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Some Estimates of Action Constants and Related Parameters

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Abstract—In this paper, we continue investigation of an action constant $\lambda_A(V)$ introduced in [1–3]. In particular, we show that, if n is the dimension of V, then $\lambda_A(V) \le \rho(A)\sqrt{n}$, which is the generalization of the well-known estimate $\lambda_I(V) \le \sqrt{n}$. We then proceed to compare the estimates for $\lambda_A(V)$ and $\lambda_I(V)$ for a variety of two-dimensional spaces. These estimates are obtained with the extensive use of computer interaction.

Keywords—Minimal projections, Action constants.

1. INTRODUCTION

Let A be $n \times n$ matrix, let $\sigma(A)$ be the set of eigenvalues of A, and let $\rho(A)$ be its spectral radius. Let V be an n-dimensional Banach space and let T be an operator on V. We say that T is similar to A ($T \sim A$) if there exists a basis in V such that the matrix of the operator T with respect to this basis is A. We now introduce a number of parameters associated with A and V. Let

$$n_A(V) = \inf\{\|T\| : T : V \to V; T \sim A\}.$$

If $X \supset V$ and T is an operator on V, we use

$$e(T,X) = \inf_{\tilde{T}} \left\{ \left\|\tilde{T}\right\| : \tilde{T} : X \to V; \tilde{T}_{|V} = T \right\},$$
$$e(T) = \sup_X \{e(T,X) : V \subset X\}.$$

The action constant $\lambda_A(V)$ is defined to be

$$\lambda_A(V) = \inf\{e(T) : T : V \to V; T \sim A\}. \tag{1}$$

Observe that for $A = I$, the action constant $\lambda_A(V)$ is the well-studied projection constant. In particular, the celebrated Kadec-Snobar Theorem gives $\lambda_I(V) \le \sqrt{n}$. In this paper, we will demonstrate that $\lambda_A(V) \le \rho(A)\sqrt{n}$ for an arbitrary matrix A, thus generalizing the Kadec-Snobar Theorem. To prove that theorem, we need an action version of the well-known two-summing norm

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of an operator (cf. [4])

$$\pi_2(T) = \sup\left\{\left[\frac{\sum_{j=1}^{N}\|Tx_j\|^2}{\sup_{\|f\|=1}\sum_{j=1}^{N}|f(x_j)|^2}\right]^{1/2}\right\},$$

where the supremum is taken over all N and all collections of vectors $x_1,\dots,x_N \in V$. We thus define

$$\pi_2^A(V) = \inf\{\pi_2(T) : T : V \to V;\ T \sim A\}.$$

We will need some well-known characterizations of $\pi_2(T : V \to V)$. For this, let V be an n-dimensional space and T be an operator from V to V. For some probability measures μ, V is isometric to a space, say V_∞, where $V_\infty \subset L_\infty(\mu)$. Now let $V_2 \subset L_2(\mu)$ be a space that consists of the same functions as V_∞ but equipped with the L_2 norm. Consider a factorization of $T : V \to V$ as follows:

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ \downarrow i & & \uparrow R \\ V_\infty & \xrightarrow{S} & V_2 \\ \cap & & \cap \\ L_\infty(\mu) & & L_2(\mu) \end{array}. \tag{2}$$

Then (cf. [5]), the following theorem holds:

$$\pi_2(T) = \inf \|S\|\,\|R\|,$$

where the infimum is taken over all such representations and (this will be important later) over all measures μ.

It follows from this description (cf. [5]) that

$$\pi_2(T) \geq e(T),$$

and hence,

$$\pi_2^A(V) \geq \lambda_A(V) \geq n_A(V) \geq \rho(A).$$

Furthermore,

$$n_A\left(\ell_\infty^{(n)}\right) = \rho(A).$$

It is also known that for two-dimensional subspaces V,

$$\lambda_I(V) \leq \frac{4}{3} < \sqrt{2}.$$

It is hence somewhat a pleasant surprise (Theorem 2) that there exists a two-dimensional space V and A with $\rho(A) = 1$ such that

$$\lambda_A(V) = \sqrt{2}.$$

The extremal value for the projectional constant is attained on the space V with the regular-hexagonal unit ball. We thus dedicate the last section of this paper to estimating $\lambda_A(V)$ and related parameters for this space V and for various rotations A. We hope that these results contribute to the general understanding of the overall problem.

