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On the density principle for rational functions

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This paper is dedicated to my advisor and good friend Richard Varga on the occasion of his 70th birthday

Let E be a subspace of $C(X)$ and define $R(E) := \{g/h\colon\ g, h \in E;\ h > 0\}$. We prove that $R(E)$ is dense in $C(X)$ if for every $X_0 \subset X$ there exists $x \in X_0$ such that E contains an approximation to a δ-function at the point x on the set X_0. We use this principle to study the density of Müntz rationals in two variables.

1. Introduction

Let X be a compact Hausdorff space. Let $C(X)$ be the space of all real-valued continuous functions on X, and let E be a subspace of $C(X)$. We define

$$R(E) = \bigl\{f/g \in C(X)\colon\ f, g \in E;\ g(x) > 0 \text{ for all } x \in X\bigr\}.$$

In order for E to be dense in $C(X)$ it both necessary and sufficient that, for every point $x \in X$, the space E contains approximations to the δ-function at the point x.

We prove that $R(E)$ is dense in $C(X)$ if the space E contains appproximations to the δ-function at only one point $x_0 \in X_0 \subset X$, but for every closed subset $X_0 \subset X$.

These two conditions may appear similar since for every $x \in X$, one may choose $X_0 \ni x$ so small that it would force E to have an approximation to the δ-function at x. The rub is that approximations have to behave as the δ-function at x only on the set X_0 and may be arbitrary on $X \setminus X_0$. In some sense this result answers a question raised by Newman [4]. We prove this result in the next section. In section 3 we apply this result to establish the density of Müntz rational functions of two variables. We will use the rest of this section to explain the relationship between our results and the “zoomers” introduced by Newman and Somorjai [4,5].

Let Λ be an infinite sequence of non-negative real numbers containing 0. Let

$$E := \operatorname{span}\bigl\{t^{\lambda},\ \lambda \in \Lambda\bigr\} \subset C[0, 1].$$

Then $R(E)$ is dense in $C[0, 1]$. The proof of this remarkable theorem is based on the existence of functions that Newman calls “zoomers”: $Z_{\varepsilon,x}(t) = (t/(x - \varepsilon))^{\lambda}$. If

$\lambda \to \infty$ then the functions $Z_{\varepsilon,x}(t)$ approximate δ-function at x on the left of x and zoom to the right of x (left-half δ-functions). In the case when $\lambda \to 0$ Bak and Newman used different constructions to prove the same result. In [4] Newman asked for the unified understanding of these distinct cases. We believe that our result answers this call. Indeed, if $\lambda \to \infty$, and X_0 is any closed subspace of [0, 1], let $x = \sup X_0$. Then the zoomers $Z_{\varepsilon,x}$ are approximation to the "full" δ-function on the set X_0 and hence our principle applies. If $\lambda \to 0$ then the construction in [1] is nothing more than an approximation to a δ-function at 0. Hence, let $X_0 \subset [0, 1]$. If $0 \in X_0$ then we have δ-function at 0. If $0 \notin X_0$ then $E \mid X_0$ is dense in $C(X_0)$ and hence contains approximation to a δ-function!

2. The density principle

Let $x_0 \in X_0 \subset X$ and let, as before, E be a subspace of $C(X)$.

Definition 1. We say that E contains a δ-approximation at x_0 with respect to X_0 if, for every open set $W \ni x_0$, there exists an open set V with

$$x_0 \in V \subset W$$

such that for every $\varepsilon > 0$ there exists a function $f_\varepsilon \in E$ with the properties:

(i) $f_\varepsilon(x) \geqslant 0$ for all $x \in X_0$,

(ii) $f_\varepsilon(x) < \varepsilon$ for all $x \in X_0 \setminus W$,

(iii) $f_\varepsilon(x) > 1/\varepsilon$ for all $x \in V$.

Theorem 1. Let E be a subspace of $C(X)$ such that for every closed $X_0 \subset X$ there exists $x_0 \in X_0$ for which E contains δ-approximation at x_0 with respect to X_0. Then $R(E)$ is dense in $C(X)$.

The proof of this theorem relies on the following duality principle proved in [2].

Theorem 2 (Duality principle). Let $E \subset C(X)$ and $f \in C(X)$. The following are equivalent:

1. $f \notin \overline{R(E)}$;
2. For every closed linear subspace $F \supset E$ such that codim $F \leqslant 2$,
$$f \notin \overline{R(F)};$$
3. There exist regular Borel measures $\mu, \nu \perp E$ not both 0 with
$$f\mu + \nu \geqslant \delta|\mu| \quad \text{for some } \delta > 0;$$
4. There exist measures $\mu, \nu \perp E$ and $\delta > 0$ such that
$$\|f - \tilde{f}\| < \delta \quad \Rightarrow \quad \tilde{f}\mu + \nu > 0.$$

