BIPARTITE HANSEL RESULTS FOR HYPERGRAPHS

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ABSTRACT. For integers $n \geq k \geq 2$, let V be an n-element set, and let $\binom{V}{k}$ denote the set of all k-element subsets of V. For disjoint $A, B \subseteq V$, we say $\{A, B\}$ covers $K \in \binom{V}{k}$ if $K \subseteq A \dot\cup B$ and K meets each of A and B, i.e., $K \cap A \neq \emptyset \neq K \cap B$. We say that a collection $\mathcal{C}$ of such pairs $\{A, B\}$ covers $\binom{V}{k}$ if every element of $\binom{V}{k}$ is covered by at least one member of $\mathcal{C}$. When $k = 2$, such a family is called a separating system of V, where this concept was introduced by Rényi [17] and studied by many authors.

Let $h(n, k)$ denote the minimum value of $\sum_{\{A,B\} \in \mathcal{C}} (|A| + |B|)$ among all covers $\mathcal{C}$ of $\binom{V}{k}$. Hansel [6] determined the bounds $\lceil n \log_2 n \rceil \leq h(n, 2) \leq n \lceil \log_2 n \rceil$, and Bollobás and Scott [1] determined an exact formula for $h(n, 2)$. We extend these results to give an exact formula for $h(n, k)$, and to guarantee that all optimal covers $\mathcal{C}$ of $\binom{V}{k}$ share a common degree-sequence. Our proofs follow lines of Bollobás and Scott, together with weight-shifting arguments in a similar vein to some of Motzkin and Straus [12].

1. INTRODUCTION

We consider a hypergraph version of a classical result of Hansel [6], and of a more recent result of Bollobás and Scott [1]. For that, fix integers $n \geq k \geq 2$ and an n-element vertex set V, and let $\binom{V}{k}$ denote the set of all k-element subsets of V. For disjoint $A, B \subseteq V$, we say $\{A, B\}$ *covers* $K \in \binom{V}{k}$ if $K \subseteq A \dot\cup B$ and K meets each of A and B, i.e., $K \cap A \neq \emptyset \neq K \cap B$. We say that a collection $\mathcal{C}$ of such pairs *covers* (is a *cover* of) $\binom{V}{k}$ if every element of $\binom{V}{k}$ is covered by at least one member of $\mathcal{C}$. Rényi [17] introduced covers $\mathcal{C}$ of $\binom{V}{2}$ as *separating systems of* V, where every pair $u \neq v \in V$ is *separated* by some $\{A, B\} \in \mathcal{C}$, in the sense that $u \in A$ and $v \in B$ or vice versa. Separating systems were since well-studied (see, e.g., [1, 3–11, 13–20]), and the following particular results motivate some of our current work.

Rényi [17] observed that $\lceil \log_2 n \rceil$ members are necessary and can suffice for $\mathcal{C}$ to be a separating system of V. For necessity, the chromatic number of the union $K_V = \bigcup_{\{A,B\} \in \mathcal{C}} K[A, B]$ satisfies

$$n = \chi(K_V) = \chi\Big(\bigcup_{\{A,B\} \in \mathcal{C}} K[A, B] \Big) \leq \prod_{\{A,B\} \in \mathcal{C}} \chi(K[A, B]) = 2^{|\mathcal{C}|}. \tag{1}$$

For sufficiency, set $m = \lceil \log_2 n \rceil$ and let $v \mapsto \boldsymbol{v}$ be any injection from V to $\{0, 1\}^m$. For each $1 \leq i \leq m$, set $A_i = \{v \in V : \boldsymbol{v}(i) = 0\}$ and $B_i = \{v \in V : \boldsymbol{v}(i) = 1\}$, where $\boldsymbol{v}(i)$ denotes the i^{th} coordinate of $\boldsymbol{v}$. Then $\mathcal{C} = \{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\}$ is a separating system of V since, for each $u \neq v \in V$, the vectors $\boldsymbol{u} \neq \boldsymbol{v}$ disagree on a coordinate $1 \leq i \leq m$, whereby $\{A_i, B_i\}$ separates u and v.

Hansel [6] considered a weighted version of Rényi's result above, where we prepare a definition for $k \geq 2$. For a cover $\mathcal{C}$ of $\binom{V}{k}$, define the *weight* $\omega(\mathcal{C})$ of $\mathcal{C}$ by $\omega(\mathcal{C}) = \sum_{\{A,B\} \in \mathcal{C}} (|A| + |B|)$, and set $h(n, k)$ to be the minimum weight $\omega(\mathcal{C})$ among all covers $\mathcal{C}$ of $\binom{V}{k}$. Hansel established the following bounds.

Theorem 1.1 (Hansel (1964), [6])**.** *For all integers $n \geq 2$, it holds that $\lceil n \log_2 n \rceil \leq h(n, 2) \leq n \lceil \log_2 n \rceil$.*

Independently, Krichevskii [10] proved a result similar to Theorem 1.1, and Katona and Szemerédi [9] rediscovered Theorem 1.1 in the context of a diameter problem in graph theory. Bollobás and Scott [1] improved Theorem 1.1 to the following exact formula for $h(n, 2)$.