PROBLEM. To what extent does the set of parameters

$$\{\lambda_A(V) : A \in M_{n\times n}\}$$

describe an n-dimensional Banach space V?

2. GENERAL THEOREMS

THEOREM 1.

$$\pi_2^A(V) \le \rho(A)\sqrt{n}.$$

PROOF OF THEOREM 1. In view of the above, we need to construct an operator $T \sim A$ and a factorization according to (2) with $\|S\|\,\|R\| \le (\sqrt{n})\rho(A)$. To do this, we use the fact (see [5]) that

$$\pi_2(I : V \to V) = \sqrt{n}.$$

Hence, from the above there exists a factorization

$$\begin{array}{ccc} V & \xrightarrow{I} & V \\ \downarrow i & & \uparrow R' \\ V_\infty & \xrightarrow{S'} & V_2 \\ \cap & & \cap \\ L_\infty(\mu) & & L_2(\mu) \end{array}$$

with $\|S'\|\,\|R'\| = \sqrt{n}$.

Since V_2 is a subspace of $L_2(\mu)$, V_2 is isometric to $\ell_2^{(n)}$, and hence (by [6]) there exists an operator $T' : V_2 \to V_2$ such that $T' \sim A$; $\|T'\| = \rho(A)$. Consider an operator given by $T = R'T'S'i = RSi$. The diagram follows:

$$\begin{array}{ccccc} V & & \xrightarrow{T} & & V \\ \downarrow i & & & & \uparrow R' \\ V_\infty & \xrightarrow{S'} & V_2 & \xrightarrow{T'} & V_2 \\ \cap & & \cap & & \cap \\ L_\infty(\mu) & & L_2(\mu) & & L_2(\mu) \end{array}.$$

Then, $T \sim A$ and $T : V \to V$;

$$\|S\|\,\|R\| = \|S'\|\,\|R'T'\| \le \|S'\|\,\|T'\|\,\|R'\| \le \rho(A)\,\|S'\|\,\|R'\| = \rho(A)\sqrt{n}.$$

Finally, observe that $T = RSi$ is the factorization of form (2), and hence

$$\pi_2(T) \le \rho(A)\sqrt{n}.$$ ∎

COROLLARY 1. *For every A and every V,*

$$\lambda_A(V) \le \rho(A)\sqrt{n}.$$

COROLLARY 2. *Let A be an $n \times n$ matrix such that*

$$\sigma(A) \subset \{z \in C : |z| = 1\}.$$

Then,

$$\pi_2^A(V) = \sqrt{n},$$

for any n-dimensional Banach space V.

PROOF. The proof follows from Theorem 1 and [7]. ∎

Corollary 2 is thus once again a direct generalization of the well-known estimate

$$\pi_2^I(V) = \sqrt{n}.$$

CONJECTURE 1. $\pi_2^A(V) = \sqrt{\sum_{j=1}^n |\lambda_j|^2}$ *where the λ_js are the eigenvalues of A.*

THEOREM 2. *Let*

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

and $V = \ell_\infty^{(2)}$ *over* $\mathbb{R}$. *Then,* $\lambda_A(V) = |\cos\theta| + |\sin\theta|$ $(\rho(A) = 1)$.

PROOF. Consider $n_A(V) = \inf_S \|S^{-1}AS\|$, where the norm is the maximum of the absolute row sums and

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is an arbitrary invertible 2×2 matrix. Further, let $A_S = S^{-1}AS$, $\Delta = ad - bc$, $E = (ab + cd)/\Delta$, $P = (b^2 + d^2)/\Delta$, $Q = (a^2 + c^2)/\Delta$, $\delta = \cot\theta$, $\sigma = \sin\theta$, and note that $1 + E^2 = PQ$. Now, without loss, assume $0 < \theta < \pi/2$. Then,

$$\frac{\|A_S\|}{\sigma} = \max\{|\delta + E| + |P|, |\delta - E| + |Q|\}.$$

CASE 1. $\delta \geq |E| \geq 0$.

