HEREDITARY PROPERTIES OF HYPERGRAPHS

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ABSTRACT. A hereditary property $\mathbf{P}^{(k)}$ is a class of k-graphs closed under isomorphism and taking induced sub-hypergraphs. Let $\mathbf{P}_n^{(k)}$ denote those k-graphs of $\mathbf{P}^{(k)}$ on vertex set $\{1, \ldots, n\}$. We prove an asymptotic formula for $\log_2 |\mathbf{P}_n^{(k)}|$ in terms of a Turán-type function concerning forbidden induced sub-hypergraphs. This result complements several existing theorems for hereditary and monotone graph and hypergraph properties.

1. Introduction

Hereditary and monotone properties are well-studied objects in the areas of extremal combinatorics and theoretical computer science. For an integer $k \geq 2$, a *property* $\mathbf{P}^{(k)}$ is a class of k-uniform hypergraphs (k-graphs, for short) closed under isomorphism. The property $\mathbf{P}^{(k)}$ is *hereditary* (*monotone*) if it is closed under taking induced (arbitrary) sub-hypergraphs. (Note that every monotone property is also hereditary.) For a property $\mathbf{P}^{(k)}$ of k-graphs and an integer n, let $\mathbf{P}_n^{(k)}$ denote the k-graphs of $\mathbf{P}^{(k)}$ defined on vertex set $[n] = \{1, \ldots, n\}$. We estimate $|\mathbf{P}_n^{(k)}|$ for an arbitrary hereditary property $\mathbf{P}^{(k)}$.

Observe that every hereditary property $\mathbf{P}^{(k)}$ admits a unique minimal family $\mathbf{F}^{(k)} = \{\mathcal{F}_i^{(k)}\}_{i \in I}$ of pair-wise non-isomorphic 'forbidden' k-graphs so that $\mathbf{P}^{(k)}$ is the set $\mathrm{Forb}_{\mathrm{ind}}(\mathbf{F}^{(k)})$ of all k-graphs containing no $\mathcal{F}_i^{(k)} \in \mathbf{F}^{(k)}$ as an induced sub-hypergraph. (Note that $\mathbf{F}^{(k)}$ may be infinite.) We shall consider hereditary properties from this perspective. For an integer n, we write $\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$ as the k-graphs of $\mathrm{Forb}_{\mathrm{ind}}(\mathbf{F}^{(k)})$ defined on vertex set $[n]$.

To estimate $|\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})|$, we consider the following Turán-type parameter (adapted from one of Prömel and Steger [21] for graphs (cf. [15])). (In the definition below, we use $|\mathcal{H}^{(k)}|$ to denote the number of edges of $\mathcal{H}^{(k)}$, and $\binom{[n]}{k}$ to denote the family of all k-element subsets of $[n]$.) For an integer n, let

$$\mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) = \max \Big\{ \big|\mathcal{H}^{(k)}\big| : \text{ there exists } \mathcal{M}^{(k)} \subseteq \tbinom{[n]}{k} \setminus \mathcal{H}^{(k)} \text{ such that} \\ \mathcal{H}_0^{(k)} \subseteq \mathcal{H}^{(k)} \quad \Longrightarrow \quad \mathcal{H}_0^{(k)} \cup \mathcal{M}^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) \Big\}, \quad (1)$$

where the maximum is taken over all k-graphs $\mathcal{H}^{(k)} \subseteq \binom{[n]}{k}$ on vertex set $[n]$. (The Reader unfamiliar with this definition may want to now consider Examples 1.2 and 1.3 below.) The definition in (1) immediately implies that $\log_2 |\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})| \geq \mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$. Indeed, let the hypergraph $\mathcal{H}^{(k)}$ be an extremal example realizing the value of $\mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$ with the corresponding hypergraph $\mathcal{M}^{(k)}$. Then each of the $2^{|\mathcal{H}_0^{(k)}|}$ subhypergraphs $\mathcal{H}_0^{(k)} \subseteq \mathcal{H}^{(k)}$ renders a distinct element $\mathcal{H}_0^{(k)} \cup \mathcal{M}^{(k)}$ in $\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$. We arrive at our main result.

Theorem 1.1. *For any family $\mathbf{F}^{(k)}$ of k-graphs,* $\log_2 |\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})| = \mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + o(n^k)$.

Theorem 1.1 extends several earlier results. Prömel and Steger [19, 20, 21, 22] proved Theorem 1.1 for $k = 2$ when $\mathbf{F}^{(2)}$ consists of a single but arbitrary graph $F^{(2)} = \mathcal{F}^{(2)}$. Alekseev [1] and Bollobás and Thomason [3] then proved Theorem 1.1 for arbitrary families $\mathbf{F}^{(2)}$ of graphs. Kohayakawa, Nagle and Rödl [15] proved Theorem 1.1 for $k = 3$.

The first author used this work as part of his Masters thesis [4] at the University of Nevada, Reno.
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Theorem 1.1 also extends earlier work on monotone properties. Define $\mathrm{Forb}(\mathbf{F}^{(k)})$ and $\mathrm{Forb}(n, \mathbf{F}^{(k)})$ analogously to the hereditary setting, this time replacing 'induced sub-hypergraphs' with 'arbitrary sub-hypergraphs'. Recall the customary Turán number $\mathrm{ex}(n, \mathbf{F}^{(k)})$ is the maximum size $|\mathcal{H}^{(k)}|$ of a k-graph $\mathcal{H}^{(k)} \in \mathrm{Forb}(n, \mathbf{F}^{(k)})$. The analogue of Theorem 1.1 for monotone properties holds: *for all $k \geq 2$ and families $\mathbf{F}^{(k)}$ of k-graphs,*

$$\log_2 \left|\mathrm{Forb}(n, \mathbf{F}^{(k)})\right| = \mathrm{ex}(n, \mathbf{F}^{(k)}) + o(n^k). \tag{2}$$

(Note that $\log_2 |\mathrm{Forb}(n, \mathbf{F}^{(k)})| \geq \mathrm{ex}(n, \mathbf{F}^{(k)})$ is immediate.) In particular, Erdős, Kleitman and Rothschild [7] proved (2) for $k = 2$ when $\mathbf{F}^{(2)} = \{K_r\}$ consists of the single graph clique $K_r = K_r^{(2)}$ on r points. To our knowledge, the work of [7] is the first in this area. Erdős, Frankl and Rödl [6] then proved (2) for an arbitrary family $\mathbf{F}^{(2)}$ of graphs[1]. The hypergraph cases $k = 3$ and $k \geq 3$ were then respectively established by Nagle and Rödl [16] and Nagle, Rödl and Schacht [18].

Further (asymptotic) evaluations of $\log_2 |\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(2)})|$ and $\log_2 |\mathrm{Forb}(n, \mathbf{F}^{(2)})|$ are available for graphs, but this is essentially the only such case. Indeed, the well-known theorems of Turán [31] and of Erdős, Stone and Simonovits (cf. [8]) give $\mathrm{ex}(n, \mathbf{F}^{(2)}) = (1 - (1/(r-1)) + o(1))\binom{n}{2}$, where $r = r(\mathbf{F}^{(2)}) = \min\{\chi(F^{(2)}) : F^{(2)} \in \mathbf{F}^{(2)}\}$ is the smallest chromatic number across the family $\mathbf{F}^{(2)}$. As proven by Prömel and Steger [19, 20, 21, 22], Alekseev [1] and Bollobás and Thomason [3], $\mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(2)})$ takes the same asymptotic form, but for a different and more technical function $r' = r'(\mathbf{F}^{(2)})$. We do not review the function $r'(\mathbf{F}^{(2)})$ here since it admits no known hypergraph analogue, but we will consider the following graph example showing $\mathrm{ex}_{\mathrm{ind}}(n, C_4) = (1 + o(1))n^2/4$.

Example 1.2. Let $\mathbf{F}^{(2)} = \{C_4\}$ consist of the quadrilateral C_4. The lower bound $\mathrm{ex}_{\mathrm{ind}}(n, C_4) \geq \lfloor n/2 \rfloor \lceil n/2 \rceil$ follows by setting $V_1 = \{1, \ldots, \lfloor n/2 \rfloor\}$ and $V_2 = [n] \setminus V_1$, and defining $H = H^{(2)} = K[V_1, V_2]$ and $M = M^{(2)} = K_{V_1}$. Now, let $\varepsilon > 0$ and $n > n_0(\varepsilon)$ be given and suppose $\mathrm{ex}_{\mathrm{ind}}(n, C_4) \geq (1 + \varepsilon)n^2/4$, and let the graphs H and M establish the value of $\mathrm{ex}_{\mathrm{ind}}(n, C_4)$. The Erdős-Stone-Simonovits theorem guarantees that H contains a copy of the complete 3-partite graph $K_{2,2,2}$, which we take, w.l.o.g., to have 3-partition $\{1,2\} \cup \{3,4\} \cup \{5,6\} \subset V(H)$. If $M \cap \{\{1,2\}, \{3,4\}, \{5,6\}\} = \varnothing$, then $H \cup M$ contains induced copies of C_4. If M overlaps precisely one of these pairs, say $\{1,2\}$, then $H_0 = H \setminus \{\{1,4\}, \{2,5\}\}$ yields an induced copy of C_4 in $H_0 \cup M$. If M overlaps at least two of these pairs, say $\{1,2\}$ and $\{3,4\}$, then $H_0 = H \setminus \{\{1,4\}, \{2,3\}\}$ yields an induced copy of C_4 in $H_0 \cup M$. $\square$

For $k \geq 3$, determining $\mathrm{ex}(n, \mathbf{F}^{(k)})$ is well-known to be a very difficult problem. It seems likely that determining $\mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$ for $k \geq 3$ and many families $\mathbf{F}^{(k)}$ is similarly difficult. Clearly, $\mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) \geq \mathrm{ex}(n, \mathbf{F}^{(k)})$ and $\mathrm{ex}_{\mathrm{ind}}(n, K_r^{(k)}) = \mathrm{ex}(n, K_r^{(k)})$, where the determination of the last parameter is Turán's original problem. Equality holds in other cases as well, one of which we sketch below from [15].