Theorem 1.2 (Bollobás and Scott (2007), [1])**.** *For an integer $n \geq 2$, set $p = \lfloor \log_2 n \rfloor$ and $R = n - 2^p$. Then, $h(n, 2) = np + 2R$.*

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Extending Theorem 1.1 to (sharp) bounds for $h(n,k)$ is not difficult (see [4]). Extending Theorem 1.2 to a formula for $h(n,k)$ seems less straightforward, and this is our current focus.

Theorem 1.3. *For integers $n \geq k \geq 2$, set $q = \lfloor n/(k-1) \rfloor$, $r = n - q(k-1)$, $p = \lfloor \log_2 q \rfloor$, and $R = q - 2^p$. Then,*

$$h(n,k) = np + 2R(k-1) + \lceil \tfrac{r}{k-1} \rceil (r+k-1). \tag{2}$$

Our proof of Theorem 1.3 gives slightly more information on *optimal* covers $\mathcal{C}$ of $\binom{V}{k}$, i.e., those with $\omega(\mathcal{C}) = h(n,k)$. For that, we define the $\mathcal{C}$*-degree* $\deg_{\mathcal{C}}(v)$ of $v \in V$ as the number of $\{A,B\} \in \mathcal{C}$ to which v is *incident*, i.e., $v \in A \dot\cup B$. Arranging these degrees in non-increasing order, we define $\boldsymbol{d}(\mathcal{C}) = (\deg_{\mathcal{C}}(v))_{v\in V}$ to be the *degree-sequence* of $\mathcal{C}$. We prove that all optimal covers $\mathcal{C}$ of $\binom{V}{k}$ share common degree-sequence $\boldsymbol{D} = \boldsymbol{D}_{n,k} \in \{p, p+1\}^V$ with j^{th} coordinate, $1 \leq j \leq n$, given by

$$\boldsymbol{D}(j) = p+1 \qquad \Longleftrightarrow \qquad 1 \leq j \leq 2R(k-1) + \lceil \tfrac{r}{k-1} \rceil (r+k-1). \tag{3}$$

Theorem 1.4. *Let $\mathcal{C}$ be an optimal cover of $\binom{V}{k}$. Then, $\boldsymbol{d}(\mathcal{C}) = \boldsymbol{D}$ is given by (3).*

We now discuss our proofs of Theorems 1.3 and 1.4. To begin, the integers k, n, p, q, r, and R from the hypothesis of Theorem 1.3 are henceforth referenced by

$$\tfrac{n-r}{k-1} = q = 2^p + R, \qquad \text{where } 0 \leq r < k-1, \qquad \text{and where } 0 \leq R < 2^p, \tag{4}$$

and the n-element set V is always fixed. To prove Theorem 1.3, we proceed along the following steps, not all of which are difficult. In Section 2, we give a straightforward extension of Rényi's construction (from earlier in the Introduction) to establish the formula in (2) as an upper bound on $h(n,k)$.

Proposition 1.5 (the upper bound)**.** *Let integers k, n, p, q, r, and R satisfy (4), and let V be an n-element set. There exists a cover $\mathcal{C}_0$ of $\binom{V}{k}$ with weight*

$$\omega(\mathcal{C}_0) = np + 2R(k-1) + \lceil \tfrac{r}{k-1} \rceil (r+k-1).$$

For the lower bound on $h(n,k)$, we split the formula in (2) into two cases, depending on whether or not $r = 0$. In Section 3, we follow an elegant approach of Bollobás and Scott [1] for Theorem 1.2 to prove the following lower bound on $h(n,k)$.

Theorem 1.6 (a lower bound)**.** *Let integers k, n, p, q, r, and R satisfy (4). Then, $h(n,k) \geq np + 2R(k-1) + 2r$, whereby Theorem 1.3 holds when $r = 0$. Moreover, Theorem 1.4 holds when $r = 0$.*

The bound in Theorem 1.6 is sharp if, and only if, $r = 0$. The majority of this paper is devoted to improving the bound of Theorem 1.6 for $r \geq 1$, which we ultimately complete in Section 5.

Theorem 1.7 (the lower bound when $r \geq 1$)**.** *Let integers k, n, p, q, $r \geq 1$, and R satisfy (4). Then, $h(n,k) \geq np + 2R(k-1) + r + k - 1$, whereby Theorem 1.3 holds. Moreover, Theorem 1.4 holds.*

Our proof of Theorem 1.7 follows lines from the proof of Theorem 1.6. We also use structural results on optimal covers given in upcoming Lemmas 4.4 and 4.6. These tools depend on weight-shifting arguments not unlike some of Motzkin and Straus [12] (see also [2]). Lemmas 4.4 and 4.6 may be of independent interest, but we were unable to avoid their use here.

