set in* $\mathbf{C}$ *that contains the interval* $[-1, 1]$. *Then*

$$\lim \|L(\Delta_n) : C([-1, 1]) \to C(\overline{\mathcal{E}})\| = \infty \qquad \textit{as} \quad n \to \infty.$$

The following generalization of Theorem 1.3 is also a direct consequence of Theorem 2.2.

Theorem 2.3. *Let* $\Delta_n \subset [-1, 1]$ *and let* $\varphi_j^{(n)}$ *be polynomials such that* $\varphi_j^{(n)}(t_k^{(n)}) = \delta_{j,k}$ *and*

$$L(\Delta_n) f = \sum_{j=1}^{n} f(t_j^{(n)})\varphi_j^{(n)} \to f \qquad \textit{for every} \quad f \in C[-1, 1]. \tag{2.16}$$

Let $t \in [-1, 1]$ *and* U *be an arbitrary complex neighborhood of* t. *Then*

$$\sup\left\{\sum^{n} |\varphi_j^{(n)}(z)| : z \in U\right\} \to \infty \qquad \textit{as} \quad n \to \infty.$$

Proof. Observe that (2.16) implies that every point of the interval $[-1, 1]$ is an accumulation point of $\{\Delta_n\}_{n=1}^{\infty}$. ■

3. Some Relevant Examples

Our first example shows that the"moreover" part of Theorem 1.2 is not valid for arbitrary polynomial interpolation.

Theorem 3.1. *There exist* $\Delta_n \subset [-1, 1]$ *and polynomials* $\varphi_k^{(n)}$ *such that* $\varphi_k^{(n)}(t_j^{(n)}) = \delta_{jk}$ *and*:

(a) $\sum_{j=1}^{n} f(t_j^{(n)})\varphi_j^{(n)} \to f$ *uniformly for every* $f \in C([-1, 1])$;
(b) *for every polynomial* p *there exists* $N \in \mathbf{N}$ *such that for all* $n \geq N$:

$$\sum_{j=1}^{n} p(t_j^{(n)})\varphi_j^{(n)}(z) = p(z) \qquad \textit{for all} \quad z \in \mathbf{C}.$$

Proof. Let H_m stand for the space of polynomials of degree at most m. It follows from the general result in Banach space theory (see [4] since $C([-1, 1])$ is a $\mathcal{L}_\infty$-space) that for every $\varepsilon > 0$ there exist

$$\Delta_n = \{-1 = t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)}\}$$

and an n-dimensional space $U_n \supset H_m$ ($n = n(m)$) such that there exists an isomorphism J_n' from U_n onto the space $U(\Delta_n)$ of piecewise linear function on Δ_n with the property that

$$\|J_n' u - u\| < \varepsilon\|u\| \qquad \text{for all} \quad u \in U_n. \tag{3.1}$$

It is also well known and easy to see that there exists a projection $P_n : C([-1,1])$ onto $U(\Delta_n)$:

$$P_n f = \sum_{j=1}^{n} f(t_j^{(n)}) u_j(\Delta_n) \tag{3.2}$$

with

$$\|P_n\| \le 1. \tag{3.3}$$

Let $1, x, \dots, x^m, w_1, \dots, w_{n-m}$ be a basis in U_n. Let $\psi_j^{(N)}$, $j = 1, \dots, n-m$, be polynomials such that

$$\psi_j^{(N)} \to w_j.$$

Then for sufficiently large N, the mapping $J_n^{(N)}$ defined by

$$\begin{aligned} J_n^{(N)} x^k &= x^k, \qquad & k \le m, \\ J_n^{(N)} w_j &= \psi_j^{(N)}, \qquad & j \le n-m, \end{aligned}$$

defines an isomorphism from the space U_n onto the space

$$V_n := \operatorname{span}\{1, x, \dots, x^m, \psi_1^{(N)}, \dots, \psi_{n-m}^{(N)}\}$$

such that

$$\|J_n^{(N)} u - u\| \le \frac{\varepsilon}{2} \|u\|. \tag{3.4}$$

Combining (3.1) and (3.4) we obtain an isomorphism J_n:

$$J_n : V_n \to U(\Delta_n t)$$

such that

$$\|J_n v - v\| < \varepsilon \|v\|. \tag{3.5}$$

Combining (3.5) with Lemma 3.2 (part (2)), we obtain a projection Q_n from $C([-1,1])$ onto V_n:

$$Q_n f = \sum f(t_j^{(n)}) \psi_j^{(n)}, \qquad \varphi_j^{(n)} \in V_n,$$

such that

$$\|Q_n\| \le (1+\varepsilon).$$

The last inequality, together with $H_m \subset V_n$, immediately imply all the conclusions of the theorem. ■

The next example implies that the conclusion of Theorem 2.4 does not hold in the neighborhoods outside the interval $[-1,1]$.