Example 1.3. When $\mathbf{F}^{(k)} = \{\mathcal{F}_{k+1}^{(k)}\}$ consists of a k-graph $\mathcal{F}_{k+1}^{(k)}$ on $k+1$ points and $f \geq (k+1)/2$ edges, then[2] $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}_{k+1}^{(k)}) = \mathrm{ex}(n, \mathcal{F}_{k+1}^{(k)})$. Indeed, suppose $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}_{k+1}^{(k)}) > \mathrm{ex}(n, \mathcal{F}_{k+1}^{(k)})$, and let $\mathcal{H}^{(k)}$ and $\mathcal{M}^{(k)}$ establish the value of $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(k)})$. Since $|\mathcal{H}^{(k)}| > \mathrm{ex}(n, \mathcal{F}_{k+1}^{(k)})$, there exists a copy of $\mathcal{F}_{k+1}^{(k)}$ in $\mathcal{H}^{(k)}$. Let this copy have vertex set X_0 and write $h_0 = |\mathcal{H}^{(k)}[X_0]|$ (for the number of edges of $\mathcal{H}^{(k)}$ induced on X_0) and $m_0 = |\mathcal{M}^{(k)}[X_0]|$. From $h_0 \geq f \geq (k+1)/2$ and $h_0 + m_0 \leq k+1$, we infer that $m_0 \leq (k+1)/2 \leq f \leq h_0$. Obtain $\mathcal{H}_0^{(k)} \subseteq \mathcal{H}^{(k)}$ by deleting any $h_0 - f + m_0$ edges from $\mathcal{H}^{(k)}[X_0]$. Then $(\mathcal{H}_0^{(k)} \cup \mathcal{M}^{(k)})[X_0]$ consists of exactly f edges, and since any two k-graphs on $k+1$ points and equally many edges are isomorphic, $\mathcal{H}_0^{(k)} \cup \mathcal{M}^{(k)}$ contains an induced copy of $\mathcal{F}_{k+1}^{(k)}$. $\square$

[1] Strictly speaking, only the case when $\mathbf{F}^{(2)} = \{F\}$ consists of a single but arbitrary graph $F = F^{(2)}$ is addressed, but it is not difficult to adapt the proof for arbitrary families $\mathbf{F}^{(2)}$.

[2] When $f \leq (k+1)/2$, one has $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}_{k+1}^{(k)}) = \mathrm{ex}(n, \overline{\mathcal{F}}_{k+1}^{(k)})$, where $\overline{\mathcal{F}}_{k+1}^{(k)}$ is the complement of $\mathcal{F}_{k+1}^{(k)}$. This equality follows from the equality in Example 1.3 together with the identity $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(k)}) = \mathrm{ex}_{\mathrm{ind}}(n, \overline{\mathcal{F}}^{(k)})$ which is easy to show for any k-graph $\mathcal{F}^{(k)}$.

Note that $\mathrm{ex}_{\mathrm{ind}}(n, K_4^{(3)} - e) = \mathrm{ex}(n, K_4^{(3)} - e)$ is then a (well-known and difficult) special case of the example above, where $K_4^{(3)} - e$ denotes the triple system consisting of 3 triples on 4 points.

Finally, we mention one other result to which Theorem 1.1 relates. Bollobás and Thomason [2] showed that $\lambda(\mathbf{F}^{(k)}) = \lim_{n\to\infty} \binom{n}{k}^{-1} \log_2 |\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})|$ exists for any $k \geq 2$ and family $\mathbf{F}^{(k)}$ (which had been a question for graphs of Scheinerman and Zito [27].) The proof in [2] did not, however, give an indication as to the possible values of $\lambda(\mathbf{F}^{(k)})$. Now, set $\widetilde{\mathrm{ex}}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) = \binom{n}{k}^{-1} \mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$. It is routine to show that the sequence

$$\big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})\big)_{n=1}^{\infty} \text{ is non-increasing, so that } \pi_{\mathrm{ind}}(\mathbf{F}^{(k)}) = \lim_{n\to\infty} \widetilde{\mathrm{ex}}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) \text{ exists.} \tag{3}$$

Theorem 1.1 then adds the perspective that $\lambda(\mathbf{F}^{(k)}) = \pi_{\mathrm{ind}}(\mathbf{F}^{(k)})$.

The reader familiar with the earlier work on Theorem 1.1 and (2) knows that 'regularity' plays a crucial role in many of the proofs. In the case of graphs, this means appealing to the celebrated Szemerédi Regularity Lemma [28, 29]; in the case of hypergraphs, this means appealing to a hypergraph extension thereof. In particular, our proof makes use of a so-called 'hypergraph regularity method', versions of which were established by a collection of authors: Nagle, Rödl, Schacht, Skokan [17, 26], Gowers [12, 13], and Tao [30] (cf. [8, 10, 11, 24, 25]). Any of these versions would suffice for our purposes here. The hypergraph regularity method consists of a hypergraph regularity lemma and a hypergraph counting lemma. We find recent versions of these lemmas due to Rödl and Schacht [24, 25] the most convenient for our argument.

Our paper is organized as follows. In Section 2, we present the hypergraph regularity lemma and the hypergraph counting lemma from [24, 25]. In Section 3, we prove Theorem 1.1.

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Note added in proof. We recently learned that Y. Ishigami [14] announced a proof of our main result based on a hypergraph regularity lemma of his.

2. Hypergraph Regularity Lemma and Counting Lemma

We present the hypergraph regularity lemma of [24, 25] in the form of Theorem 2.7 and the hypergraph counting lemma of [24, 25] in the form of Theorem 2.8. We organize this section as follows. In Section 2.1, we present definitions and notation needed for Theorems 2.7 and 2.8. In Section 2.2, we state Theorems 2.7 and 2.8.

2.1. **Definitions.** We start with some basic concepts and notation.

Basic concepts. For integers $\ell \geq j \geq 1$, let $[\ell]^j = \binom{[\ell]}{j}$ denote the set of unordered j-tuples from $[\ell]$. (Note, $[\ell]^j$ does not represent a cross-product. Moreover, for a set X, $\binom{X}{j}$ always denotes the set of j-tuples from X.) Given vertex sets $V_1, \dots, V_\ell$, denote by $K^{(j)}(V_1, \dots, V_\ell)$ the *complete* ℓ-partite, j-uniform hypergraph (i.e., the family of all j-element subsets $J \subseteq \bigcup_{i\in[\ell]} V_i$ satisfying $|V_i \cap J| \leq 1$ for every $i \in [\ell]$). If $|V_i| = m$ for every $i \in [\ell]$, then an (m, ℓ, j)*-cylinder* $\mathcal{H}^{(j)}$ on $V_1 \cup \dots \cup V_\ell$ is any subset of $K^{(j)}(V_1, \dots, V_\ell)$. For $j \leq i \leq \ell$ and set $\Lambda_i \in [\ell]^i$, we denote by $\mathcal{H}^{(j)}[\Lambda_i] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda\in\Lambda_i} V_\lambda\big]$ the sub-hypergraph of the (m, ℓ, j)-cylinder $\mathcal{H}^{(j)}$ induced on $\bigcup_{\lambda\in\Lambda_i} V_\lambda$ (so that $\mathcal{H}^{(j)}[\Lambda_i]$ is an (m, i, j)-cylinder).

For an (m, ℓ, j)-cylinder $\mathcal{H}^{(j)}$ and an integer $j \leq i \leq \ell$, we denote by $\mathcal{K}_i(\mathcal{H}^{(j)})$ the family of all i-element subsets of $V(\mathcal{H}^{(j)})$ which span complete sub-hypergraphs in $\mathcal{H}^{(j)}$. Note that $|\mathcal{K}_i(\mathcal{H}^{(j)})|$ is the number of all copies of $K_i^{(j)}$ in $\mathcal{H}^{(j)}$. Given an $(m, \ell, j-1)$-cylinder $\mathcal{H}^{(j-1)}$ and an (m, ℓ, j)-cylinder $\mathcal{H}^{(j)}$, we say $\mathcal{H}^{(j-1)}$ *underlies* $\mathcal{H}^{(j)}$ if $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$. This brings us to to the (important) notion of a *complex.*

Definition 2.1 (**(m, ℓ, h)-complex**)**.** *Let $m \geq 1$ and $\ell \geq h \geq 1$ be integers. An (m, ℓ, h)*-complex $\boldsymbol{\mathcal{H}}$ *is a collection of (m, ℓ, j)-cylinders $\{\mathcal{H}^{(j)}\}_{j=1}^h$ such that*

(*a*) *$\mathcal{H}^{(1)}$ is an $(m, \ell, 1)$-cylinder, i.e., $\mathcal{H}^{(1)} = V_1 \cup \dots \cup V_\ell$ with $|V_i| = m$ for $i \in [\ell]$, and* (*b*) $\mathcal{H}^{(j-1)}$ *underlies* $\mathcal{H}^{(j)}$ *for* $2 \le j \le h$, *i.e.,* $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$.