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2. Proof of Proposition 1.5

Fix integers k, n, p, q, r, and R satisfying (4). We extend the approach of Rényi of mapping vertices $v \mapsto \boldsymbol{v}$ to vectors. Fix any partition $\Pi : V = X_1 \dot\cup \ldots \dot\cup X_{2^p}$ into 2^p many classes, where $|X_i| = 2(k-1)$ if $1 \leq i \leq R$, $|X_i| = r + k - 1$ if $i = R+1$, and $|X_i| = k-1$ if $R+2 \leq i \leq 2^p$. Since

$$\big(2(k-1) \times R\big) + r + k - 1 + \big((k-1) \times (2^p - (R+1))\big) \overset{(4)}{=} n,$$

such a partition Π exists. Fix an arbitrary bijection $X_i \mapsto \boldsymbol{x}_i$ from Π to $\{0,1\}^p$, and define $v \mapsto \boldsymbol{v}$ by setting $\boldsymbol{v} = \boldsymbol{x}_i$ if, and only if, $v \in X_i$. We next use $v \mapsto \boldsymbol{v}$ to define the promised collection $\mathcal{C}_0$.

For each $1 \leq j \leq p$, set $A_j = \{v \in V : \boldsymbol{v}(j) = 0\}$ and $B_j = \{v \in V : \boldsymbol{v}(j) = 1\}$. Then $|A_j| + |B_j| = n$. For each $1 \leq i \leq R+1$, let X_i be subdivided as $X_i = Y_i \dot\cup Z_i$, where $|Y_i| = k-1$. Set

$$A_{p+1} = \begin{cases} Y_1 \dot\cup \ldots \dot\cup Y_R & \text{when } R \neq 0, \\ \emptyset & \text{when } R = 0, \end{cases} \quad \text{and} \quad B_{p+1} = \begin{cases} Z_1 \dot\cup \ldots \dot\cup Z_R & \text{when } R \neq 0, \\ \emptyset & \text{when } R = 0. \end{cases}$$

Then $|A_{p+1}| + |B_{p+1}| = 2R(k-1)$. Set

$$A_{p+2} = \begin{cases} Y_{R+1} & \text{when } r \neq 0, \\ \emptyset & \text{when } r = 0, \end{cases} \quad \text{and} \quad B_{p+2} = \begin{cases} Z_{R+1} & \text{when } r \neq 0, \\ \emptyset & \text{when } r = 0. \end{cases}$$

Then, $|A_{p+2}| + |B_{p+2}| = \lceil r/(k-1) \rceil (r+k-1)$. Define $\mathcal{C}_0 = \{\{A_1, B_1\}, \ldots, \{A_{p+2}, B_{p+2}\}\}$, where

$$\omega(\mathcal{C}_0) = \sum_{j=1}^{p+2} (|A_j| + |B_j|) = np + 2R(k-1) + \left\lceil \tfrac{r}{k-1} \right\rceil (r+k-1).$$

To see that $\mathcal{C}_0$ covers $\binom{V}{k}$, fix $K \in \binom{V}{k}$ and consider two cases. First, assume K meets distinct X_h and X_i from Π. Fix $u \in K \cap X_h$ and $v \in K \cap X_i$, whereby $\boldsymbol{u} \neq \boldsymbol{v}$ disagree on some coordinate $1 \leq j \leq p$. Then $\{A_j, B_j\} \in \mathcal{C}_0$ separates u and v, and since $A_j \dot\cup B_j = V \supseteq K$, the same $\{A_j, B_j\} \in \mathcal{C}_0$ covers K. Second, assume $K \subseteq X_i$ for some $1 \leq i \leq 2^p$. Then $1 \leq i \leq R+1$, since otherwise $|X_i| = k-1$ is too small. Then $K \subseteq X_i = Y_i \dot\cup Z_i$, where $|Y_i| = k-1$ and $|Z_i| \in \{r, k-1\}$ are each too small for either $Y_i \supseteq K$ or $Z_i \supseteq K$. Thus, $\{Y_i, Z_i\}$ covers K, and hence so do one of $\{A_{p+1}, B_{p+1}\}, \{A_{p+2}, B_{p+2}\} \in \mathcal{C}_0$.

Remark 2.1. In the context of Proposition 1.5, the cover $\mathcal{C}_0$ is not unique. For example, when $R \geq 2$ replace $\{A_{p+1}, B_{p+1}\}$ with $\{Y_1, Z_1\}, \ldots, \{Y_R, Z_R\}$.

3. Proof of Theorem 1.6

In Section 3.1, we prove the former conclusion of Theorem 1.6, that $h(n,k) \geq np + 2R(k-1) + 2r$. In Section 3.2, we isolate some details of this proof that we wish to apply later in this paper. In Section 3.3, we prove the latter conclusion of Theorem 1.6, that Theorem 1.4 holds when $r = 0$. Throughout this section, integers k, n, p, q, r, and R satisfy (4), and V is a fixed n-element set.

3.1. **Former conclusion of Theorem 1.6.** We follow an elegant approach of Bollobás and Scott [1]. Fix an arbitrary cover $\mathcal{C}$ of $\binom{V}{k}$, and for simplicity of notation in this argument, write $d_v = \deg_{\mathcal{C}}(v)$ for the $\mathcal{C}$-degree of $v \in V$, and write $\boldsymbol{d} = \boldsymbol{d}(\mathcal{C})$ for the degree-sequence of $\mathcal{C}$. Standard double counting gives

$$\sum_{v \in V} d_v = \sum_{\{A,B\} \in \mathcal{C}} (|A| + |B|) = \omega(\mathcal{C}), \qquad \text{and} \qquad \alpha = \alpha(\mathcal{C}) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{v \in V} d_v = \frac{\omega(\mathcal{C})}{n} \tag{5}$$

denotes the *average degree in* $\mathcal{C}$. For sake of argument, we assume that

$$\alpha < p + 1, \tag{6}$$

since otherwise we would have

$$\begin{aligned} \omega(\mathcal{C}) \stackrel{(5)}{=} \alpha n \stackrel{\neg(6)}{\geq} (p+1)n = np + n \stackrel{(4)}{=} np + (2^p + R)(k-1) + r \\ \stackrel{(4)}{\geq} np + (2R+1)(k-1) + r = np + 2R(k-1) + r + k - 1, \end{aligned} \tag{7}$$

which already exceeds $np + 2R(k-1) + 2r$ on account of $r < k-1$ in (4).