Theorem 3.2. *Let* $z \in \mathbf{C}$; $z \notin [-1, 1]$. *Then there exists a neighborhood* $U \ni z$ *and* $\Delta_n \subset [-1, 1]$ *and polynomials* $\varphi_j^{(n)}$ *such that*:

(a) $\varphi_j^{(n)}(t_k^{(n)}) = \delta_{jk}$;
(b) $\sum f(t_k^{(n)})\varphi_k^{(n)}(x) \to f(x)$ *uniformly for every* $f \in C([-1, 1])$; *and*
(c) $\sup\{\sum_{k=1}^{n} |\varphi_k^{(n)}(z)| : z \in U\} \leq 1$.

Proof. Let Δ_n be Chebyshev points on $[-1, 1]$ and let $\psi_j^{(n)}$ be the fundamental polynomials for the Hermite–Fejér interpolation. Let $M_n = \sup\{\sum |\psi_k^{(n)}(z)| : z \in \overline{U}\}$ where U is chosen such that $\overline{U} \cap [-1, 1] = \emptyset$. Let $\varepsilon_n \to 0$ be such that

$$\varepsilon_n M_n < \tfrac{1}{2}.$$

The Mergelian theorem furnishes the polynomials $p_n(z)$ such that

$$|p_n(x) - 1| < \varepsilon_n \qquad \text{for all} \quad x \in [-1, 1], \tag{3.6}$$

$$|p_n(x)| < \varepsilon_n \qquad \text{for all} \quad x \in \overline{U}. \tag{3.7}$$

Let $U_n = \operatorname{span}\{\psi_j^{(n)}\}$ and $V_n = p_n \cdot U_n$ be subspaces of $C([-1, 1])$. Then $J_n : U_n \to V_n$ defined by

$$J_n u = p_n \cdot u$$

satisfies

$$\| J_n u - u \| < \varepsilon_n \|u\|.$$

Hence, by Lemma 2.1, there exist polynomials $\varphi_j^{(n)} \in V_n$ that, together with the Hermite–Fejér interpolation, verify (a) and (b). Finally, for $z \in U$:

$$\sum_{j=1}^{n} |\varphi_j^{(n)}(z)| \leq (1 + \varepsilon_n)|p_n(z)| \sum_{j=1}^{n} |\psi_j^{(n)}(z)| < (1 + \varepsilon_n)\varepsilon_n M < 1. \qquad \blacksquare$$

Let us observe the following "local version" of the Somorjai theorem (see [6]), that claims that even in the arbitrary neighborhood U something must go wrong.

Theorem 3.3. *Let* U *and* $\varphi_j^{(n)}$ *satisfy the conclusion of Theorem* 3.2. *Then there exists a polynomial* $p(z)$ *such that the interpolants*

$$\sum p_n(t_j^{(n)})\varphi_j^{(n)}$$

do not converge to $p(z)$.

Proof. Using the Mergelian theorem once again, let $p(z)$ be a polynomial such that

$$|p(x)| \leq 1 \qquad \text{for} \quad x \in [-1, 1]$$

and

$$|p(z)| \geq 10 \qquad \text{for} \quad z \in U.$$

Then

$$\sup\left\{\left|\sum^{n} p(t_j^{(n)})\varphi_j^{(n)}(z)\right| : z \in U\right\} \leq 1.$$

and hence

$$\sup\left\{\left|p(z) - \sum_{k=1}^{n} p(t_j^{(n)})\varphi_j^{(n)}(z)\right| : x \in U\right\} \geq 9. \qquad \blacksquare$$

Corollary 3.1. *Let $\varphi_j^{(n)}$ be polynomials constructed in Theorem* 3.1. *Then, for any open set U in the complex plain*

$$\sup\left\{\sum_{j=1}^{n} |\varphi_j^{(n)}(z)| : z \in U\right\} \to \infty.$$

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