Relative density and hypergraph regularity. We begin by defining a relative density of a j-uniform hypergraph w.r.t. a $(j-1)$-uniform hypergraph.

Definition 2.2 (relative density). *Let* $\mathcal{H}^{(j)}$ *be a* j*-uniform hypergraph and let* $\mathcal{H}^{(j-1)}$ *be a* $(j-1)$*-uniform hypergraph. We define the* density of $\mathcal{H}^{(j)}$ w.r.t. $\mathcal{H}^{(j-1)}$ *as*

$$d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)}) = \begin{cases} \frac{|\mathcal{H}^{(j)} \cap \mathcal{K}_j(\mathcal{H}^{(j-1)})|}{|\mathcal{K}_j(\mathcal{H}^{(j-1)})|} & \text{if } \quad \big|\mathcal{K}_j(\mathcal{H}^{(j-1)})\big| > 0 \\ 0 & \text{otherwise}. \end{cases}$$

The following definition provides a notion of regularity for cylinders and for complexes. (In the following definition, and throughout the entire paper, the notation $a \pm b$ represents a quantity within the interval $[a-b, a+b]$.)

Definition 2.3 (**$(\varepsilon, \boldsymbol{d})$-regular**). *Let* $d \ge 0$, *vector* $\boldsymbol{d} = (d_2, \dots, d_h)$ *of non-negative reals and* $\varepsilon > 0$ *be given. We say that*

(1) *an* (m, j, j)*-cylinder* $\mathcal{H}^{(j)}$ *is* (ε, d)-regular *w.r.t. an underlying* $(m, j, j-1)$*-cylinder* $\mathcal{H}^{(j-1)}$ *if whenever* $\mathcal{Q}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$ *satisfies* $\big|\mathcal{K}_j(\mathcal{Q}^{(j-1)})\big| \ge \varepsilon \big|\mathcal{K}_j(\mathcal{H}^{(j-1)})\big|$, *then* $d(\mathcal{H}^{(j)}|\mathcal{Q}^{(j-1)}) = d \pm \varepsilon$;

(2) *an* (m, ℓ, j)*-cylinder* $\mathcal{H}^{(j)}$ *is* (ε, d)-regular *w.r.t. an underlying* $(m, \ell, j-1)$*-cylinder* $\mathcal{H}^{(j-1)}$ *if for every* $\Lambda_j \in [\ell]^j$, $\mathcal{H}^{(j)}[\Lambda_j]$ *is* (ε, d)*-regular w.r.t.* $\mathcal{H}^{(j-1)}[\Lambda_j]$;

(3) *an* (m, ℓ, h)*-complex* $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ *is* $(\varepsilon, \boldsymbol{d})$-regular *if, for every* $j = 2, \dots, h$, $\mathcal{H}^{(j)}$ *is* (ε, d_j)*-regular w.r.t.* $\mathcal{H}^{(j-1)}$.

Partitions. The regularity lemma for k-uniform hypergraphs provides a well-structured family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ of vertices, pairs, ..., and $(k-1)$-tuples of a given vertex set. We now discuss the structure of these partitions recursively, following the approach of [26].

Let k be a fixed integer and V be a set of vertices. Let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{|\mathscr{P}^{(1)}|}\}$ be a partition of V. For every $1 \le j \le |\mathscr{P}^{(1)}|$, let $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ be the family of all crossing j-tuples J, i.e., the set of j-tuples which satisfy $|J \cap V_i| \le 1$ for every $1 \le i \le |\mathscr{P}^{(1)}|$.

Suppose that partitions $\mathscr{P}^{(i)}$ of $\mathrm{Cross}_i(\mathscr{P}^{(1)})$ for $1 \le i \le j-1$ have been defined. Then for every $(j-1)$-tuple I in $\mathrm{Cross}_{j-1}(\mathscr{P}^{(1)})$, there exists a unique class $\mathcal{P}^{(j-1)} = \mathcal{P}^{(j-1)}(I) \in \mathscr{P}^{(j-1)}$ so that $I \in \mathcal{P}^{(j-1)}$. For every j-tuple J in $\mathrm{Cross}_j(\mathscr{P}^{(1)})$, we define the *polyad of* J by $\hat{\mathcal{P}}^{(j-1)}(J) = \bigcup \big\{\mathcal{P}^{(j-1)}(I) \colon I \in [J]^{j-1}\big\}$. In other words, $\hat{\mathcal{P}}^{(j-1)}(J)$ is the unique collection of j partition classes of $\mathscr{P}^{(j-1)}$ each containing a $(j-1)$-subset I of J. ($\hat{\mathcal{P}}^{(j-1)}(J)$ is, in fact, a $(j, j-1)$-cylinder.) We define the family of all polyads $\hat{\mathscr{P}}^{(j-1)} = \big\{\hat{\mathcal{P}}^{(j-1)}(J) \colon J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})\big\}$, which we view as a set (as opposed to a multiset, since $\hat{\mathcal{P}}^{(j-1)}(J)$ and $\hat{\mathcal{P}}^{(j-1)}(J')$ are not necessarily distinct for $J \ne J'$). To simplify notation, we shall often write the elements of $\hat{\mathscr{P}}^{(j-1)}$ as $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ (dropping the argument J), since the families $\mathscr{P}$ with which we work always satisfy $\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \ne \varnothing$.

Now, observe that $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$. The structural requirement on the partition $\mathscr{P}^{(j)}$ of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ is

$$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}, \tag{4}$$

where '$\prec$' denotes the refinement relation of set partitions. In other words, we require that the set of cliques spanned by a polyad in $\hat{\mathscr{P}}^{(j-1)}$ is sub-partitioned in $\mathscr{P}^{(j)}$ and every partition class in $\mathscr{P}^{(j)}$ belongs (corresponds) to precisely one polyad in $\hat{\mathscr{P}}^{(j-1)}$. Note that (4) implies (inductively) that

$$\boldsymbol{\mathcal{P}}(J) = \big\{\hat{\mathcal{P}}^{(i)}(J)\big\}_{i=1}^{j-1}, \text{ where } \hat{\mathcal{P}}^{(i)}(J) = \bigcup \big\{\mathcal{P}^{(i)}(I) \colon I \in [J]^i\big\}, \tag{5}$$

is a $(j, j-1)$-complex (since each $\hat{\mathcal{P}}^{(i)}(J)$ is a (j, i)-cylinder).

In the context of applications, we want to control the number of partition classes from $\mathscr{P}^{(j)}$ contained in $\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})$ for a fixed polyad $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$. The following definition makes this precise.

Definition 2.4 (**family of partitions**). *Suppose V is a set of vertices, $k \geq 2$ is an integer and $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ is a vector of positive integers. We say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ is a* family of partitions on V, *if it satisfies the following:*

(*a*) *$\mathscr{P}^{(1)}$ is a partition of V into a_1 classes,*

(*b*) *$\mathscr{P}^{(j)}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ refining $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ where, for every $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$, $|\{\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}\colon \mathcal{P}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\}| = a_j$.*

Moreover, we say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is t-bounded, if $\max\{a_1, \dots, a_{k-1}\} \leq t$.

Moreover, we want the families $\mathscr{P}$ with which we work to be 'equitable', in the following sense.

Definition 2.5 (**$(\eta, \varepsilon, \boldsymbol{a})$-equitable**). *Suppose V is a set of n vertices, η and ε are positive reals and $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ is a vector of positive integers, where a_1 divides n.*

We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on V is $(\eta, \varepsilon, \boldsymbol{a})$-equitable *if it satisfies the following:*

(*a*) *$\big|[V]^k \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)})\big| \leq \eta\binom{n}{k}$,*

(*b*) *$\mathscr{P}^{(1)} = \{V_i\colon i \in [a_1]\}$ is an equitable vertex partition, i.e., $|V_i| = |V|/a_1$ for $i \in [a_1]$, and*

(*c*) *for every $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$, the $(n/a_1, k, k-1)$-complex $\boldsymbol{\mathcal{P}}(K)$ (see (5)) is $(\varepsilon, (1/a_2, \dots, 1/a_{k-1}))$-regular.*

2.2. **Hypergraph Regularity Lemma and Counting Lemma.** Theorems 2.7 and 2.8 pivot on the following concept, pioneered by Frankl and Rödl [9] for $k = 3$.

Definition 2.6 (**(δ_k, r)-regular**). *Let $\delta_k > 0$ and positive integer r be given. We say that a k-graph $\mathcal{H}^{(k)}$ is*

(1) (δ_k, r)-regular *w.r.t. a given $(k-1)$-graph $\mathcal{H}^{(k-1)}$ if every collection $\boldsymbol{\mathcal{Q}}^{(k-1)} = \{\mathcal{Q}_i^{(k-1)}\}_{i=1}^r$ of sub-hypergraphs of $\mathcal{H}^{(k-1)}$ satisfying $\big|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\big| > \delta_k\big|\mathcal{K}_k(\mathcal{H}^{(k-1)})\big|$ also satisfies*

$$\big|\mathcal{H}^{(k)} \cap \bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\big| = \big(d(\mathcal{H}^{(k)}|\mathcal{H}^{(k-1)}) \pm \delta_k\,\big)\big|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\big|\,,$$

and otherwise, we say $\mathcal{H}^{(k)}$ is (δ_k, r)-irregular *w.r.t. $\mathcal{H}^{(k-1)}$;*

(2) (δ_k, r)-regular *w.r.t. a given family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on $V = V(\mathcal{H}^{(k)})$ if*

$$\big|\bigcup\big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \text{ s.t. } \mathcal{H}^{(k)} \text{ is } (\delta_k, r)\text{-irregular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\big\}\big| \leq \delta_k\tbinom{|V|}{k}\,.$$

The regularity lemma of [24, 25] is given as follows.