The following ideas have roots in [1, 9, 13]: independently for each $\{A, B\} \in \mathcal{C}$, set

$$Z_{\{A,B\}} = \begin{cases} V \setminus A & \text{with probability } 1/2, \\ V \setminus B & \text{with probability } 1/2. \end{cases} \tag{8}$$

Set $Z = \bigcap_{\{A,B\} \in \mathcal{C}} Z_{\{A,B\}}$, which is a random subset of V whose expected size we now analyze. On the one hand, $\mathcal{C}$ covers $\binom{V}{k}$, so no k-tuple $K \in \binom{V}{k}$ can forever survive (8), i.e., belong to Z. Consequently,

$|Z| \leq k-1$ and thus $\mathbb{E}[|Z|] \leq k-1$. On the other hand, linearity of expectation gives $\mathbb{E}[|Z|] = \sum_{v \in V} \mathbb{P}[v \in Z]$, where the event $v \in Z$ holds if, and only if, the independent events $v \in Z_{\{A,B\}}$ (cf. (8)) hold for each of the d_v many elements $\{A, B\} \in \mathcal{C}$ to which v is incident. Thus,

$$\mathbb{E}[|Z|] = \sum_{v \in V} \left(\frac{1}{2}\right)^{d_v} \leq k-1. \tag{9}$$

Applying the Arithmetic-Geometric Mean Inequality to (9) yields

$$\frac{k-1}{n} \geq \frac{1}{n} \sum_{v \in V} \left(\frac{1}{2}\right)^{d_v} \geq \left(2^{-\sum_{v \in V} d_v}\right)^{1/n} \overset{(5)}{=} 2^{-\alpha} \qquad \Longrightarrow \qquad \alpha \geq \log_2 \left(\frac{n}{k-1}\right) \overset{(4)}{\geq} p. \tag{10}$$

We continue with a key idea of Bollobás and Scott [1]: in (9), replace $\boldsymbol{d} = (d_v)_{v \in V}$ with a positive integer sequence $\boldsymbol{e} = (e_v)_{v \in V}$ satisfying the following properties:

(a) $\sum_{v \in V} e_v = \sum_{v \in V} d_v$;
(b) $\sum_{v \in V} \left(\frac{1}{2}\right)^{e_v} \leq \sum_{v \in V} \left(\frac{1}{2}\right)^{d_v}$;
(c) $|e_w - e_x| \leq 1$ for all $w, x \in V$.

To construct $\boldsymbol{e} = (e_v)_{v \in V}$, fix $w, x \in V$. An easy calculation reveals that

$$d_x \geq d_w + 1 \qquad \Longleftrightarrow \qquad \left(\frac{1}{2}\right)^{d_x} + \left(\frac{1}{2}\right)^{d_w} \geq \left(\frac{1}{2}\right)^{d_x - 1} + \left(\frac{1}{2}\right)^{d_w + 1}, \tag{11}$$

where equality holds in one iff equality holds in both. Now, if $d_x \geq d_w + 2$, we replace d_x in $\boldsymbol{d}$ with $d'_x = d_x - 1$, and we replace d_w in $\boldsymbol{d}$ with $d'_w = d_w + 1$. The resulting sequence $\boldsymbol{d'}$ clearly satisfies Property (a), and by (11) it also satisfies Property (b). Iterating such replacements on $\boldsymbol{d'}$ eventually yields a sequence $\boldsymbol{e}$ which also satisfies Property (c).

We claim $\boldsymbol{e}$ assumes only the values p and $p+1$. Indeed, Property (c) guarantees $\boldsymbol{e}$ assumes at most two values e and $e+1$ (and e when $\boldsymbol{e}$ is constant). Property (a) gives

$$\frac{1}{n} \sum_{v \in V} e_v = \frac{1}{n} \sum_{v \in V} d_v \overset{(5)}{=} \alpha, \tag{12}$$

so $e = \lfloor (1/n) \sum_{v \in V} e_v \rfloor = \lfloor \alpha \rfloor$. Since (6) and (10) give $p \leq \alpha < p+1$,

$$e = \lfloor \alpha \rfloor = p. \tag{13}$$

We now conclude the proof of Theorem 1.6. Set $V^- = \{v \in V : e_v = p\}$ and $V^+ = \{v \in V : e_v = p+1\}$. Property ($b$) and (9) yield

$$|V^-| \left(\frac{1}{2}\right)^p + |V^+| \left(\frac{1}{2}\right)^{p+1} \leq \sum_{v \in V} \left(\frac{1}{2}\right)^{d_v} \leq k-1 \qquad \Longrightarrow \qquad 2|V^-| + |V^+| \leq 2^{p+1}(k-1), \tag{14}$$

or equivalently (using $|V^-| = n - |V^+|$)

$$|V^+| \geq 2n - 2^{p+1}(k-1) = 2\left(n - 2^p(k-1)\right) \overset{(4)}{=} 2\left(R(k-1) + r\right) = 2R(k-1) + 2r. \tag{15}$$