$$\frac{\|A_S\|}{\sigma} = \delta + \max\{E + |P|, -E + |Q|\}.$$

For E fixed, determine, by use of $|Q| = (1 + E^2)/|P|$, that the two arguments of the max are equal to $\sqrt{1+2E^2}$, when $|P| = \sqrt{1+2E^2} - E$. On the other hand, if $|P| > \sqrt{1+2E^2} - E$, then obviously $E + |P| > \sqrt{1+2E^2}$, whereas if $|P| < \sqrt{1+2E^2} - E$, then $-E + |Q| = -E + (1 + E^2)/|P| > -E+(1+E^2)/(\sqrt{1+2E^2}-E) = (1+2E^2-E\sqrt{1+2E^2})/(\sqrt{1+2E^2}-E) = \sqrt{1+2E^2}$. We conclude that, for E fixed, $\inf \max\{E + |P|, -E + |Q|\} = \sqrt{1+2E^2}$. Thus, the inf (over all S) is achieved when $E = 0$, and thus when $|P| = |Q| = 1$, i.e., when $ab + cd = 0$ and $b^2 + d^2 = a^2 + c^2 = |ad - bc|$, and in particular, when $d = a$ and $b = c = 0$.

CASE 2. $|E| \geq \delta > 0$. Assume first that $E > 0$. Then,

$$\frac{\|A_S\|}{\sigma} = E + \max\{\delta + |P|, -\delta + |Q|\}.$$

Analogously, as in Case 1, determine that the two arguments of the max are equal to $\sqrt{1+E^2+\delta^2}$ when $|P| = \sqrt{1+E^2+\delta^2} - \delta$. Thus, the inf (over all S) is achieved when $E = \delta$.

Assume, finally, that $E < 0$. Then,

$$\frac{\|A_S\|}{\sigma} = -E + \max\{-\delta + |P|, \delta + |Q|\},$$

and the argument is completely symmetric and the conclusion is the same as in the case $E > 0$.

Note, however, that $\delta + \sqrt{1+2\delta^2} > \delta + 1$ and thus the inf over all S from both cases is achieved in Case 1. That is, we find that the infimum is achieved for $S = I$, and thus

$$n_A(V) = \inf \|A_S\| = \sigma(1 + \delta) = \sin\theta + \cos\theta.$$

But then, since $V = \ell_\infty^{(2)}$ is its own maximal overspace, $\lambda_A(V) = n_A(V)$. ∎

COROLLARY 3. *Let*

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then, there exists a two-dimensional Banach space V *such that* $n_A(V) = \sqrt{2}$. *Hence, we have*

$$\sqrt{2} = n_A(V) \leq \lambda_A(V) \leq \pi_2^A(V) \leq \sqrt{2},$$

and thus

$$\sqrt{2} = n_A(V) = \lambda_A(V) = \pi_2(V) = \sqrt{2}.$$

The operator $T \sim A$ for which $\sqrt{2} = \|T\|$ is an operator for which the norms

$$\|T\|,\ \pi_2(T),\ e(T),$$

coincide.

Theorem 2 does not generalize to large n. In fact, we have the following result.

THEOREM 3. *For every A with $\rho(A) = 1$,*

$$\lambda_A\left(\ell_\infty^{(n)}\right) \leq \sqrt{2}.$$

PROOF. Let $\epsilon > 0$. Let A be an $n \times n$ matrix written in the "real" Jordan form

$$A = \begin{bmatrix} A_{11} & \eta_1 I & & \\ & A_{22} & \eta_2 I & \\ & & \ddots & \\ & & & A_{kk} \end{bmatrix},$$

where A_{jj} are two-by-two or one-by-one real matrices and η_j are ϵ or 0. Consider this matrix as an operator acting on ℓ_∞ direct sum $\oplus_{j=1}^k \ell_\infty^{i(j)} \simeq \ell_\infty^{(n)}$ where $i(j) = 1$ or 2 depending on the size of the matrix A_{jj}

$$\|A\| = \left(\max\left\|A_{jj} : \ell_\infty^{i(j)} \to \ell_\infty^{i(j)}\right\|\right) + \max \eta_j \leq \sqrt{2} + \epsilon.$$

Since $\ell_\infty^{(n)}$ is an injective space

$$e(A) = \|A\| \leq \sqrt{2} + \epsilon,$$

and $\lambda(\ell_\infty^{(n)}) \leq \sqrt{2} + \epsilon$ for any $\epsilon > 0$. ∎

By use of Dvoretzky's Theorem and the fact that

$$n_A\left(\ell_\infty^{(2)}\right) = \rho(A),$$

the stronger version of this result can probably be obtained.

