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Minimal Boolean Sum and Blending-Type Projections and Extensions

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Abstract—In this paper, useful characterization theorems are presented for minimal Boolean sum and blending-type projections and extensions.

Keywords—Minimal, Projection, Boolean sum, Blending-type.

In the following, V and W are finite-dimensional subspaces of a Banach space X; let

$$\begin{aligned} A &: V \to [v_1, \dots, v_n] = V, \\ B &: W \to [w_1, \dots, w_m] = W, \end{aligned}$$

be fixed operators on V and W, respectively, and let

$$P : X \to V, \qquad Q : X \to W$$

denote two bounded extension operators of A and B, respectively, i.e., $P_{|V} = A$ and $Q_{|W} = B$. In the cases $A = I$ and $B = I$, P and Q are of course projections.

1. BOOLEAN SUM OF OPERATORS

Consider the Boolean sum of P and Q

$$P \oplus Q = P + Q - PQ : X \to V + W = Z.$$

NOTE. If P and Q are projections, then $P \oplus Q$ is a projection $\Leftrightarrow PQP = QP \Leftarrow PQ = QP$.

Suppose now that $Q = Q_0$ is fixed. Let $\mathcal{R} = \{P \oplus Q_0\}$. We wish to characterize the solution to the following extremal problem:

$$\min_{R \in \mathcal{R}} \|R\| = \min_{P} \|P \oplus Q_0\|_{X \to Z}.$$

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NOTE. If P and Q are projections, then the operators $R = P \oplus Q$ are not necessarily projections in Theorems 1 and 2 but, if P and Q are projections, then R is a projection in Theorem 3 and throughout Section 2 (blending-type projections and extensions).

Consider the set of "extremal pairs" of R

$$\mathcal{E}(R) = \{(x,y) \in S(X^{**}) \times S(X^*) : \langle R^{**}x, y\rangle = \|R\|\}.$$

In the following, let $K = B(X^{**}) \times B(X^*)$ and note that K is compact if we take the weak*-topologies on $B(X^{**})$ and $B(X^*)$. Furthermore, each $R \in \mathcal{R}$ can be identified with a continuous (bilinear) function $\hat{R}$ on K in the obvious way.

THEOREM 1. CHARACTERIZATION. *$R = P \oplus Q_0$ has minimal norm in $\mathcal{R}$ $\Leftrightarrow$ the closed convex hull of $\{y \otimes (I - Q_0^{**})x\}_{(x,y)\in\mathcal{E}(R)}$ contains an operator for which V is an invariant subspace.*

PROOF. Following the method of proof of Theorem 1 in [1], best approximate $R_0 = P_0(I - Q_0) + Q_0 \in \mathcal{R} \subset \mathcal{B}(X, Z)$ from

$$\mathcal{D} = \left\{\Delta(I - Q_0) : \Delta \in \operatorname{sp}\left\{\delta \otimes v : \delta \in V^{\perp},\ v \in V\right\}\right\}.$$

Equivalently, perturb $\widehat{R_0} \in C(K)$ by functions $\hat{D}$ in the subspace $\hat{\mathcal{D}}$. $\hat{R} = \widehat{R_0} - \hat{D}_0$, where $D_0 = \Delta_0(I - Q_0)$, is of minimal norm $\Leftrightarrow$ $\exists$ a (total mass one) measure $\mu \geq 0$ supported in $\mathcal{E}(R)$ (μ may be taken positive since the functions $\hat{R}$ are homogeneous) such that $\mu \perp \hat{\mathcal{D}}$, i.e.,

$$\begin{aligned}
0 &= \int_{\mathcal{E}(R)} \hat{D}\, d\mu = \int_{\mathcal{E}(R)} \langle \Delta\,(I - Q_0^{**})\,x, y\rangle\; d\mu(x,y) \\
&= \int_{\mathcal{E}(R)} \langle (\delta \otimes v)\,(I - Q_0^{**})\,x, y\rangle\; d\mu(x,y) = \int_{\mathcal{E}(R)} \langle (I - Q_0^{**})\,x, \delta\rangle\,\langle v, y\rangle\, d\mu(x,y) \qquad (1)\\
&= \left\langle \int_{\mathcal{E}(R)} \langle v, y\rangle\,(I - Q_0^{**})\,x\, d\mu(x,y), \delta \right\rangle, \qquad \forall \Delta = \delta \otimes v \Leftrightarrow \\
E_R &= \int_{\mathcal{E}(R)} y \otimes (I - Q_0^{**})\,x\, d\mu(x,y) : V \to V.
\end{aligned}$$

∎

In a similar fashion, we obtain the following theorem.