Theorem 2.7 (regularity lemma). *Let $k \geq 2$ be a fixed integer. For all positive constants η and δ_k and functions $r\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta\colon \mathbb{N}^{k-1} \to (0, 1]$, there exist integers t and n_0 so that the following holds.*

For every k-uniform hypergraph $\mathcal{H}^{(k)}$ with $|V(\mathcal{H}^{(k)})| = n \geq n_0$, where $t!$ divides n, there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that

(*i*) *$\mathscr{P}$ is $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and t-bounded;*

(*ii*) *$\mathcal{H}^{(k)}$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$.*

The corresponding counting lemma of [24, 25] is given as follows.

Theorem 2.8 (counting lemma). *For all integers $\ell \geq k \geq 2$ and positive constants $\gamma > 0$ and $d_k > 0$, there exists $\delta_k > 0$ such that for all integers $a_{k-1}, \dots, a_2$, there exist $\delta > 0$ and positive integers r and m_0 so that the following holds.*

Suppose

(*i*) *$\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\delta, (1/a_2, \dots, 1/a_{k-1}))$-regular $(m, \ell, k-1)$-complex with $m \geq m_0$.*

(*ii*) *$\mathcal{H}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ is a k-graph which is, for every $\Lambda_k \in [\ell]^k$, (δ_k, r)-regular w.r.t. $\mathcal{R}^{(k-1)}[\Lambda_k]$ with density $d(\mathcal{H}^{(k)}|\mathcal{R}^{(k-1)}[\Lambda_k]) \geq d_k$.*

Then

$$\big|\mathcal{K}_\ell(\mathcal{H}^{(k)})\big| \geq (1-\gamma)d_k^{\binom{\ell}{k}} \prod_{j=2}^{k-1} \big(\tfrac{1}{a_j}\big)^{\binom{\ell}{j}} \times m^\ell\,.$$

3. Proof of Theorem 1.1

Our proof breaks into three parts, the first of which sets up and outlines our argument. The remaining two sections fill in technical details.

3.1. **Setup and Outline of Argument.** It suffices to prove Theorem 1.1 for n divisible by a fixed but arbitary integer T. In particular, suppose that for each $\nu > 0$, fixed integer T and integer $m > m_0(k, \nu, T)$, we have $\log_2 |\mathrm{Forb}_{\mathrm{ind}}(mT, \mathbf{F}^{(k)})| \leq \mathrm{ex}_{\mathrm{ind}}(mT, \mathbf{F}^{(k)}) + \nu(mT)^k$. Then for a given integer $n > n_0(k, \nu, T)$ not divisible by T, set $m = \lceil n/T \rceil$ so that

$$\begin{aligned}\log_2 |\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})| &\leq \log_2 |\mathrm{Forb}_{\mathrm{ind}}(mT, \mathbf{F}^{(k)})| \leq \mathrm{ex}_{\mathrm{ind}}(mT, \mathbf{F}^{(k)}) + \nu(mT)^k \\ &= \widetilde{\mathrm{ex}}_{\mathrm{ind}}(mT, \mathbf{F}^{(k)})\tbinom{mT}{k} + \nu(mT)^k \overset{(3)}{\leq} \widetilde{\mathrm{ex}}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})\tbinom{n+T}{k} + \nu(n+T)^k \\ &= \widetilde{\mathrm{ex}}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})\tbinom{n}{k} + \nu n^k + O(n^{k-1}) = \mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + \nu n^k + O(n^{k-1})\,.\end{aligned}$$

We now prove that for every $\nu > 0$, there exist integers $T = T(\nu)$ and $n_0 = n_0(\nu, T)$ so that for every $n \geq n_0$ divisible by T,

$$\log_2 |\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})| \leq \mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + \nu n^k\,. \tag{6}$$

As our proof depends on Theorems 2.7 and 2.8, we first discuss a sequence of auxiliary constants.

Constants. Let $\nu > 0$ be given. Let integer $b_0 \geq k$ be large enough so that (cf. (3))

$$\widetilde{\mathrm{ex}}_{\mathrm{ind}}(b_0, \mathbf{F}^{(k)}) < \pi_{\mathrm{ind}}(\mathbf{F}^{(k)}) + \tfrac{\nu}{11}. \tag{7}$$

With integer b_0 fixed above, choose $0 < \eta = d_0 < 1/b_0$ so that

$$8d_0 \log_2 \tfrac{\mathrm{e}}{d_0} \leq \tfrac{\nu}{4}. \tag{8}$$

For fixed integers $\ell = b_0 \geq k$ and fixed constants $\gamma = 1/2$ and $d_k = d_0$, let $\delta_k^{(2.8)} = \delta_k^{(2.8)}(b_0, k, 1/2, d_0)$ be the constant guaranteed by Theorem 2.8. Set

$$\delta_k = \min\left\{\delta_k^{(2.8)}, \left(\tfrac{\nu}{200}\right)^5, \left(1 - \tfrac{k}{b_0}\right)^{4k}, b_0^{-20k}\right\}. \tag{9}$$

For fixed integers $\ell = b_0 \geq k$, fixed constants $\gamma = 1/2$, $d_k = d_0$ and δ_k, and for positive integer variables $y_{k-1}, \dots, y_2$, let

$$\delta = \delta^{(2.8)}(b_0, k, 1/2, d_0, \delta_k, y_{k-1}, \dots, y_2) \quad \text{and} \quad r = r^{(2.8)}(b_0, k, 1/2, d_0, \delta_k, y_{k-1}, \dots, y_2) \tag{10}$$

be the functions guaranteed by Theorem 2.8.

We now define further constants in terms of the Regularity Lemma, Theorem 2.7. With input constants $\eta = d_0$ and δ_k and functions δ and r, all defined above, Theorem 2.7 guarantees integer constants

$$t = t^{(2.7)}(\eta, \delta_k, \delta, r) \quad \text{and} \quad n_0 = n_0^{(2.7)}(\eta, \delta_k, \delta, r).$$

The constant T advertised in (6) is set to be $T = t!$.

Now, for $n > n_0$ divisible by T and sufficiently large (wherever needed), we verify (6).

Proof of (6)**.** According to Theorem 2.7, every k-graph $\mathcal{G}^{(k)}$ on n vertices (n defined above) admits an $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable t-bounded family of partitions $\mathscr{P}$ with respect to which $\mathcal{G}^{(k)}$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular. As such, with each $\mathcal{G}^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$, we may associate a family of partitions $\mathscr{P}_{\mathcal{G}^{(k)}}$. (If $\mathcal{G}^{(k)}$ admits multiple such partitions, we arbitrarily choose one of them.) Accordingly, we may impose an equivalence relation $\sim$ on $\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$ according to the following rule: for $\mathcal{G}_1^{(k)}, \mathcal{G}_2^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$,

$$\mathcal{G}_1^{(k)} \sim \mathcal{G}_2^{(k)} \quad \Longleftrightarrow \quad \mathscr{P}_{\mathcal{G}_1^{(k)}} = \mathscr{P}_{\mathcal{G}_2^{(k)}}\,.$$

Let $\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) = \Pi_1 \cup \dots \cup \Pi_N$ be the partition of $\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$ induced by $\sim$. To prove (6), we first seek to bound the parameter $N = N(n)$.

Clearly, N is at most the number of t-bounded families of partitions on the vertex set $[n]$. For a fixed vector $\boldsymbol{a} = (a_1, \dots, a_{k-1})$, there are at most $\prod_{j=1}^{k-1} a_j^{n^j}$ families of partitions $\mathscr{P}(k-1, \boldsymbol{a})$ on the vertex set $[n]$. Consequently,

$$N \le \sum_{\boldsymbol{a}} \Big\{ \prod_{j=1}^{k-1} a_j^{n^j} : 1 \le a_j \le t \text{ for } j = 1, \dots, k-1 \Big\} = t^{O(n^{k-1})} = 2^{O(n^{k-1})} . \tag{11}$$

We now seek to bound $|\Pi_\alpha|$ for every $\alpha = 1, \dots, N$. Fix $1 \le \alpha \le N$ and, correspondingly, family of partitions $\mathscr{P}_\alpha = \{\mathscr{P}_\alpha^{(1)}, \dots, \mathscr{P}_\alpha^{(k-1)}\}$, i.e., the family associated to every $\mathcal{G}^{(k)} \in \Pi_\alpha$. With each $\mathcal{G}^{(k)} \in \Pi_\alpha$, associate the vector

$$\boldsymbol{x}_{\mathcal{G}^{(k)}} = \big(x_{\hat{\mathcal{P}}^{(k-1)}} : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_\alpha^{(k-1)} \big) \in \{0,1\}^{|\hat{\mathscr{P}}_\alpha^{(k-1)}|} ,$$

where, for fixed $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_\alpha^{(k-1)}$,

$$x_{\hat{\mathcal{P}}^{(k-1)}} = \begin{cases} 1 & \text{if } \mathcal{G}^{(k)} \text{ is } (\delta_k, r(\boldsymbol{a}^{\mathscr{P}_\alpha}))\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)} \text{ and } d(\mathcal{G}^{(k)} | \hat{\mathcal{P}}^{(k-1)}) \in [d_0, 1-d_0], \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Then $|\{\boldsymbol{x}_{\mathcal{G}^{(k)}} : \mathcal{G}^{(k)} \in \Pi_\alpha\}| \le 2^{|\hat{\mathscr{P}}_\alpha^{(k-1)}|}$, where the t-boundedness of $\mathscr{P}_\alpha$ gives

$$\big|\hat{\mathscr{P}}_\alpha^{(k-1)}\big| = \binom{a_1}{k} \prod_{j=2}^{k-1} a_j^{\binom{k}{j}} \le t^{2^k} = O(1). \tag{13}$$

With $\alpha \in [N]$ still fixed, now fix vector $\boldsymbol{x} \in \{0,1\}^{|\hat{\mathscr{P}}_\alpha^{(k-1)}|}$ and define $\Pi_{\alpha,\boldsymbol{x}} = \{\mathcal{G}^{(k)} \in \Pi_\alpha : \boldsymbol{x}_{\mathcal{G}^{(k)}} = \boldsymbol{x}\}$. We will prove the following lemma.