Thus, by (5) and Property (a), we conclude with

$$\omega(\mathcal{C}) = \sum_{v \in V} d_v = \sum_{v \in V} e_v = p|V^-| + (p+1)|V^+| = np + |V^+| \overset{(15)}{\geq} np + 2R(k-1) + 2r. \tag{16}$$

3.2. **Notes.** We have now proven both Proposition 1.5 and the former conclusion of Theorem 1.6. These combine to say (with k, n, p, q, r, and R satisfying (4)) that

$$np + 2R(k-1) + 2r \leq h(n,k) \leq np + 2R(k-1) + \left\lceil \tfrac{r}{k-1} \right\rceil (r + k - 1). \tag{17}$$

Recall that $\mathcal{C}$ in Section 3.1 was an arbitrary cover of $\binom{V}{k}$. Below, we revisit (6), (7), and (9) when $\mathcal{C}$ is assumed to be optimal (but where k, n, p, q, r, and R remain fixed by (4)).

Fact 3.1. *Let $\mathcal{C}$ optimally cover $\binom{V}{k}$. Then the average degree α of $\mathcal{C}$ (cf. (5)) satisfies $p \le \alpha \le p+1$. Moreover, $\alpha = p \iff r = R = 0$, and $\alpha = p+1 \iff R = 2^p - 1$ and $h(n,k) = np + 2R(k-1) + r + k - 1$.*

Proof of Fact 3.1. We shall often use the identities $\alpha n = \omega(\mathcal{C}) = h(n,k)$, which hold by (5) and the optimality of $\mathcal{C}$. We showed $\alpha \ge p$ in (10), and $\alpha = p \iff r = R = 0$ is immediate from (17). If $\alpha > p+1$, then (7) would contradict (17). Similarly, if $\alpha = p+1$, then (7) would have equality throughout, which gives $R = 2^p - 1$ and $h(n,k) = \omega(\mathcal{C}) = np + 2R(k-1) + r + k - 1$. Finally, if $R = 2^p - 1$ and $h(n,k) = \omega(\mathcal{C}) = np + 2R(k-1) + r + k - 1$, then

$$\alpha n = h(n,k) = np + 2R(k-1) + r + k - 1 \quad \Longrightarrow \quad \frac{(\alpha - p)n - r}{k-1} = 2R + 1 = 2^p + R \overset{(4)}{=} \frac{n-r}{k-1},$$

from which $\alpha = p+1$ follows. $\square$

Fact 3.2. *Let $\mathcal{C}$ optimally cover $\binom{V}{k}$. Then,*

$$\sum_{v \in V} \left(\frac{1}{2}\right)^{d_v} \ge r\left(\frac{1}{2}\right)^{p+1} + (k-1)\left(1 - \left(\frac{1}{2}\right)^{p+1}\right).$$

Proof of Fact 3.2. We separate the cases of $\alpha \le p+1$ (cf. Fact 3.1). If $\alpha = p+1$, then Fact 3.1 says $R = 2^p - 1$, and so in (4) we have $q = (n-r)/(k-1) = 2^{p+1} - 1$. Thus (10) yields

$$\sum_{v \in V} \left(\frac{1}{2}\right)^{d_v} \ge n\left(\frac{1}{2}\right)^{p+1} = (k-1)q\left(\frac{1}{2}\right)^{p+1} + r\left(\frac{1}{2}\right)^{p+1} = (k-1)\left(1 - \left(\frac{1}{2}\right)^{p+1}\right) + r\left(\frac{1}{2}\right)^{p+1},$$

as desired. Henceforth, we assume $p \le \alpha < p+1$ (cf. Fact 3.1), but we suppose

$$\sum_{v \in V} \left(\frac{1}{2}\right)^{d_v} < r\left(\frac{1}{2}\right)^{p+1} + (k-1)\left(1 - \left(\frac{1}{2}\right)^{p+1}\right). \tag{18}$$

Construct $\boldsymbol{e} = (e_v)_{v \in V}$ precisely as in (11) so that $\boldsymbol{e}$ assumes at most two values, which are still p and $p+1$ (by Fact 3.1, (12), and (13)). By Property (b) and (18), we infer

$$|V^-|\left(\frac{1}{2}\right)^{p} + |V^+|\left(\frac{1}{2}\right)^{p+1} \le \sum_{v \in V}\left(\frac{1}{2}\right)^{d_v} < r\left(\frac{1}{2}\right)^{p+1} + (k-1)\left(1 - \left(\frac{1}{2}\right)^{p+1}\right),$$

or equivalently (cf. (15)), $|V^+| > 2R(k-1) + k - 1 + r$. By Property (a) (cf. (16)),