Proof of theorem 1. We first show that if λ is a measure on X such that $\lambda \perp E$ (i.e., $\int f \,\mathrm{d}\lambda = 0$ for all $f \in E$) and if $x_0 \in \operatorname{supp}\lambda$ such that E contains a δ-approximation at x_0 with respect to $\operatorname{supp}\lambda$ then

$$x_0 \in \operatorname{supp}\lambda^+ \cap \operatorname{supp}\lambda^-.$$

Indeed, say $x_0 \in \operatorname{supp}\lambda^+$; $x_0 \notin \operatorname{supp}\lambda^-$. Then there exists an open set $W \subset X$ such that $\lambda|W > 0$. Let V and f_ε be as in definition 1. Then $\lambda(V) > 0$ and

$$\begin{aligned}0 = \int f_\varepsilon \,\mathrm{d}\lambda &= \int_{(\operatorname{supp}\lambda)\setminus W} f_\varepsilon \,\mathrm{d}\lambda + \int_W f_\varepsilon \,\mathrm{d}\lambda \leqslant \int_{(\operatorname{supp}\lambda)\setminus W} f_\varepsilon \,\mathrm{d}\lambda + \int_V f_\varepsilon \,\mathrm{d}\lambda \\ &\leqslant \frac{1}{\varepsilon}(V) - \varepsilon\lambda(X).\end{aligned}$$

By letting $\varepsilon \to 0$ we obtain a contradiction.

Next we will appeal to the duality principle of theorem 2. Suppose $f \in C(X)$ such that $f \notin R(X)$. Let $\mu, \nu \perp E$ and $\delta > 0$. We intend to show that there exists $\tilde{f} \in C(X)$ such that $\|f - \tilde{f}\| < \delta$ and yet the measure

$$\tilde{f}\mu + \nu$$

is not positive, thus contradicting theorem 2, condition 4.

To this end, let $x_0 \in \operatorname{supp}\mu \cup \operatorname{supp}\nu$ be such that E contains a δ-approximation to x_0. Let U be an open set containing x_0 such that there exists a function f_1 with

$$\|f - f_1\| < \frac{\delta}{2} \quad \text{and} \quad f_1|U \equiv f(x_0).$$

If $x_0 \in \operatorname{supp}(f_1\mu + \nu)$ we set $\tilde{f} = f_1$. If $x_0 \notin \operatorname{supp}(f_1\mu + \nu)$, then there exists a neighborhood U_1 of x_0 such that $U_1 \subset U$ and $f(x_0)\mu|U_1 = -\nu|U_1$. In that case, set $\tilde{f} := f_1 + \delta/2$. In either case

$$x_0 \in \operatorname{supp}\big(\tilde{f}\mu + \nu\big)$$

and thus

$$x_0 \in \operatorname{supp}(c\mu + \nu)$$

where $c = \tilde{f}(x_0)$. Since $(c\mu + \nu) \perp E$ we have

$$x_0 \in \operatorname{supp}(c\mu + \nu)^+ \cap \operatorname{supp}(c\mu + \nu)^-. \tag{2.1}$$

Observe that $(c\mu + \nu)|U = \tilde{f}\mu + \nu|U$ and together with (2.1) that implies that $\tilde{f}\mu + \nu$ is not a positive measure on U and hence it is not a positive measure. This contradicts condition 4 of theorem 2 and we conclude that $f \in R(E)$. $\square$

Remark 1. We suspect that the assumptions of theorem 1 are very close to being necessary. However, in the form presented here, the density of $R(E)$ does not imply the assumptions of the theorem as the following example shows.

Example 1. Let $C[0,\pi]$ be the space of all continuous periodic functions, i.e., $f(0) = f(\pi)$. Let $E = \operatorname{span}\{1, \cos kx, \sin kx\}_{k+1}$. It was shown in [3] that $R(E)$ is dense in $C[0,\pi]$. We now intend to show that the space E does not satisfy the assumptions of theorem 2. Let $X_0 = [0,\pi]$ and let $x_0 \in [0,\pi]$. Observe that either $\sin x_0$ or $\cos x_0$ is not zero. Assume that $\cos x_0 > 0$. Then there exists a neighborhood $W \ni x_0$ such that $\cos x > 0$ for all $x \in W$. Since $(\cos x) \perp E$, then for any f_ε from definition 2 we have

$$0 = \int_0^{2\pi} f_\varepsilon(x) \cos x \, \mathrm{d}x \geqslant -\varepsilon + \frac{1}{\varepsilon} \int_V \cos x \, \mathrm{d}x \to \infty.$$

It is interesting to point out that $[0,\pi]$ is the only interval X_0 that does not satisfy definition 2, since E is dense on every proper subinterval of $[0,\pi]$.

3. Density of Müntz rationals

As was mentioned in section 1, the set $R(E)$, where

$$E = \operatorname{span}\bigl[1, t^{\lambda_j}\bigr]_{j=1}^{\infty} \subset C[0,1]$$

is dense as long as the sequence of integers (λ_j) is infinite. Consider the same question for the functions of two variables.