THEOREM 2. *$R = P_0 \oplus Q$ has minimal norm $\Leftrightarrow$ there exists a (total mass one) measure μ such that the operator*

$$E_R = \int_{\mathcal{E}(R)} (y \otimes x)(I - P_0)\, d\mu(x,y) : W \to W.$$

PROOF. Again following the method of proof of Theorem 1 in [1], best approximate $R_0 = P_0 + (I - P_0)Q_0 \in \mathcal{R} \subset \mathcal{B}(X, Z)$ from

$$\mathcal{D} = \left\{(I - P_0)\Delta : \Delta \in \operatorname{sp}\left\{\epsilon \otimes w : \epsilon \in W^{\perp},\ w \in W\right\}\right\}.$$

Equivalently, perturb $\widehat{R_0} \in C(K)$ by functions $\hat{D}$ in the subspace $\hat{\mathcal{D}}$. $\hat{R} = \widehat{R_0} - \hat{D}_0$, where $D_0 = (I - P_0)\Delta_0$ is of min norm $\Leftrightarrow$ $\exists$ a (total mass one) measure $\mu \geq 0$ supported in $\mathcal{E}(R)$ (μ may be taken positive since the functions in $\hat{R}$ are homogeneous) such that $\mu \perp \hat{\mathcal{D}}$, i.e.,

$$\begin{aligned}
0 &= \int_{\mathcal{E}(R)} \hat{D}\, d\mu = \int_{\mathcal{E}(R)} \langle (I - P_0)\Delta x, y\rangle\, d\mu(x,y) \\
&= \int_{\mathcal{E}(R)} \langle (I - P_0)(\epsilon \otimes w)(x,y)\rangle\, d\mu(x,y) = \int_{\mathcal{E}(R)} \langle x, \epsilon\rangle \langle (I - P_0)w, y\rangle\, d\mu(x,y) \qquad (2)\\
&= \left\langle \int_{\mathcal{E}(R)} \langle (I - P_0)w, y\rangle\, x\, d\mu, \epsilon \right\rangle, \qquad \forall \Delta = \epsilon \otimes w \Leftrightarrow \\
E_R &= \int_{\mathcal{E}(R)} (y \otimes x)(I - P_0)\, d\mu(x,y) : W \to W.
\end{aligned}$$

∎

Note that (1) translates to ($\vec{v} = v_1, \dots, v_n$; N some $n \times n$ matrix)

$$\int_{\mathcal{E}(R)} \langle \vec{v}, y \rangle (I - Q_0)^{**} x \, d\mu = N\vec{v}. \tag{3}$$

Note that (2) translates to ($\vec{w} = w_1, \dots, w_m$; M some $m \times m$ matrix)

$$\int_{\mathcal{E}(R)} \langle (I - P_0)\vec{w}, y \rangle x \, d\mu = M\vec{w}. \tag{4}$$

NOTE 1. (See, e.g., [2] or [3] for definitions and notation.) Writing the operator of Theorem 1 as $E_R = (I - Q_0) \circ E_P$, where $E_P = \int_{\mathcal{E}(R)} (y \otimes x) \, d\mu(x, y)$, we see that E_P can be viewed as a norm-one integral operator in $(X^* \check{\otimes} X)^*$ separating R from $\mathcal{D} = \{\Delta(I - Q_0) : \Delta \in \mathcal{B}(X, V);\ \Delta = 0 \text{ on } V\}$. That is,

$$\begin{aligned} \langle R, E_P \rangle = \operatorname{trace}(E_P \circ (P \oplus Q_0)) &= \int_{\mathcal{E}(R)} \langle R^{**}x, y \rangle \, d\mu(x, y) \\ &= \|R\| \int_{\mathcal{E}(R)} d\mu(x, y) = \|R\| \nu(E_P) = \|R\|, \end{aligned}$$

where ν denotes the norm of E_P in the space of integral operators $I_1(X, X^{**})$ ($\nu(E_P) \leq \int_{\mathcal{E}(R)} \|y\| \, \|x\| \, d\mu(x, y) = 1$), and $\langle \mathcal{D}, E_R \rangle \equiv 0$ as in the proof of Theorem 1.