$$h(n,k) = \omega(\mathcal{C}) = p|V^-| + (p+1)|V^+| = np + |V^+| > np + 2R(k-1) + k - 1 + r,$$

which contradicts (17). $\square$

3.3. **Latter conclusion of Theorem 1.6.** The proof is similar to that of Fact 3.2. Assume now that $r = 0$, but that k, n, p, q, and R are otherwise fixed by (4). We use that $h(n,k) = np + 2R(k-1)$, which follows from (17). Now, let $\mathcal{C}$ optimally cover $\binom{V}{k}$ with degree-sequence $\boldsymbol{d} = \boldsymbol{d}(\mathcal{C}) = (d_v)_{v \in V}$, but assume for contradiction that $\boldsymbol{d} \ne \boldsymbol{D}$ (cf. (3)). Using Fact 3.1, $\mathcal{C}$ has average degree $p \le \alpha < p+1$, where $\alpha = p+1$ is forbidden by $h(n,k) = np + 2R(k-1)$. We again construct $\boldsymbol{e} = (e_v)_{v \in V}$ precisely as in (11), and observe that (once appropriately ordered) $\boldsymbol{e} = \boldsymbol{D}$. Indeed, revisiting (16),

$$np + 2R(k-1) = h(n,k) = \omega(\mathcal{C}) = np + |V^+|,$$

so that $\boldsymbol{e} \in \{p, p+1\}^V$ has precisely $2R(k-1)$ many $(p+1)$-digits. Since $\boldsymbol{d} \ne \boldsymbol{D} = \boldsymbol{e}$, there must exist $x, w \in V$ with $d_x \ge d_w + 2$. As such, strict inequality holds throughout (11), and so strict inequality holds throughout (14)–(16). Now, $\omega(\mathcal{C}) > np + 2R(k-1) = h(n,k)$, contradicting the optimality of $\mathcal{C}$.

4. Tools for Proving Theorem 1.7

Our proof of Theorem 1.7 follows lines from Section 3, where the set Z from (8) continues to play a critical role. Here, we will work with specialized optimal covers $\mathcal{C}$ which admit structural information on these sets Z. This structure is described formally in upcoming Lemmas 4.4 and 4.6, but we first describe it informally in (i) and (ii) below. For that, we again consider the context of Section 3, where Z is from (8), and where we fix an arbitrary $\{A, B\} \in \mathcal{C}$. Suppose Z meets $A \dot\cup B$, i.e., $Z \cap (A \dot\cup B) \neq \emptyset$. It is not possible for Z to meet both A and B, but it is possible for $A \dot\cup B$ to miss some of Z. We prefer to avoid the concurrence $Z \cap (A \dot\cup B) \neq \emptyset \neq Z \setminus (A \dot\cup B)$, and therefore seek optimal covers $\mathcal{C}$ whose *every* outcome Z from (8) satisfies

(i) for each $\{A, B\} \in \mathcal{C}$, either $Z \subseteq A$, or $Z \subseteq B$, or $Z \cap (A \dot\cup B) = \emptyset$.

In this context, $\deg_{\mathcal{C}}(Z)$ is well-defined, but we wish to say more. We want for $\mathcal{C}$ to also satisfy that

(ii) whenever $\deg_{\mathcal{C}}(Z) < \alpha = \alpha(\mathcal{C})$ is below the average, then $|Z| = k - 1$.

In Section 4.1, we restate (i) and (ii) formally in order to prove that such optimal covers exist, and to apply them to the context of Theorem 1.7. In Section 4.2, we give a few other related but easy facts.

4.1. **Formalizing *(i)* and *(ii)*.** To develop (i), we fix a cover $\mathcal{C}$ of $\binom{V}{k}$, and describe the sample space of (8) as follows. Fix symbols a and b, and let $\{a, b\}^{\mathcal{C}}$ denote the set of all functions $\psi : \mathcal{C} \to \{a, b\}$. For $\psi \in \{a, b\}^{\mathcal{C}}$ and $\{A, B\} \in \mathcal{C}$, define

$$Z^{\psi}_{\{A,B\}} = \begin{cases} V \setminus A & \text{if } \psi(\{A, B\}) = a, \\ V \setminus B & \text{if } \psi(\{A, B\}) = b, \end{cases} \qquad \text{and define} \qquad Z_{\psi} = \bigcap_{\{A,B\} \in \mathcal{C}} Z^{\psi}_{\{A,B\}}. \tag{19}$$

Then (8) arises when $\psi \in \{a, b\}^{\mathcal{C}}$ is chosen uniformly at random. Below, we consider all such instances.

Definition 4.1 (surviving sets)**.** Let V, $\mathcal{C}$, and $\{a, b\}^{\mathcal{C}}$ be given as above. For $\psi \in \{a, b\}^{\mathcal{C}}$, we call Z_{ψ} in (19) the *surviving set* (w.r.t. ψ) of $\mathcal{C}$. We call $\mathcal{Z} = \mathcal{Z}(\mathcal{C}) = \{Z_{\psi} : \psi \in \{a, b\}^{\mathcal{C}}\}$ the *surviving family* of $\mathcal{C}$. Since $\emptyset \in \mathcal{Z}$ is possible, we write $\mathcal{Z}^* = \mathcal{Z} \setminus \{\emptyset\}$ for the non-empty surviving sets of $\mathcal{C}$.