Lemma 3.1. $\log_2 |\Pi_{\alpha,\boldsymbol{x}}| \le \mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + \frac{\nu}{2} n^k$.

Now, Lemma 3.1, combined with (11) and (13), implies (6). Indeed

$$\begin{aligned} |\mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})| = \sum_{\alpha=1}^{N} |\Pi_\alpha| &= \sum_{\alpha=1}^{N} \sum \big\{ |\Pi_{\alpha,\boldsymbol{x}}| : \boldsymbol{x} \in \{0,1\}^{|\hat{\mathscr{P}}_\alpha^{(k-1)}|} \big\} \\ &\le 2^{O(n^{k-1})} \times O(1) \times 2^{\mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + \frac{\nu}{2} n^k} \le 2^{\mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + \nu n^k} , \end{aligned}$$

where the last inequality holds for sufficiently large n. It remains to prove Lemma 3.1.

3.2. **Proof of Lemma 3.1.** Fix $\alpha \in [N]$ and, correspondingly, $\mathscr{P}_\alpha = \mathscr{P}_\alpha(k-1, \boldsymbol{a}^{\mathscr{P}_\alpha})$ with $\boldsymbol{a}^{\mathscr{P}_\alpha} = (a_1, \dots, a_{k-1})$. Fix $\boldsymbol{x} = (x_{\hat{\mathcal{P}}^{(k-1)}} : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_\alpha^{(k-1)})$. Define $\mathcal{O}_{\alpha,\boldsymbol{x}}^{(k)}$ to be the set of k-tuples $K \in \mathrm{Cross}_k(\mathscr{P}_\alpha^{(1)})$ for which every $\mathcal{G}^{(k)} \in \Pi_{\alpha,\boldsymbol{x}}$ is 'regular' w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$ and 'moderately dense':

$$\mathcal{O}_{\alpha,\boldsymbol{x}}^{(k)} = \bigcup \Big\{ \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_\alpha^{(k-1)} \text{ satisfies } x_{\hat{\mathcal{P}}^{(k-1)}} = 1 \text{ (cf. 12))} \Big\} . \tag{14}$$

(The $\mathcal{O}$-notation signifies to us that these are the k-tuples whose polyads earn a 'One' from $\boldsymbol{x}$.) Note that every k-tuple $K \in \binom{[n]}{k} \setminus \mathcal{O}_{\alpha,\boldsymbol{x}}^{(k)}$ satisfies that either

(I) $K \notin \mathrm{Cross}_k(\mathscr{P}_\alpha^{(1)})$, or
(II) $x_{\hat{\mathcal{P}}^{(k-1)}(K)} = 0$, which means (cf. (12)) that for *every* $\mathcal{G}^{(k)} \in \Pi_{\alpha,\boldsymbol{x}}$:
  (a) $\mathcal{G}^{(k)}$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}_\alpha}))$-irregular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$, or
  (b) $d(\mathcal{G}^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) < d_0$, or
  (c) $d(\mathcal{G}^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) > 1 - d_0$.

Note that conditions (b) and (c) are exclusive (since $d_0 \leq 1/2$) but neither (a) and (b) nor (a) and (c) are necessarily. As such, we set

$$\begin{aligned}
\hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},\mathrm{irr}} &= \Big\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\alpha} : \ \forall \mathcal{G}^{(k)} \in \Pi_{\alpha,\boldsymbol{x}}, \ \mathcal{G}^{(k)} \text{ is } (\delta_k, r(\boldsymbol{a}^{\mathscr{P}_\alpha}))\text{-irregular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\Big\}, \\
\hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},-} &= \Big\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\alpha} : \ \forall \mathcal{G}^{(k)} \in \Pi_{\alpha,\boldsymbol{x}}, \ d(\mathcal{G}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) < d_0\Big\} \setminus \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},\mathrm{irr}}, \\
\hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},+} &= \Big\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\alpha} : \ \forall \mathcal{G}^{(k)} \in \Pi_{\alpha,\boldsymbol{x}}, \ d(\mathcal{G}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) > 1-d_0\Big\} \setminus \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},\mathrm{irr}}.
\end{aligned} \tag{15}$$

Note that, as defined, $\hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},\mathrm{irr}}$, $\hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},-}$ and $\hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},+}$ are pairwise disjoint.

Observe that every $\mathcal{G}^{(k)} \in \Pi_{\alpha,\boldsymbol{x}}$ can be written as

$$\mathcal{G}^{(k)} = \mathcal{G}^{(k)}_{\alpha,\boldsymbol{x}} \cup \mathcal{G}^{(k)}_{\alpha,\text{non-cross}} \cup \mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},\mathrm{irr}} \cup \mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},-} \cup \mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},+} \tag{16}$$

where $\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x}} = \mathcal{G}^{(k)} \cap \mathcal{O}^{(k)}_{\alpha,\boldsymbol{x}}$, $\mathcal{G}^{(k)}_{\alpha,\text{non-cross}} = \mathcal{G}^{(k)} \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)}_\alpha)$, and for each $* \in \{\mathrm{irr}, -, +\}$,

$$\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},*} = \mathcal{G}_k \cap \bigcup \Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},*}\Big\}.$$

To bound the number of $\mathcal{G}^{(k)} \in \Pi_{\alpha,\boldsymbol{x}}$, we estimate the total number of k-graphs of each of the five forms in the union in (16), and multiply.

Estimations for the middle three forms are easy. Indeed, every $\mathcal{G}^{(k)} \in \Pi_{\alpha,\boldsymbol{x}}$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. the $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}_\alpha}), \boldsymbol{a}^{\mathscr{P}_\alpha})$-equitable partition $\mathscr{P}_\alpha$. Hence, all possible k-graphs of the form $\mathcal{G}^{(k)}_{\alpha,\text{non-cross}}$, $\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},\mathrm{irr}}$ and $\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},-}$ have respective sizes

$$|\mathcal{G}^{(k)}_{\alpha,\text{non-cross}}| \leq \eta\tbinom{n}{k}, \quad |\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},\mathrm{irr}}| \leq \delta_k\tbinom{n}{k}, \quad |\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},-}| \leq d_0\tbinom{n}{k}.$$

Therefore, with $\eta, \delta_k, d_0 < 1/2$ and n sufficiently large, there are, respectively, at most

$$\sum_{i=0}^{\eta\binom{n}{k}} \binom{\binom{n}{k}}{i} \leq 2^{2\eta n^k \log_2 \frac{\mathrm{e}}{\eta}}, \qquad \sum_{i=0}^{\delta_k\binom{n}{k}} \binom{\binom{n}{k}}{i} \leq 2^{2\delta_k n^k \log_2 \frac{\mathrm{e}}{\delta_k}}, \qquad \sum_{i=0}^{d_0\binom{n}{k}} \binom{\binom{n}{k}}{i} \leq 2^{2d_0 n^k \log_2 \frac{\mathrm{e}}{d_0}}, \tag{17}$$

k-graphs of the form $\mathcal{G}^{(k)}_{\alpha,\text{non-cross}}$, $\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},\mathrm{irr}}$ and $\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},-}$.

Bounding the number of k-graphs of the form $\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},+}$ is similar to the work above. Such a k-graph $\mathcal{G}^{(k)}_+$ must satisfy $\mathcal{G}^{(k)}_+ \subseteq \bigcup\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},+}\}$, where for each $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},+}$, $d(\mathcal{G}^{(k)}_+|\hat{\mathcal{P}}^{(k-1)}) > 1-d_0$. Setting

$$\sigma = \big|\bigcup\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},+}\}\big| = \sum\{\big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big| : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},+}\},$$

we see that $\mathcal{G}^{(k)}_+$ is one of the sub-hypergraphs of $\bigcup\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},+}\}$ with size $(1-d_0)\sigma \leq |\mathcal{G}^{(k)}_+| \leq \sigma$. The number of such sub-hypergraphs is

$$\sum_{i=(1-d_0)\sigma}^{\sigma} \binom{\sigma}{i} \leq 2^{2d_0\sigma \log_2 \frac{\mathrm{e}}{d_0}} \leq 2^{2d_0 n^k \log_2 \frac{\mathrm{e}}{d_0}}, \tag{18}$$

which bounds the number of k-graphs of the form $\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x},+}$.