The operator E_R of Theorem 2 can be viewed analogously.

DEFINITION. *We say $W = [\vec{w}]$ is P-related to $V = [\vec{v}]$ if*

$$\vec{v} = (I - P)\vec{w},$$

for some bases $\vec{v}$ and $\vec{w}$. (Of course, then $m = n$ in this case.)

NOTE 2. If W is P-related to V, then $V \subset \ker P$.

COROLLARY 1. *Let $X = L^p(T)$, $1 \leq p < \infty$ or $X = C(T)$, $p = \infty$, and W be piecewise continuously differentiable and P_0-related to V in the setting of Theorem 2, where $P_0 = \vec{u}_0 \otimes \vec{v}$ and $\vec{u}_0$ is piecewise continuously differentiable. Then, if $Q = \sum_{i=1}^{n} r_i \otimes w_i$ provides a minimal R in $\mathcal{R}$, the following linear (first-order differential, if $p < \infty$) equation for $\vec{r} = (r_1, \dots, r_n)$ holds:*

$$\frac{1}{p} (\vec{r}\,' + \vec{u}_0') \cdot M\vec{w} = \frac{1}{q} (\vec{r} + \vec{u}_0) \cdot M\vec{w}\,', \qquad \text{on } T, \tag{5}$$

where M is the matrix in (4), "$'$" denotes differentiation along an arbitrary vector field in T, and $1/q + 1/p = 1$.

PROOF. From Theorem 2, R is minimal if and only if

$$\int_{\mathcal{E}(R)} \langle (I - P_0)\vec{w}, y \rangle x \, d\mu = M\vec{w},$$

as noted in (4). But now $(x, y) \in \mathcal{E}(R)$ implies that

$$\|R\| = \langle x, \vec{u}_0 \rangle \cdot \langle \vec{v}, y \rangle + \langle x, \vec{r} \rangle \cdot \langle (I - P_0)\vec{w}, y \rangle.$$

Let $\vec{d} = \langle (I - P_0)\vec{w}, y \rangle$. Then, since W is P_0-related to V, we have that

$$x(t) = \operatorname{ext}\left(\vec{d} \cdot \vec{\rho}(t)\right) =: f\left(\vec{d} \cdot \vec{\rho}(t)\right),$$

where $\vec{\rho}(t) = \vec{u}_0(t) + \vec{r}(t)$. Then (suppressing unnecessary notation), we have

$$G(\vec{\rho}(t)) := \int \vec{d} f\left(\vec{d}\cdot\vec{\rho}(t)\right) d\mu = M\vec{w}(t). \tag{6}$$

As in [4], by examining $\vec{\rho} = G^{-1}(M\vec{w})$, we can see that $\vec{\rho}$ is almost everywhere differentiable. Assuming f is differentiable and differentiating both sides of the above equation with respect to t, we have (by the chain rule)

$$\int \vec{d} f'\left(\vec{d}\cdot\vec{\rho}(t)\right)\vec{d}\cdot\vec{\rho}\,'(t)\, d\mu = M\vec{w}\,'(t).$$

Next, "dot" both sides of the above equation with $\vec{\rho}(t)$ to obtain

$$\int \vec{d}\cdot\vec{\rho}(t) f'\left(\vec{d}\cdot\vec{\rho}(t)\right)\vec{d}\cdot\vec{\rho}\,'(t)\, d\mu = \vec{\rho}(t)\cdot M\vec{w}\,'(t).$$

Next, "factor out" $\vec{\rho}\,'(t)$ from the left-hand side of the above (and shift left the associated $\vec{d}$ in the integrand)

$$\vec{\rho}\,'(t)\int \vec{d}\vec{d}\cdot\vec{\rho}(t) f'\left(\vec{d}\cdot\vec{\rho}(t)\right) d\mu = \vec{\rho}(t)\cdot M\vec{w}\,'(t). \tag{7}$$