The following remark suggests some relevance of properties (i) and (ii) and Definition 4.1.

Remark 4.2. In the context of (8), we may use the notation from Definition 4.1 to rewrite (9) as

$$\mathbb{E}[|Z|] = 2^{-|\mathcal{C}|} \sum_{\psi \in \{a,b\}^{\mathcal{C}}} |Z_{\psi}| = \sum_{Z_{\Psi} \in \mathcal{Z}} \big(|Z_{\Psi}| \cdot \mathbb{P}[Z = Z_{\Psi}] \big), \tag{20}$$

where for each $Z_{\Psi} \in \mathcal{Z}$, the quantity $2^{|\mathcal{C}|}\mathbb{P}[Z = Z_{\Psi}]$ counts the number of functions $\psi \in \{a, b\}^{\mathcal{C}}$ for which $Z_{\psi} = Z_{\Psi}$. Thus, if $\mathcal{C}$ satisfies property (i) above, we may further infer

$$\mathbb{E}[|Z|] = \sum_{Z_{\Psi} \in \mathcal{Z}} \big(|Z_{\Psi}| \cdot 2^{-\deg_{\mathcal{C}}(Z_{\Psi})} \big), \tag{21}$$

because for each $Z_{\Psi} \in \mathcal{Z}$, precisely $2^{|\mathcal{C}| - \deg_{\mathcal{C}}(Z_{\Psi})}$ many functions $\psi \in \{a, b\}^{\mathcal{C}}$ satisfy $Z_{\psi} = Z_{\Psi}$. Note that (ii) adds that terms $Z_{\Psi} \in \mathcal{Z}$ of degree below average each contribute $|Z_{\Psi}| = k - 1$ to (21). □

To continue developing (i), we define the equivalence relation $\sim_{\mathcal{C}}$ on V by setting $u \sim_{\mathcal{C}} v$ if, and only if, for each $\{A, B\} \in \mathcal{C}$,

$$u, v \in A, \qquad \text{or} \qquad u, v \in B, \qquad \text{or} \qquad \{u, v\} \cap (A \dot\cup B) = \emptyset. \tag{22}$$

We use the following terminology and notation for the equivalence classes S of $\sim_{\mathcal{C}}$.

Definition 4.3 (bones, skeleton)**.** Let V, $\mathcal{C}$, and $\sim_{\mathcal{C}}$ be given as in (22). We call the family $\mathcal{S} = \mathcal{S}(\mathcal{C})$ of equivalence classes of $\sim_{\mathcal{C}}$ the *skeleton* of $\mathcal{C}$. We call the elements $S \in \mathcal{S}$ the *bones* of $\mathcal{C}$.

The following lemma (proven in Section 6) implies (i) in the language of Definitions 4.1 and 4.3.

Lemma 4.4. *For every cover $\mathcal{C}$ of $\binom{V}{k}$, there exists a cover $\hat{\mathcal{C}}$ of $\binom{V}{k}$ so that the following hold:*

(1) *for each $v \in V$, we have $\deg_{\hat{\mathcal{C}}}(v) \leq \deg_{\mathcal{C}}(v)$, and so $\omega(\hat{\mathcal{C}}) \leq \omega(\mathcal{C})$ (cf. (5));*

(2) *the skeleton* $\hat{\mathcal{S}}$ *and surviving family* $\hat{\mathcal{Z}}$ *of* $\hat{\mathcal{C}}$ *satisfy* $\hat{\mathcal{Z}}^* = \hat{\mathcal{S}}$.

In light of Lemma 4.4, we shall call any *optimal* cover $\mathcal{C}$ of $\binom{V}{k}$ with $\mathcal{Z}^*(\mathcal{C}) = \mathcal{S}(\mathcal{C})$ a *strong cover.*

Remark 4.5. Every finite set V admits a strong cover $\mathcal{C}$ of $\binom{V}{k}$. Indeed, applying Lemma 4.4 to an optimal cover $\mathcal{C}$ yields a strong cover $\hat{\mathcal{C}}$ since $\omega(\hat{\mathcal{C}}) \leq \omega(\mathcal{C}) = h(n,k)$ must have equality. In particular, $\hat{\mathcal{C}}$ must satisfy $\deg_{\hat{\mathcal{C}}}(v) = \deg_{\mathcal{C}}(v)$ for each $v \in V$, in which case $\hat{\mathcal{C}}$ and $\mathcal{C}$ are *degree-equivalent.* □

We now restate (ii) in the terminology above (which we prove in Section 9).

Lemma 4.6. *For a finite set* V*, let* $\mathcal{C}$ *be a strong cover of* $\binom{V}{k}$*. Then every bone* $S \in \mathcal{S}$ *with* $\deg_{\mathcal{C}}(S) < \alpha = \alpha(\mathcal{C})$ *below the average has maximum size* $|S| = k-1$.

4.2. **Shifting facts.** To prove Theorem 1.7, we also use the following elementary 'shifting' mechanisms (defined for arbitrary covers $\mathcal{C}$), together with some elementary consequences.