CONJECTURE 2. *Let A_n be an arbitrary sequence of $n \times n$ matrices. Then*

$$\lim_{n\to\infty} \frac{\lambda_{A_n}(\ell_\infty^n)}{\rho(A_n)} = 1.$$

3. NOTES ON HEXAGONAL SPACES

In this part, we will estimate some extension and action constants on the two-dimensional real spaces with the hexagonal unit ball. These estimates are motivated by the fact that the hexagonal spaces are extremal for the estimates of the projectional constants. The study of the two-dimensional case is also motivated by the still unsolved problem.

CONJECTURE 3. *Suppose that V is a two-dimensional Banach space such that $\lambda_A(V) = \lambda_A(\ell_2^{(2)})$ for all 2×2 matrices A. Then, V is isometric to $\ell_2^{(2)}$.*

PROPOSITION 1. *For $0 < a = \cot\theta \le 1$ $(0 < \theta \le \pi/4)$, consider the hexagonal space $V = [v_1, v_2] \subset \ell_1^{(3)}$, where $v_1, v_2 = (1, a, 0), (0, a, 1)$ and let*

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Noting that A is similar to the following matrix:

$$A' = \frac{1}{2\sqrt{1+a^2}} \begin{bmatrix} -a^2 - a + 2 & a(1+a) \\ -\dfrac{a\left(a^2+2a+5\right)}{(1+a)} & a^2 + a + 2 \end{bmatrix},$$

let $T : V \to V$ be the operator given by the matrix A' with respect to the basis v_1, v_2. Then, $e(T) = (1+a)/\sqrt{1+a^2} = |\cos\theta| + |\sin\theta|$.

PROOF. Define the operator $\tilde{T}$ from $\ell_1^{(3)}$ onto V given (in matrix form with respect to the standard basis) by

$$\frac{1}{\sqrt{1+a^2}} \begin{bmatrix} 1 & -\dfrac{1+a}{2} & -\dfrac{2a}{1+a} \\ a & 0 & \dfrac{a(1-a)}{1+a} \\ 0 & \dfrac{1+a}{2} & 1 \end{bmatrix}.$$

Then, check that $\tilde{T}$ is an extension of the operator T on V given by the matrix A with respect to the basis v_1, v_2. It is then straightforward to compute (use the absolute column sums (all equal) to compute $\|\tilde{T}\|$) that

$$\left\|\tilde{T}\right\| = \frac{1+a}{\sqrt{1+a^2}} = \|T\|, \tag{3}$$

since

$$\|T\| = \sup_{\alpha,\beta} \frac{\|T(\alpha v_1 + \beta v_2\|)}{\|\alpha v_1 + \beta v_2\|} = \sup_{\gamma} \frac{|\tilde{a} + \tilde{c}\gamma| + a\left|\tilde{a} + \tilde{b} + \left(\tilde{c} + \tilde{d}\right)\gamma\right| + \left|\tilde{b} + \tilde{d}\gamma\right|}{1 + a|1+\gamma| + |\gamma|},$$

and, at $\gamma = -1$, $= (|\tilde{c} - \tilde{a}| + |\tilde{b} - \tilde{d}| + a|\tilde{b} - \tilde{c} + \tilde{a} - \tilde{d}|)/2 = (1+a)/\sqrt{1+a^2}$ in all cases where

$$A' = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix}.$$

Note that, since the unit ball of V is a hexagon (with six extreme points), V is not isometric to $\ell_\infty^{(2)}$ (whose unit ball has four extreme points). The result now follows because (3) implies $e(T) = \|\tilde{T}\|$, since L_1 is a "maximal overspace" [8,9], $e(T) = e(T, \ell_1^{(3)})$. ∎

PROPOSITION 2. *For $0 \le a \le 1$, consider the hexagonal space $V = [v_1, v_2] \subset \ell_1^{(3)}$, where $v_1, v_2 = (1, a, 0), (0, a, 1)$, and let A be the (rotation by $\pi/4$) matrix $(1/\sqrt{2})[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}]$.*

Noting that A is similar to the following matrix:

$$A' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1-a & 1+a \\ -\dfrac{a^2+1}{1+a} & 1+a \end{bmatrix},$$

let $T : V \to V$ be the operator given by the matrix A' with respect to the basis v_1, v_2. Then, $e(T) = \sqrt{2} = \cos(\pi/4) + \sin(\pi/4)$.