But now, let $X = L^p$, for $1 < p < \infty$; then

$$f(z) = (\operatorname{sgn} z)|z|^{q/p} \qquad \text{and} \qquad zf'(z) = \frac{q}{p} z|z|^{q/p-1} = \frac{q}{p} f(z).$$

Thus, we have

$$\vec{\rho}\,'\cdot\frac{q}{p}\int \vec{d} f\left(\vec{d}\cdot\vec{\rho}\right) d\mu = \vec{\rho}\cdot M\vec{w}\,',$$

and finally, we conclude by use of (6) that

$$\frac{1}{p}\vec{\rho}\,'\cdot M\vec{w} = \frac{1}{q}\vec{\rho}\cdot M\vec{w}\,', \qquad \text{on } T.$$

We obtain the result for $p = 1, \infty$ either by a limiting process or by referring to [5,6]. ∎

NOTE 3. An equation similar to (5) cannot be derived from (1) of Theorem 1 since the extremals x do not appear exposed.

If P and Q commute, however, then the equation (5) also holds where Q is replaced by P.

COROLLARY 2. *Let $\tilde{\mathcal{R}} = \{P \oplus Q = P + Q - PQ\}$. $P \oplus Q$ is minimal $\Rightarrow$ there exist (total mass one) measures μ_1 and μ_2 such that*

$$\begin{aligned} E^{(1)}_{P\oplus Q} &= \int_{\mathcal{E}(P\oplus Q)} y \otimes (I - Q^{**})x \, d\mu_1(x,y) : V \to V, \qquad \text{and} \\ E^{(2)}_{P\oplus Q} &= \int_{\mathcal{E}(P\oplus Q)} (y \otimes x)(I - P)\, d\mu_2(x,y) : W \to W. \end{aligned} \tag{8}$$

PROOF. $\tilde{\mathcal{R}}$ is not the translate of a subspace, but apply Theorems 1 and 2 to $\mathcal{R}_1 = \{R \in \tilde{\mathcal{R}} : Q$ is fixed$\}$ and to $\mathcal{R}_2 = \{R \in \tilde{\mathcal{R}} : P$ is fixed$\}$. ∎

Condition (8) in Corollary 2 is probably not sufficient to provide a converse. As mentioned above, if P and Q are projections, then $P \oplus Q$ may not be a projection (cf. example below).

EXAMPLE. In [7], a sequence of pairs of n-dimensional subspaces V_n and W_n in X were constructed such that there exist projections $P_n : X \to V_n$; $Q_n : X \to W_n$ with $\|P_n\| = 1$; $\|Q_n\| \le 2$ and such that for every projection $R_n : X_n \to V_n + W_n$ we have $\|R_n\| \to \infty$. This shows that the minimal-norm Boolean sum may not be a projection. Indeed, if it were, then its norm would be $\leq 1 + 2 + 1 \cdot 2 = 5$.

It is tempting, however, to make the following conjecture.

CONJECTURE 1. *If P_0 and Q_0 are projections and $P_0 \oplus Q_0$ satisfies (8) and $P_0 \oplus Q_0$ is a projection, then the converse of Corollary 2 is true, i.e., $P_0 \oplus Q_0$ is minimal in $\tilde{\mathcal{R}}$.*

THEOREM 3. *Let $V \cap W = \{0\}$, $V \subset \ker Q_0$ and consider the set of operators into $V + W$ given by $\tilde{\mathcal{R}} = \{P \oplus Q_0 : W \subset \ker P\}$. Then, the operator $R \in \tilde{R}$ is minimal $\Leftrightarrow$ there exists a (total mass one) measure μ such that*

$$E_R = \int_{\mathcal{E}(R)} y \otimes (I - Q_0^{**})\, x\, d\mu(x, y) : V \to V + W. \tag{9}$$

PROOF. Mimic the proof of Theorem 1 where now $\delta \in V^{\perp} \cap W^{\perp} = (V + W)^{\perp}$. ∎

THEOREM. EXISTENCE. *Minimal operators exist in all the theorems of this paper.*

PROOF. The proof follows from a standard argument using the fact that $\mathcal{D}$ is closed and any closed bounded subset of $\mathcal{B}(X, Z^*)$ is compact in the weak* operator topology. ∎

As immediate examples, we obtain well-known characterizations of minimal Boolean sum projections in the following cases.

EXAMPLE 1. $n = m = 1$.