To bound the number of k-graphs of the form $\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x}}$, we require the following proposition, the proof of which we give momentarily.

Proposition 3.2. $\big|\mathcal{O}^{(k)}_{\alpha,\boldsymbol{x}}\big| \leq \mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + \frac{\nu}{4}n^k$.

Proposition 3.2 implies there are at most

$$2^{|\mathcal{O}^{(k)}_{\alpha,\boldsymbol{x}}|} \leq 2^{\mathrm{ex}_{\mathrm{ind}}(n,\mathbf{F}^{(k)}) + \frac{\nu}{4}n^k} \tag{19}$$

k-graphs of the form $\mathcal{G}^{(k)}_{\alpha,\boldsymbol{x}}$.

To conclude the proof of Lemma 3.1, we combine (16)–(19) and use (8) and[3] $\delta_k \le d_0 = \eta$ from (9) to conclude

$$\log_2 |\Pi_{\alpha,\boldsymbol{x}}| \le \mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + \big(2\eta \log_2 \tfrac{\mathrm{e}}{\eta} + 2\delta_k \log_2 \tfrac{\mathrm{e}}{\delta_k} + 4d_0 \log_2 \tfrac{\mathrm{e}}{d_0} + \tfrac{\nu}{4}\big) n^k \le \mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + \tfrac{\nu}{2} n^k,$$

as promised. It remains to prove Proposition 3.2.

3.3. **Proof of Proposition 3.2.** In this section, we use the following notation. With α and $\boldsymbol{x}$ fixed, now fix an arbitrary crossing set $A \in \mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)})$. For $* \in \{\mathrm{irr}, -, +\}$, define auxiliary k-graphs (cf. (12) and (15))

$$\tbinom{A}{k}_{\alpha,\boldsymbol{x}} = \big\{K \in \tbinom{A}{k} \colon\, x_{\hat{\mathcal{P}}^{(k-1)}(K)} = 1\big\}, \quad \tbinom{A}{k}^{*}_{\alpha,\boldsymbol{x}} = \big\{K \in \tbinom{A}{k} : \hat{\mathcal{P}}^{(k-1)}(K) \in \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},*}\big\},$$

and observe

$$\tbinom{A}{k} = \tbinom{A}{k}_{\alpha,\boldsymbol{x}} \cup \tbinom{A}{k}^{\mathrm{irr}}_{\alpha,\boldsymbol{x}} \cup \tbinom{A}{k}^{-}_{\alpha,\boldsymbol{x}} \cup \tbinom{A}{k}^{+}_{\alpha,\boldsymbol{x}} \tag{20}$$

is a partition. Define

$$\mathrm{Cross}^{\mathrm{irr}}_{a_1}(\mathscr{P}_\alpha^{(1)}) = \big\{A \in \mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)}) : |\tbinom{A}{k}^{\mathrm{irr}}_{\alpha,\boldsymbol{x}}| \ge \delta_k^{1/4}\tbinom{a_1}{k}\big\} \tag{21}$$

and $\mathrm{Cross}^{\mathrm{reg}}_{a_1}(\mathscr{P}_\alpha^{(1)}) = \mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)}) \setminus \mathrm{Cross}^{\mathrm{irr}}_{a_1}(\mathscr{P}_\alpha^{(1)})$. The following two facts will imply Proposition 3.2.

Fact 3.3. $|\mathrm{Cross}^{\mathrm{irr}}_{a_1}(\mathscr{P}_\alpha^{(1)})| < \delta_k^{1/2}(\frac{n}{a_1})^{a_1}$.

Fact 3.4. $\max\{|\binom{A}{k}_{\alpha,\boldsymbol{x}}| : A \in \mathrm{Cross}^{\mathrm{reg}}_{a_1}(\mathscr{P}_\alpha^{(1)})\} < (\widetilde{\mathrm{ex}}_{\mathrm{ind}}(a_1, \mathbf{F}^{(k)}) + \frac{\nu}{9})\binom{a_1}{k}$.

We defer the proofs of Facts 3.3 and 3.4 in favor of first showing how they imply Proposition 3.2.

Recall that Proposition 3.2 seeks to bound $|\mathcal{O}^{(k)}_{\alpha,\boldsymbol{x}}|$ (cf. (14)). To that end, we count (in two ways) pairs (A, K), where $A \in \mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)})$ and $K \in \binom{A}{k}_{\alpha,\boldsymbol{x}}$, and use Facts 3.3 and 3.4 to infer

$$\begin{aligned}\big|\mathcal{O}^{(k)}_{\alpha,\boldsymbol{x}}\big|\big(\tfrac{n}{a_1}\big)^{a_1-k} &= \sum \big\{\big|\tbinom{A}{k}_{\alpha,\boldsymbol{x}}\big| :\, A \in \mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)})\big\} = \sum_{*\in\{\mathrm{irr},\mathrm{reg}\}} \sum \big\{\big|\tbinom{A}{k}_{\alpha,\boldsymbol{x}}\big| :\, A \in \mathrm{Cross}^{*}_{a_1}(\mathscr{P}_\alpha^{(1)})\big\}\\ &\le \big|\mathrm{Cross}^{\mathrm{irr}}_{a_1}(\mathscr{P}_\alpha^{(1)})\big|\tbinom{a_1}{k} + \big|\mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)})\big| \max\big\{\big|\tbinom{A}{k}_{\alpha,\boldsymbol{x}}\big| : A \in \mathrm{Cross}^{\mathrm{reg}}_{a_1}(\mathscr{P}_\alpha^{(1)})\big\}\\ &\le \delta_k^{1/2}\big(\tfrac{n}{a_1}\big)^{a_1}\tbinom{a_1}{k} + \big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(a_1, \mathbf{F}^{(k)}) + \tfrac{\nu}{9}\big)\tbinom{a_1}{k}\big(\tfrac{n}{a_1}\big)^{a_1}\end{aligned}$$

so that (using (9))

$$\big|\mathcal{O}^{(k)}_{\alpha,\boldsymbol{x}}\big| \le \big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(a_1, \mathbf{F}^{(k)}) + \tfrac{\nu}{9} + \delta_k^{1/2}\big)\tfrac{n^k}{k!} \le \big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(a_1, \mathbf{F}^{(k)}) + \tfrac{\nu}{8} + o(1)\big)\tbinom{n}{k} \le \big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(a_1, \mathbf{F}^{(k)}) + \tfrac{\nu}{7}\big)\tbinom{n}{k}.$$

Since[4] $a_1 \ge b_0$ and the sequence $(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(s, \mathbf{F}^{(k)}))_{s=1}^{\infty}$ is non-increasing with limit $\pi_{\mathrm{ind}}(\mathbf{F}^{(k)})$ (cf. (3)), we have (using (7))

$$\big|\mathcal{O}^{(k)}_{\alpha,\boldsymbol{x}}\big| \le \big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(b_0, \mathbf{F}^{(k)}) + \tfrac{\nu}{7}\big)\tbinom{n}{k} < \big(\pi_{\mathrm{ind}}(\mathbf{F}^{(k)}) + \tfrac{\nu}{11} + \tfrac{\nu}{7}\big)\tbinom{n}{k} < \big(\pi_{\mathrm{ind}}(\mathbf{F}^{(k)}) + \tfrac{\nu}{4}\big)\tbinom{n}{k} \le \big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + \tfrac{\nu}{4}\big)\tbinom{n}{k},$$

so that $|\mathcal{O}^{(k)}_{\alpha,\boldsymbol{x}}| \le \mathrm{ex}_{\mathrm{ind}}(n, \mathbf{F}^{(k)}) + \frac{\nu}{4}n^k$ follows, proving Proposition 3.2.

We now prove Facts 3.3 and 3.4.

Proof of Fact 3.3. To bound $|\mathrm{Cross}^{\mathrm{irr}}_{a_1}(\mathscr{P}_\alpha^{(1)})|$, we count (in two ways) the number of pairs (A, K), where $A \in \mathrm{Cross}^{\mathrm{irr}}_{a_1}(\mathscr{P}_\alpha)$ and $K \in \binom{A}{k}^{\mathrm{irr}}_{\alpha,\boldsymbol{x}}$, to infer (cf. (21))

$$\begin{aligned}\big|\mathrm{Cross}^{\mathrm{irr}}_{a_1}(\mathscr{P}_\alpha)\big| \times \delta_k^{1/4}\tbinom{a_1}{k} &\le \big|\bigcup\big\{K : \hat{\mathcal{P}}^{(k-1)}(K) \in \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},\mathrm{irr}}\big\}\big| \times \big(\tfrac{n}{a_1}\big)^{a_1-k}\\ &= \big|\bigcup\big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\alpha,\boldsymbol{x},\mathrm{irr}}\big\}\big| \times \big(\tfrac{n}{a_1}\big)^{a_1-k}.\end{aligned}$$

[3] We also use the fact that the function $x\log_2 \frac{\mathrm{e}}{x}$ is increasing on $(0, 1]$.