Let $0 < \alpha_1 < \alpha_2 \leqslant 1$. Let

$$\begin{aligned} D &:= \bigl\{(x,y) \in [0,1] \times [0,1]\colon\ \alpha_1 x \leqslant y \leqslant \alpha_2 x\bigr\}, \\ E &:= \operatorname{span}\bigl[1, x^{\lambda_j} y^{\mu_j}\bigr]_{j=1}^{\infty} \subset C(D). \end{aligned} \tag{3.1}$$

In this case having infinitely many lattice points (λ_j, μ_j) is no longer sufficient for the density of $R(E)$. Indeed, choose $(\lambda_j, \mu_j) = (j, j)$; $j = 1, \ldots, \infty$. Then for every $f(x,y) \in E$; $f(x,y) = f(y,x)$ and hence $R(E)$ is not dense in $C(D)$.

In this section we present a condition on the set $\bigl\{(\lambda_j, \mu_j)\bigr\}$ that guarantees the density of $R(E)$.

Theorem 3. Let the set $\{(\lambda_j, \mu_j)\} \subset \mathbb{N} \times \mathbb{N}$ be such that the set of the ratios

$$\left\{ \frac{\lambda_j}{\mu_j} \right\} \tag{3.2}$$

is dense in $\mathbb{R}_+$. Then $R(E)$ is dense in $C(D)$.

Proof. Let X_0 be a closed subset of D and let $x_0 \in X_0 = (a,b)$ such that

$$\|x_0\| = \sqrt{a^2 + b^2} \geqslant \|x\| \quad \text{for all } x \in X_0.$$

We claim that the space E contains a δ-approximation at x_0 with respect to the set X_0. By theorem 2, this claim would prove the theorem.

It now remains to prove the claim. If $x_0 = (0,0)$ then the functions

$$f_\varepsilon(x) \equiv \frac{1}{\varepsilon} \in E \text{ provide the } \delta\text{-approximation at } x_0 \text{ w.r.t. } X_0 = (0,0).$$

Without loss of generality, let $x_0 = (a,b)$ and $b \neq 0$. Let $\alpha = a^2/b^2$ and consider curves

$$\gamma_1\colon\ x^2 + y^2 = a^2 + b^2 \quad \text{and} \quad \gamma_2\colon\ yx^\alpha = a^\alpha b.$$

Both curves pass through (a,b) and the tangent line through (a,b) to both curves has the same slope $(-a/b)$. Since γ_1 is convex and γ_2 is a concave curve, we conclude that the function $g(x,y)$, defined by

$$g(x,y) = \frac{yx^\alpha}{a^\alpha b},$$

has the following properties:

$$\begin{aligned} &g(x,y) \leqslant 1 \quad \text{if } (x,y) \in X_0 \subset \bigl\{(x,y)\colon\ x^2+y^2 \leqslant \|x_0\|^2\bigr\},\\ &g(a,b) = 1,\\ &g(x,y) > 1 \quad \text{if } yx^\alpha > a^\alpha b. \end{aligned}$$

Let $W = \{(x,y)\colon\ (x-a)^2 + (y-b)^2 < r^2\}$. Then there exists $\eta > 0$ such that, for every β with $|\beta - \alpha| < \eta$, the functions

$$g_\beta(x,y) = \left(\frac{y}{b-\eta}\right)\left(\frac{x}{a-\eta}\right)^\beta$$

satisfy

$$\begin{aligned} &g_\beta(x,y) > 0,\\ &g_\beta(x,y) < 1 \quad \text{if } (x,y) \in X_0 \setminus W,\\ &g_\beta(x,y) > 1 \quad \text{if } (x,y) \in V \end{aligned}$$

for some open set $V \subset W$ containing (a,b).

Let $\mathbb{N}_0 \subset \mathbb{N}$ be such that

$$\frac{\lambda_j}{\mu_j} \to \alpha\colon\ j \in \mathbb{N}_0 \quad \text{and} \quad \mu_j \to \infty.$$

The functions

$$f_j(x,y) = \left(\frac{x}{a-\eta}\right)^{\lambda_j}\left(\frac{y}{b-\eta}\right)^{\mu_j} = \bigl(g_\beta(x,y)\bigr)^{\mu_j} \quad \text{with } \beta = \lambda_j/\mu_j$$

are elements of the space E and provide δ-approximation at x_0 with respect to X_0. $\square$

Remark 2. The condition (3.2) is attractive since it relates the density of the ratios of exponents to the density of the ratios of functions. Unfortunately, it is not sufficient. It was shown in [3] that for

$$E = \operatorname{span}\left\{1, x^{(2^k)} y^{(2^j)};\ (k, j) \in \mathbb{N} \times \mathbb{N}\right\}$$

$R(E)$ is dense in $C(D)$. However, the only limit points of the set $\{2^k/2^j\}$ are zero and infinity. It would be interesting to find a condition which is both necessary and sufficient.

References

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