PROOF. Define the operator $\tilde{T}$ from $\ell_1^{(3)}$ onto V given (in matrix form with respect to the standard basis) by

$$\frac{1}{\sqrt{2}} \begin{bmatrix} \dfrac{1-a}{1+a} & \dfrac{1-a}{1+a} & -1 \\ \dfrac{2a}{1+a} & \dfrac{2a}{1+a} & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then, check that $\tilde{T}$ is an extension of the operator T on V given by the matrix A with respect to the basis v_1, v_2 and $\|\tilde{T}\| = \sqrt{2}$. It is then straightforward to compute (use the absolute column sums (all equal) to compute $\|\tilde{T}\|$) that

$$\left\|\tilde{T}\right\| = \sqrt{2} = \|T\|, \tag{4}$$

since

$$\|T\| \geq \frac{\|Tv_1\|}{\|v_1\|} = \frac{1}{\sqrt{2}} \frac{\|(1-a)v_1 + (1+a)v_2\|}{\|v_1\|} = \frac{\sqrt{2}(1+a)}{1+a} = \sqrt{2}.$$

Once again, note that the unit ball of V is a hexagon, and thus V is not isometric to $\ell_\infty^{(2)}$. The result now follows because (4) implies $e(T) = \|\tilde{T}\|$, since, as in Proposition 1, L_1 is a "maximal overspace". ∎

NOTE 1. Consider the regular hexagonal space $H = [v_1, v_2] \subset \ell_1^{(3)}$, where $v_1, v_2 = (1, 1/2, 1/2)$, $(0, \sqrt{3}/2, \sqrt{3}/2)$. The ball is then the regular hexagon with all six sides the same length, as is easily seen. Let A be the (rotation by $\pi/4$) matrix $(1/\sqrt{2})[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}]$, and let $T : H \to H$ be the operator given by the matrix A with respect to the basis v_1, v_2. Then, we have the estimate

$$e(T) \leq \lambda_I(H)\|T\| = \frac{4}{3}\frac{\sin(5\pi/12)}{\sin(\pi/3)} = \frac{4}{3}\frac{\sqrt{3}+1}{\sqrt{6}} > \sqrt{2}.$$

PROPOSITION 3. *In the setting of Note 1 above, we have in fact that* $e(T) < 4/3 < \sqrt{2}$.

PROOF. Define the operator $\tilde{T}$ from $\ell_1^{(3)}$ onto V given (in matrix form with respect to the standard basis) by

$$\frac{1}{6\sqrt{2}}\begin{bmatrix} 2 & 1-\sqrt{3} & -1-\sqrt{3} \\ 1+\sqrt{3} & 2 & 1-\sqrt{3} \\ -1+\sqrt{3} & 1+\sqrt{3} & 2 \end{bmatrix}.$$

Then, check that $\tilde{T}$ is an extension of the operator T on V given by the matrix A with respect to the basis v_1, v_2. It is then immediate (use the absolute column sums (all equal) to compute $\|\tilde{T}\|$) that

$$\left\|\tilde{T}\right\| = \frac{\sqrt{2}}{3}\left(1+\sqrt{3}\right) < \frac{4}{3}.$$

In this case, we can use elementary geometric considerations to conclude that

$$\|T\| = \frac{\sin(5\pi/12)}{\sin(pi/3)} = \frac{\sqrt{3}+1}{\sqrt{6}} < \frac{\sqrt{2}}{3}\left(1+\sqrt{3}\right) = \left\|\tilde{T}\right\|.$$

Thus, we cannot so easily conclude that $e(T) = \|\tilde{T}\|$ as we did in Propositions 1 and 2 (where $\|\tilde{T}\| = \|T\|$). We can, however, use the theory of [10] to check that $\tilde{T}$ is indeed a minimal extension of the operator T. But, nevertheless, since once again L_1 is a "maximal overspace" for V, the result follows. ∎

NOTE 2. Consider again the regular hexagonal space $H = [v_1, v_2] \subset \ell_1^{(3)}$, where $v_1, v_2 = (1, 1/2, 1/2)$, $(0, \sqrt{3}/2, \sqrt{3}/2)$, let A be the (rotation by $\pi/6$) matrix $1/2[\begin{smallmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{smallmatrix}]$, and let $T : H \to H$ be the operator given by the matrix A with respect to the basis v_1, v_2. By elementary geometric considerations, we see that $\|T\| = 2/\sqrt{3}$, and thus, we have the immediate estimate $e(T) \leq \lambda_I(H)\|T\| = (4/3)(2\sqrt{3}) > \sqrt{2}$.