[4] Indeed, bound the fraction $|\mathrm{Cross}_k(\mathscr{P}_\alpha^{(1)})|\binom{n}{k}^{-1}$. Since $|\mathrm{Cross}_k(\mathscr{P}_\alpha^{(1)})| = \binom{a_1}{k}(\frac{n}{a_1})^k$, this fraction equals $(n/a_1)^k(a_1)_k/(n)_k$, and with n sufficiently large, this fraction is at most $(1 - a_1^{-1})^{k-1}$. Since $\mathscr{P}_\alpha$ is an $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}_\alpha}), \boldsymbol{a}^{\mathscr{P}_\alpha})$-equitable family of partitions, this fraction is at least $1 - \eta$. Now $a_1 \ge 1/\eta \ge b_0$ follows from (8).

Since every $\mathcal{G}^{(k)} \in \Pi_{\alpha,\boldsymbol{x}}$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}_\alpha}))$-regular w.r.t. the family of partitions $\mathscr{P}_\alpha$, the union above cannot be larger than $\delta_k\binom{n}{k}$ (see Definition 2.6). We therefore infer

$$\left|\mathrm{Cross}^{\mathrm{irr}}_{a_1}(\mathscr{P}_\alpha)\right| \le \delta_k^{3/4}\tbinom{n}{k}\big(\tfrac{n}{a_1}\big)^{a_1-k}\tbinom{a_1}{k}^{-1} \le \delta_k^{3/4}\big(\tfrac{n}{a_1}\big)^{a_1}\big(1-\tfrac{k-1}{a_1}\big)^{1-k} < \delta_k^{1/2}\big(\tfrac{n}{a_1}\big)^{a_1},$$

where the last inequality follows from $a_1 \ge b_0$ (see the last footnote) and our choice of δ_k in (9). □

Proof of Fact 3.4. Assume, on the contrary, that there exists an a_1-element set $A_0 \in \mathrm{Cross}^{\mathrm{reg}}_{a_1}(\mathscr{P}^{(1)}_\alpha)$ for which

$$\big|\tbinom{A_0}{k}_{\alpha,\boldsymbol{x}}\big| \ge \big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(a_1,\mathbf{F}^{(k)}) + \tfrac{\nu}{9}\big)\tbinom{a_1}{k}. \tag{22}$$

This assumption implies the following.

Claim 3.5. *There exists a b_0-element set $B_0 \in \binom{A_0}{b_0}$ for which the induced sub-hypergraphs $\binom{B_0}{k}_{\alpha,\boldsymbol{x}} = \binom{A_0}{k}_{\alpha,\boldsymbol{x}} \cap \binom{B_0}{k}$ and $\binom{B_0}{k}^{\mathrm{irr}}_{\alpha,\boldsymbol{x}} = \binom{A_0}{k}^{\mathrm{irr}}_{\alpha,\boldsymbol{x}} \cap \binom{B_0}{k}$ satisfy $|\binom{B_0}{k}_{\alpha,\boldsymbol{x}}| \ge \mathrm{ex}_{\mathrm{ind}}(b_0,\mathbf{F}^{(k)})+1$ and $\binom{B_0}{k}^{\mathrm{irr}}_{\alpha,\boldsymbol{x}} = \varnothing$.*

We defer the proof of Claim 3.5 momentarily in favor of first completing the proof of Fact 3.4.

Fix the b_0-element set $B_0 \in \binom{A_0}{b_0}$ from Claim 3.5. For $* \in \{+,-\}$, set $\binom{B_0}{k}^*_{\alpha,\boldsymbol{x}} = \binom{A_0}{k}^*_{\alpha,\boldsymbol{x}} \cap \binom{B_0}{k}$ so that (20) and $\binom{B_0}{k}^{\mathrm{irr}}_{\alpha,\boldsymbol{x}} = \varnothing$ from Claim 3.5 imply that

$$\tbinom{B_0}{k} = \tbinom{B_0}{k}_{\alpha,\boldsymbol{x}} \cup \tbinom{B_0}{k}^+_{\alpha,\boldsymbol{x}} \cup \tbinom{B_0}{k}^-_{\alpha,\boldsymbol{x}} \tag{23}$$

is a partition. Since (by Claim 3.5) the k-graph $\binom{B_0}{k}_{\alpha,\boldsymbol{x}}$ (on b_0 vertices) has at least $\mathrm{ex}_{\mathrm{ind}}(b_0,\mathbf{F}^{(k)})+1$ many edges, it must be the case that there exist a sub-hypergraph $\binom{B_0}{k}'_{\alpha,\boldsymbol{x}} \subseteq \binom{B_0}{k}_{\alpha,\boldsymbol{x}}$ and an $\mathcal{F}^{(k)}_0 \in \mathbf{F}^{(k)}$ so that $\mathcal{F}^{(k)}_0$ appears as an induced sub-hypergraph of $\binom{B_0}{k}'_{\alpha,\boldsymbol{x}} \cup \binom{B_0}{k}^+_{\alpha,\boldsymbol{x}}$. Indeed, recall the definition in (1) and set $\mathcal{H}^{(k)} = \binom{B_0}{k}_{\alpha,\boldsymbol{x}}$ and take $\mathcal{M}^{(k)} = \binom{B_0}{k}^+_{\alpha,\boldsymbol{x}}$ (which are disjoint by (23)). Since $|\mathcal{H}^{(k)}| > \mathrm{ex}_{\mathrm{ind}}(b_0,\mathbf{F}^{(k)})$, there must exist (for our choice of $\mathcal{M}^{(k)}$) a subhypergraph $\mathcal{H}^{(k)}_0 \subseteq \mathcal{H}^{(k)}$ (which we've denoted in this context by $\binom{B_0}{k}'_{\alpha,\boldsymbol{x}}$) for which $\mathcal{H}^{(k)}_0 \cup \mathcal{M}^{(k)} \notin \mathrm{Forb}_{\mathrm{ind}}(b_0,\mathbf{F}^{(k)})$. This means that there exists a hypergraph $\mathcal{F}^{(k)}_0 \in \mathbf{F}^{(k)}$ which appears as an induced subhypergraph of $\mathcal{H}^{(k)}_0 \cup \mathcal{M}^{(k)} = \binom{B_0}{k}'_{\alpha,\boldsymbol{x}} \cup \binom{B_0}{k}^+_{\alpha,\boldsymbol{x}}$, as asserted. We shall now show that this same $\mathcal{F}^{(k)}_0$ appears as an induced sub-hypergraph of every $\mathcal{G}^{(k)} \in \Pi_{\alpha,\boldsymbol{x}} \subseteq \mathrm{Forb}_{\mathrm{ind}}(n,\mathbf{F}^{(k)})$. This contradiction shows that the assumption in (22) is wrong, and therefore proves Fact 3.4.

Indeed, fix $\mathcal{F}^{(k)}_0 \in \mathbf{F}^{(k)}$ as above and fix any $\mathcal{G}^{(k)}_0 \in \Pi_{\alpha,\boldsymbol{x}}$. For convenience, we also write $\mathcal{F}^{(k)}_0$ as the induced copy appearing in $\binom{B_0}{k}'_{\alpha,\boldsymbol{x}} \cup \binom{B_0}{k}^+_{\alpha,\boldsymbol{x}}$. We write $F_0 = V(\mathcal{F}^{(k)}_0) \subseteq B_0$ as the vertex set of this copy and set $f_0 = |F_0|$. For notational simplicity, we shall assume, w.l.o.g., that $F_0 = [f_0] \subset [n]$ and that, moreover,

$$1 \in V_1,\ 2 \in V_2,\ \ldots,\ f_0 \in V_{f_0}. \tag{24}$$

(Recall $\hat{\mathscr{P}}^{(1)}_\alpha = \{V_1,\ldots,V_{a_1}\}$, i.e., $[n] = V_1 \cup \cdots \cup V_{a_1}$.) We write $\widetilde{\mathcal{F}}^{(k)}_0 = \binom{F_0}{k} \setminus \mathcal{F}^{(k)}_0$ as the complement of $\mathcal{F}^{(k)}_0$. Then

$$\mathcal{F}^{(k)}_0 \subseteq \tbinom{B_0}{k}'_{\alpha,\boldsymbol{x}} \cup \tbinom{B_0}{k}^+_{\alpha,\boldsymbol{x}} \subseteq \tbinom{B_0}{k}_{\alpha,\boldsymbol{x}} \cup \tbinom{B_0}{k}^+_{\alpha,\boldsymbol{x}}, \quad \text{and since } \mathcal{F}^{(k)}_0 \text{ is } \textit{induced} \text{ (cf. (23))},$$

$$\widetilde{\mathcal{F}}^{(k)}_0 \subseteq \Big(\tbinom{B_0}{k}_{\alpha,\boldsymbol{x}} \setminus \tbinom{B_0}{k}'_{\alpha,\boldsymbol{x}}\Big) \cup \tbinom{B_0}{k}^-_{\alpha,\boldsymbol{x}} \subseteq \tbinom{B_0}{k}_{\alpha,\boldsymbol{x}} \cup \tbinom{B_0}{k}^-_{\alpha,\boldsymbol{x}}. \tag{25}$$

We now arrive at a juncture of the proof. For $K \in \binom{F_0}{k}$, define

$$\mathcal{H}^{(k)}_K = \begin{cases} \mathcal{G}^{(k)}_0 \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K)) & \text{if} \quad K \in \mathcal{F}^{(k)}_0, \\ \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K)) \setminus \mathcal{G}^{(k)}_0 & \text{if} \quad K \in \widetilde{\mathcal{F}}^{(k)}_0, \end{cases} \qquad \text{and} \qquad \mathcal{H}^{(k)} = \bigcup\big\{\mathcal{H}^{(k)}_K :\ K \in \tbinom{F_0}{k}\big\}. \tag{26}$$