EXAMPLE 2. Let T (with "+") be a compact Abelian group with Haar measure ν, $\hat{T}$ its dual, $\{v_\gamma\}_{\gamma \in \hat{T}}$ the set of all characters, and let $X = L^p(T)$, $1 \leq p < \infty$ or $X = C(T)$, $p = \infty$.

COROLLARY 3. *Let $m = 0$, i.e., $W = \{0\}$. Then,*

$$E_R = E_P = \int_{\mathcal{E}(R)} y \otimes x\, d\mu(x, y) : V \to V$$

is the characterization of a minimal operator P given in [1].

THEOREM 4. CHARACTERIZATION. *Let $Q_0 : X \to W$ be a fixed operator and let $P : X \to V$ be arbitrary. Then, $R = P \oplus Q_0$ has minimal norm in $\mathcal{R} = \{P \oplus Q_0\} \Leftrightarrow$ there exists a (total mass one) measure μ such that*

$$E_R = \int_{\mathcal{E}(R)} y \otimes (I - Q_0^{**})\, x\, d\mu(x, y) : V \to \{0\}.$$

PROOF. Modify the proof of Theorem 1 so that $\delta \in \{0\}^{\perp}$, i.e., δ has no restrictions. ∎

2. BLENDING-TYPE OPERATORS

$X = L^p$, $1 \leq p \leq \infty$, "L^∞"$= C$. Consider the blending-type operator

$$P^s \oplus Q^t = P^s + Q^t - P^s Q^t : L^p\left(T^2\right) \to V^s + W^t = Z,$$

where $P^s = P \otimes I$ and $Q^t = I \otimes Q$ with V^s being the range of P^s and W^t being the range of Q^t. $v(s, t) = v^t(s) = P(f^t(s))$, where $f^t(s) = f(s, t)$.

NOTE 4. $P^s Q^t = Q^t P^s$ (approximate f by a finite sum of separable functions on which clearly $P^s Q^t = Q^t P^s$), and this implies that if P^s and Q^t are projections, then $P^s \oplus Q^t$ is a projection onto Z.

From Theorem 1, we obtain the following result.

THEOREM 5. CHARACTERIZATION. *Fix $Q = Q_0$. Then $R = P^s \oplus Q_0^t$ has minimal norm $\Leftrightarrow$ there exists a (total mass one) measure μ such that*

$$E_R = \int_{\mathcal{E}(R)} y \otimes \left(I - \left(Q_0^t\right)^{**}\right) x \, d\mu(x,y) : V^s \to V^s.$$

(Here, for each t, $(y(s,t) \otimes x(\cdot,t))(w(s)) = \langle x(\cdot,t), \langle w(s), y(s,t)\rangle_s\rangle \in L^p(T)^{**}$.)

PROOF. Modify the proof of Theorem 1 as follows. First, check that $\delta \otimes w$ defined on $(x,y) \in L^p(T^2) \times L^q(T^2)$ by $\delta \otimes w(x,y) = \langle\langle x(r,t), \delta(r)\rangle_r, \langle w(s), y(s,t)\rangle_s\rangle_t$ is continuous on $X^{**} \times X^*(= L^p(T^2) \times L^q(T^2))$, and then follow the proof of Theorem 1. (Also, use $L^p(T^2) = L^p(T) \otimes L^p(T)$.)

Likewise, from Theorem 2 we obtain the following result (the proof is the analogue of that of Theorem 5).

THEOREM 6. CHARACTERIZATION. *Fix $P = P_0$. Then, $R = P_0^s \oplus Q^t$ has minimal norm $\Leftrightarrow$ there exists a (total mass one) measure μ such that*

$$E_R = \int_{\mathcal{E}(R)} (y \otimes x)\left(I - P_0^s\right) d\mu(x,y) : W^t \to W^t.$$

COROLLARY 4. *In Theorem 6, where $P = \vec{u}_0 \otimes \vec{v}$, write $Q^t = \vec{r}(s,t) \otimes \vec{w}(s,t)$ and assume that W^t is piecewise continuously differentiable and P_0^s-related to V^s. Then, if $Q^t = \sum_{i=1}^n r_i \otimes w_i$ provides a minimal R in $\mathcal{R}$, then the following linear (first-order differential if $p < \infty$) equation for $\vec{r} = (r_1, \dots, r_n)$ holds:*

$$\frac{1}{p}\left(\frac{\partial}{\partial s}\vec{r} + \frac{\partial}{\partial s}\vec{u}_0\right) M\vec{w} = \frac{1}{q}(\vec{r} + \vec{u}_0) \cdot M \frac{\partial}{\partial s}\vec{w}, \qquad \text{on } T, \tag{10}$$

where M is the matrix in (4), "$\frac{\partial}{\partial s}$" denotes partial-differentiation along an arbitrary vector field in T and $1/q + 1/p = 1$.