PROPOSITION 4. *In the setting of Note 2 above, we have in fact that* $e(T) = 2/\sqrt{3} < 4/3 < \sqrt{2}$.

PROOF. Define the operator $\tilde{T}$ from $\ell_1^{(3)}$ onto V given (in matrix form with respect to the standard basis) by

$$\frac{\sqrt{3}}{3}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then, check that $\tilde{T}$ is an extension of the operator T on V given by the matrix A with respect to the basis v_1, v_2. It is then immediate (use the absolute column sums (all equal) to compute $\|\tilde{T}\|$) that

$$\left\|\tilde{T}\right\| = \frac{2}{\sqrt{3}}.$$

In this case, we can use elementary geometric considerations to conclude that $\|T\| = 2/\sqrt{3} = \|\tilde{T}\|$. Thus, we can conclude that $e(T) = \|\tilde{T}\|$ (as we did in Propositions 1 and 2). ∎

NOTE 3. Consider again the regular hexagonal space, $H = [v_1, v_2] \subset \ell_1^{(3)}$, where $v_1, v_2 = (1, 1/2, 1/2), (0, \sqrt{3}/2, \sqrt{3}/2)$, let A be the (rotation by $\pi/3$) matrix $(1/2)\left[\begin{smallmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{smallmatrix}\right]$, and let $T : H \to H$ be the operator given by the matrix A with respect to the basis v_1, v_2. Then, we have the estimate $e(T) \le \lambda_I(H)\|T\| = (4/3) \cdot 1 < \sqrt{2}$.

PROPOSITION 5. *In the setting of Note 2 above, we have in fact that* $e(T) = 4/3 < \sqrt{2}$.

PROOF. Define the operator $\tilde{T}$ from $\ell_1^{(3)}$ onto V given (in matrix form with respect to the standard basis) by

$$\frac{1}{3}\begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Then, check that $\tilde{T}$ is an extension of the operator T on V given by the matrix A with respect to the basis v_1, v_2. It is then immediate (use the absolute column sums (all equal) to compute $\|\tilde{T}\|$) that

$$\left\|\tilde{T}\right\| = \frac{4}{3}.$$

In this case, we can use elementary geometric considerations to conclude that $\|T\| = 1$. Thus, we cannot so easily conclude that $e(T) = \|\tilde{T}\|$ as we did in Propositions 1 and 2 (where $\|\tilde{T}\| = \|T\|$). We can, however, use the theory of [10] to check that $\tilde{T}$ is indeed a minimal extension of the operator T. Once again, L_1 is a "maximal overspace" for V and the result follows. ∎

DEFINITION 1. *Define* $\tilde{\lambda}_A(V) := \inf_T e(T)$ *where* T *yields*

$$\inf\left\{\frac{e(T)}{\|T\|} : T : V \to V;\ T \sim A\right\}.$$

CONJECTURE 4. *In the settings of Propositions 1 and 2, we conjecture that* $e(T) = \tilde{\lambda}_A(V)$. *In particular, let* A *be the (rotation by* $\pi/4$*) matrix*

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

and consider the regular hexagonal space $H = [v_1, v_2] \subset \ell_1^{(3)}$, *where* $v_1, v_2 = (1,1,0), (0,1,1)$. *That is, we conjecture that* $\tilde{\lambda}_A(H) = \sqrt{2}$.

CONJECTURE 5. *Since the regular hexagonal space* H *is also extreme for the case* $A = I$ *(with* $\lambda_I(H) = 4/3$ *(see [11])), we conjecture that, for all* n, *those spaces* V *which yield the extreme of* $\lambda_I(V)$ *also yield the spaces which are extreme for* $\tilde{\lambda}_A(V)$ *in Conjecture 5. (For example, in the case* $n = 3$, *the extremal space is conjectured to be the "icosahedron" space, although the authors at this time have no idea what the "natural" extremal* A *(corresponding to the rotation by* $\pi/4$ *in the case* $n = 2$*) might be.)*

For related work, see also [12–14].

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