It follows from this construction that every element of $\mathcal{K}_{f_0}(\mathcal{H}^{(k)})$ corresponds to an induced copy of $\mathcal{F}_0^{(k)}$ appearing in $\mathcal{G}_0^{(k)}$. Indeed, fix a copy $K_{f_0}^{(k)}$ in $\mathcal{H}^{(k)}$, and write its vertices as $1' \in V_1, \dots, f_0' \in V_{f_0}$. (Recall we took $F_0 = V(\mathcal{F}_0^{(k)}) = \{1, \dots, f_0\}$, where $1 \in V_1, \dots, f_0 \in V_{f_0}$.) Every element $K' \in \binom{\{1', \dots, f_0'\}}{k}$ appears in $\mathcal{H}^{(k)}$. When K' corresponds to a k-tuple $K \in \mathcal{F}_0^{(k)} \subseteq \binom{[f_0]}{k}$ (by removing the primes from the vertex labels), then K' appears in $\mathcal{G}_0^{(k)}$ (by (26)). Otherwise, when K' corresponds to a k-tuple $K \in \widetilde{\mathcal{F}}_0^{(k)} = \binom{[f_0]}{k} \setminus \mathcal{F}_0^{(k)}$, then K' does not appear in $\mathcal{G}_0^{(k)}$ (again, by (26)). We will therefore show that $|\mathcal{K}_{f_0}(\mathcal{H}^{(k)})| > 0$ to produce the contradiction to $\mathcal{G}_0^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(n, \mathbf{F}^{(k)})$ that we promised earlier. To prove $|\mathcal{K}_{f_0}(\mathcal{H}^{(k)})| > 0$, we appeal to the Counting Lemma, Theorem 2.8.

Define $(n/a_1, f_0, k-1)$-complex $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ by setting, for each $j = 1, \dots, k-1$,

$$\mathcal{R}^{(j)} = \bigcup \left\{ \mathcal{P}^{(j)}(J) \colon\ J \in \tbinom{F_0}{j} \right\}. \tag{27}$$

We apply Theorem 2.8 to the k-graph $\mathcal{H}^{(k)}$ and complex $\boldsymbol{\mathcal{R}}$, but first check that the hypotheses of Theorem 2.8 are met:

(*i*) $\boldsymbol{\mathcal{R}}$ is a $(\delta(\boldsymbol{a}^{\mathscr{P}}), (1/a_2, \dots, 1/a_{k-1}))$-regular $(n/a_1, f_0, k-1)$-complex automatically, since $\mathscr{P}_\alpha$ is an $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable family of partitions. Note that the function δ is chosen appropriately in (10) for an application of Theorem 2.8.

(*ii*) $\mathcal{H}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ follows from (26) and (27). We claim that, for each $K \in \binom{F_0}{k}$, $\mathcal{H}^{(k)}$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathcal{R}^{(k-1)}[K] = \hat{\mathcal{P}}^{(k-1)}(K)$ (cf. (24)) with $d(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) \geq d_0$. Indeed, we first recall that $\mathcal{G}_0^{(k)}$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$ (since $\binom{B_0}{k}_{\alpha, \boldsymbol{x}}^{\mathrm{irr}} = \varnothing$), and now consider two cases.

Case 1 ($K \in \mathcal{F}_0^{(k)}$). Here, $\mathcal{H}_K^{(k)} = \mathcal{G}_0^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K))$ (cf. (26)) is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$. If $K \in \binom{B_0}{k}_{\alpha, \boldsymbol{x}}$ (cf. (25)), then $d(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) \geq d_0$, and if $K \in \binom{B_0}{k}_{\alpha, \boldsymbol{x}}^{+}$, then $d(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) \geq 1 - d_0 \geq d_0$.

Case 2 ($K \in \widetilde{\mathcal{F}}_0^{(k)}$). Here, $\mathcal{H}_K^{(k)} = \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K)) \setminus \mathcal{G}_0^{(k)}$ (cf. (26)) is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$ because its 'complement' $\mathcal{G}_0^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K))$ is. If $K \in \binom{B_0}{k}_{\alpha, \boldsymbol{x}}$ (cf. (25)), then $d(\mathcal{G}_0^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) \leq 1 - d_0$, and if $K \in \binom{B_0}{k}_{\alpha, \boldsymbol{x}}^{-}$, then $d(\mathcal{G}_0^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) \leq d_0 \leq 1 - d_0$. Either way, $d(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) = 1 - d(\mathcal{G}_0^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) \geq d_0$.

Note that the constant $\delta_k > 0$ and function r were chosen appropriately in (9) and (10) for an application of Theorem 2.8;

It is appropriate to apply the Counting Lemma to $\mathcal{H}^{(k)}$ and $\boldsymbol{\mathcal{R}}$, and so we conclude

$$\left| \mathcal{K}_{f_0}(\mathcal{H}^{(k)}) \right| \geq \tfrac{1}{2} d_0^{\binom{f_0}{k}} \prod_{j=2}^{k-1} \left(\tfrac{1}{a_j} \right)^{\binom{f_0}{j}} \left(\tfrac{n}{a_1} \right)^{f_0} > 0.$$

This concludes our proof of Fact 3.4. $\square$

Proof of Claim 3.5. The proof is a routine application of the first moment method. Fix any a_1-element set $A_0 \in \mathrm{Cross}_{a_1}^{\mathrm{reg}}(\mathscr{P}_\alpha^{(1)})$ satisfying (22) and write $|\binom{A_0}{k}_{a, \boldsymbol{x}}| = c\binom{a_1}{k}$ for some constant $0 \leq c \leq 1$. Our assumption in (22) is that

$$c \geq \widetilde{\mathrm{ex}}_{\mathrm{ind}}(a_1, \mathbf{F}^{(k)}) + \tfrac{\nu}{9}. \tag{28}$$

For $B \in \binom{A_0}{b_0}$ chosen uniformly at random, consider the random variables $X_B = \binom{b_0}{k} - |\binom{B}{k}_{\alpha,\boldsymbol{x}}|$ and $Y_B = |\binom{B}{k}^{\mathrm{irr}}_{\alpha,\boldsymbol{x}}|$. Then

$$\mathbb{E}\, X_B = \left(\binom{a_1}{k} - \left|\binom{A_0}{k}_{\alpha,\boldsymbol{x}}\right|\right) \frac{\binom{a_1-k}{b_0-k}}{\binom{a_1}{b_0}} = (1-c)\binom{b_0}{k} \quad \text{and} \quad \mathbb{E}\, Y_B = |\binom{A_0}{k}^{\mathrm{irr}}_{\alpha,\boldsymbol{x}}| \frac{\binom{a_1-k}{b_0-k}}{\binom{a_1}{b_0}} < \delta_k^{1/4}\binom{b_0}{k} < \delta_k^{1/5},$$

where the inequalities above hold by virtue of $A_0 \in \mathrm{Cross}^{\mathrm{reg}}_{a_1}(\mathscr{P}^{(1)}_\alpha)$ (cf. (21)) and our choice of δ_k in (9). The Markov inequality then yields

$$\mathrm{Prob}\big[X_B \geq \mathbb{E}\, X_B + 2\delta_k^{1/5}\binom{b_0}{k}\big] + \mathrm{Prob}[Y_B \geq 1] \leq \big(1 + 2\delta_k^{1/5}\big)^{-1} + \delta_k^{1/5} < 1,$$

where the last inequality holds by our choice of δ_k in (9). Thus, there exists a set $B_0 \in \binom{A_0}{b_0}$ for which $\binom{B}{k}^{\mathrm{irr}}_{\alpha,\boldsymbol{x}} = \varnothing$ and for which

$$\begin{aligned}\left|\binom{B_0}{k}_{\alpha,\boldsymbol{x}}\right| &\geq \big(c - 2\delta_k^{1/5}\big)\binom{b_0}{k} \overset{(28)}{\geq} \big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(a_1, \mathbf{F}^{(k)}) + \tfrac{\nu}{9} - 2\delta_k^{1/5}\big)\binom{b_0}{k} \overset{(9)}{\geq} \big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(a_1, \mathbf{F}^{(k)}) + \tfrac{\nu}{10}\big)\binom{b_0}{k} \\ &\overset{(3)}{\geq} \big(\pi_{\mathrm{ind}}(\mathbf{F}^{(k)}) + \tfrac{\nu}{10}\big)\binom{b_0}{k} \overset{(7)}{\geq} \big(\widetilde{\mathrm{ex}}_{\mathrm{ind}}(b_0, \mathbf{F}^{(k)}) - \tfrac{\nu}{11} + \tfrac{\nu}{10}\big)\binom{b_0}{k} = \mathrm{ex}_{\mathrm{ind}}(b_0, \mathbf{F}^{(k)}) + \tfrac{\nu}{110}\binom{b_0}{k}.\end{aligned}$$

Since $\left|\binom{B_0}{k}_{\alpha,\boldsymbol{x}}\right|$ and $\mathrm{ex}_{\mathrm{ind}}(b_0, \mathbf{F}^{(k)})$ are integers, we have that $\left|\binom{B_0}{k}_{\alpha,\boldsymbol{x}}\right| \geq \mathrm{ex}_{\mathrm{ind}}(b_0, \mathbf{F}^{(k)}) + 1$, as promised. □

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