Similarly, write $P^s = \vec{u}(s,t) \otimes \vec{v}(s,t)$ and assume that V^s is piecewise continuously differentiable and Q_0^t-related to W^s, where $Q_0 = \vec{r}_0 \otimes \vec{w}$. Then, if $P^s = \sum_{i=1}^n u_i \otimes v_i$ provides a minimal R in $\mathcal{R}$, then the following linear (first-order differential if $p < \infty$) equation for $\vec{u} = (u_1, \dots, u_n)$ holds:

$$\frac{1}{p}\left(\frac{\partial}{\partial t}\vec{r}_0 + \frac{\partial}{\partial t}\vec{u}\right) N\vec{v} = \frac{1}{q}(\vec{r}_0 + \vec{u}) \cdot N \frac{\partial}{\partial t}\vec{v}, \qquad \text{in } T, \tag{11}$$

where N is the analogue of the matrix in (3), "$\frac{\partial}{\partial t}$" denotes partial-differentiation along an arbitrary vector field in T and $1/q + 1/p = 1$.

COROLLARY 5. *$R = P^s \oplus Q^t$ has minimal norm $\Leftrightarrow$ there exist (total mass one) measures μ_1, μ_2, μ_3, and μ_4 such that*

$$\begin{aligned}
E_R^{(1a)} &= \int_{\mathcal{E}(R)} y \otimes \left(I - Q^t\right)^{**} x \, d\mu_1(x,y) : V^s \to V^s, \\
E_R^{(1b)} &= \int_{\mathcal{E}(R)} y \otimes \left(I - P^s\right)^{**} x \, d\mu_2(x,y) : W^t \to W^t, \\
E_R^{(2a)} &= \int_{\mathcal{E}(R)} (y \otimes x)\left(I - P^s\right) d\mu_3(x,y) : W^t \to W^t, \\
E_R^{(2b)} &= \int_{\mathcal{E}(R)} (y \otimes x)\left(I - Q^t\right) d\mu_4(x,y) : V^s \to V^s.
\end{aligned} \tag{12}$$

PROOF. See the proof of Corollary 2 and use Theorems 5 and 6 and the fact that P^s and Q^t commute. ∎

CONJECTURE 2. *The converse of Corollary 5 is true. (See Conjecture 1.)*

CONJECTURE 3. *$P^s \oplus Q^t$ is minimal precisely when P and Q are minimal.*

CONJECTURE 4. *If $V^s = W^t$, then $R = P^s \oplus Q^t$ is minimal $\Rightarrow P = Q$, and so, for some positive (total mass one) measure μ,*

$$E_{P^s \oplus P^t} = \int y \otimes \left(I - P^t\right)^{**} x \, d\mu(x,y) : V^s \to V^s. \tag{13}$$

COROLLARY 6. *If $m = 0$ (i.e., $W^t = \{0\}$), then P^s is minimal $\Leftrightarrow$ there exists a positive (total mass one) measure μ such that*

$$E_{P^s} = \int_{\mathcal{E}(P^s)} y \otimes x \, d\mu(x,y) : V^s \to V^s. \tag{14}$$

Note that (14) translates to ($\vec{v} = v_1, \dots, v_n$; N some $n \times n$ matrix)

$$\int_{\mathcal{E}(P^s)} \langle \vec{v}(s), \langle x(r,t), y(s,t) \rangle_t \rangle_s \, d\mu(x,y) = M\vec{v}.$$

As a consequence of Corollary 6, we have the following result due to Franchetti and Cheney.

COROLLARY 7. *(See [8].) In Corollary 6, then $P^s = P \otimes I$ where P is a minimal operator. Further, P^s is minimal among all operators onto V^s.*

PROOF. From [1] or Corollary 3, $E_p = \int_{\mathcal{E}(P)} y \otimes x \, d\mu(x,y) : V^s \to V^s$. Then, $\|P^s\| = \|P\| \, \|I\|$ and so $(x(s,t), y(s,t)) = (x(s), y(s))$ is an extremal pair for P^s, $\forall (x,y) \in \mathcal{E}(P)$. Thus, $E_{P^s} = E_P$ and P^s is minimal by Corollary 6. Moreover, check that $E_{P^s} : V^s \to V^s$, and so P^s is minimal among all projections onto V^s.

EXAMPLE 3. $n = 1$. $W = V = [v_1]$. $P = u_1 \otimes v_1$. Then, $(P^s \oplus P^t)f = \langle f^t(\cdot), u_1 \rangle v_1(s) + \langle f^s(\cdot), u_1 \rangle v_1(y) - \langle \langle \langle f^*(\cdot), u_1(\cdot) \rangle (x), u_1(x) \rangle v_1(x) \rangle v_1(y)$. Then, it can be checked that $P = \mathrm{ext}(v_1) \otimes v_1$ is minimal where $\mathrm{ext}(v)$ is an extremal of v (e.g., if $v \in L^p$, $1 \le p < \infty$, then $\mathrm{ext}(v) = \kappa \operatorname{sign}(v)|v|^{p-1}$), in particular, if

$$v_1 = 1 : \left(P^s \oplus P^t\right) f = \int_0^1 f(x,y) \, dx + \int_0^1 f(x,y) \, dy - \int_0^1 \int_0^1 f(x,y) \, dx \, dy.$$

EXAMPLE 4. Let T (with "+") be a compact Abelian group with Haar measure ν, $\hat{T}$ its dual, $\{v_\gamma\}_{\gamma \in \hat{T}}$ the set of all characters, N a finite part of $\hat{T}$, V the linear span of the characters v_τ, $\tau \in N$, and let $X = L^p(T)$, $1 \le p < \infty$ or $X = C(T)$, $p = \infty$. Then, the Fourier projection $F = \sum_{\tau \in N} v_\tau \otimes v_\tau$ yields a minimal blending projection.

EXAMPLE 5. $P = F$ is minimal in Corollary 6.

PROOF. SKETCH. Let $(x(s,t), y(r,u))$ be a fixed extremal pair for P^s,

$$\int_T \left[\int_T \left[\sum_{t \in N} \langle x^t(s), v_\tau(s) \rangle_s v_\tau(r) y(r,t) \right] dv(r) \right] dv(t) = \|P^s\|.$$

Then, analogously as in [9], show $(x_\sigma(s,t), y_\sigma(r,u)) = (x(\sigma + s, t), y(\sigma + r, u))$ is an extremal pair for P^s for each $\sigma \in T$ (by use of $v_\tau(\sigma) v_\tau(-\sigma) = 1$).

Next, verify (14) as follows: $E_{P^s} = \int_T y_\sigma \otimes x_\sigma \, dv(\sigma) : V \to V$ since

$$\begin{aligned}
\langle E_{P^s} v_\tau, u_\gamma \rangle &= \int_T \langle \langle v_\tau(s), \langle x_\sigma(r,t), y_\sigma(s,t) \rangle_t \rangle_s, v_\gamma(r) \rangle_r \, dv(\sigma) \\
&= \int_T \langle \langle v_\tau(s), \langle x(\sigma + r, t), y(\sigma + s, t) \rangle_t \rangle_s, v_\tau(r) \rangle_r \, dv(\gamma) \\
&= \int_T \langle \langle v_\tau(s - \sigma), \langle x(r,t), y(s,t) \rangle_t \rangle_s, v_\gamma(r + \gamma) \rangle_r \, dv(\gamma) \\
&= \langle (y \otimes x)(v_\tau), v_\gamma \rangle \langle v_\gamma, v_\tau \rangle = 0, \qquad \text{if } \gamma \ne \tau.
\end{aligned}$$

∎

EXAMPLE 6. Let $W = V$; then $R = F^s \oplus F^t$ is minimal with F^t fixed (and, by symmetry, is minimal with F^s fixed).

PROOF. The proof follows analogously as in the proof of Example 5. ∎

For related work, please see the references [10–17].

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