Definition 4.7 (shifting)**.** Let V, $\mathcal{C}$, and $\mathcal{S}$ be given as in Definition 4.3. Fix a bone $S \in \mathcal{S}$, and fix a subset $U \subset V \setminus S$. For $\{A, B\} \in \mathcal{C}$, the following sets are well-defined by (22):

$$A_{U,S} = \begin{cases} A \cup U & \text{if } S \subseteq A, \\ A \setminus U & \text{if } S \cap A = \emptyset, \end{cases} \qquad \text{and} \qquad B_{U,S} = \begin{cases} B \cup U & \text{if } S \subseteq B, \\ B \setminus U & \text{if } S \cap B = \emptyset. \end{cases}$$

Define $\mathcal{C}^*_{U,S} = \{\{A_{U,S}, B_{U,S}\} : \{A,B\} \in \mathcal{C}\}$ and $\mathcal{C}_{U,S} = \big\{\{U,S\}\big\} \cup \mathcal{C}^*_{U,S}$, to be *$S$-shifts of U in $\mathcal{C}$.*

Remark 4.8. It may happen that elements $\{A_{U,S}, B_{U,S}\} \in \mathcal{C}^*_{U,S}$ repeat, making $\mathcal{C}^*_{U,S}$ a multiset, or that $|A_{U,S}|+|B_{U,S}| < k$ (or $|U|+|S| < k$), making such elements ineffective toward covering $\binom{V}{k}$. Nonetheless, we leave $\mathcal{C}^*_{U,S}$ and $\mathcal{C}_{U,S}$ as is, and maintain the obvious and herein pervasively used identities

$$\deg_{\mathcal{C}^*_{U,S}}(U) = \deg_{\mathcal{C}^*_{U,S}}(S) = \deg_{\mathcal{C}}(S) = \deg_{\mathcal{C}_{U,S}}(S) - 1 = \deg_{\mathcal{C}_{U,S}}(U) - 1, \tag{23}$$

and $\deg_{\mathcal{C}^*_{U,S}}(v) = \deg_{\mathcal{C}_{U,S}}(v) = \deg_{\mathcal{C}}(v)$ for all $v \in V \setminus (U \dot\cup S)$. □

Moreover, sometimes we 'shift' when the set U (originally disjoint from S) is, in fact, *disjoint* from V.

Definition 4.9 (immersion)**.** Let V, $\mathcal{C}$, and $\mathcal{S}$ be given as in Definition 4.3. Fix a bone $S \in \mathcal{S}$, and let W be a set which is disjoint from V. For $\{A, B\} \in \mathcal{C}$, define

$$A^{W,S} = \begin{cases} A \cup W & \text{if } S \subseteq A, \\ A & \text{if } S \cap A = \emptyset, \end{cases} \qquad \text{and} \qquad B^{W,S} = \begin{cases} B \cup W & \text{if } S \subseteq B, \\ B & \text{if } S \cap B = \emptyset. \end{cases}$$

Define $\mathcal{C}^{W,S} = \big\{\{W,S\}\big\} \cup \big\{\{A^{W,S}, B^{W,S}\} : \{A,B\} \in \mathcal{C}\big\}$ to be the *S-immersion of W into $\mathcal{C}$.*

The following elementary fact (verified in the Appendix) is easy to prove from the definitions above.

Fact 4.10. *Let* $\mathcal{C}$, $\mathcal{C}_{U,S}$, $\mathcal{C}^*_{U,S}$ *and* $\mathcal{C}^{W,S}$ *be given as in Definitions 4.7 and 4.9. The following hold:*

(a) $\mathcal{C}_{U,S}$ *covers* $\binom{V}{k}$ *whenever* $1 \leq |U| \leq k-1 = |S|$.
(b) *When* $\mathcal{C}_{U,S}$ *covers* $\binom{V}{k}$*, it has weight* $\omega(\mathcal{C}) + |U| + |S| + \sum_{u \in U} \big(\deg_{\mathcal{C}}(S) - \deg_{\mathcal{C}}(u)\big)$.
(c) *When* $\mathcal{C}^*_{U,S}$ *covers* $\binom{V}{k}$*, it has weight* $\omega(\mathcal{C}^*_{U,S}) = \omega(\mathcal{C}_{U,S}) - |U| - |S|$.
(d) $\mathcal{C}^{W,S}$ *covers* $\binom{V \cup W}{k}$ *whenever* $1 \leq |W| \leq k-1 = |S|$*, and has weight* $\omega(\mathcal{C}) + |W|\big(1 + \deg_{\mathcal{C}}(S)\big) + |S|$.

5. Proof of Theorem 1.7

Let integers k, n, p, q, $r \geq 1$, and R satisfy (4), and let V be a fixed n-element set. We prove Theorem 1.7 by induction on $2^p - R \geq 1$, and begin with the base case.

5.1. **base case: $2^p - R = 1$.** In Proposition 5.1 below, we prove all conclusions of Theorem 1.7 simultaneously for $2^p - R = 1$ and $r \geq 1$. For this, all optimal covers $\mathcal{C}$ of $\binom{V}{k}$ have common weight $\omega(\mathcal{C}) = h(n,k)$, and thus common average degree $\alpha = \alpha(n,k) = (1/n)h(n,k)$ (cf. (5)). Fact 3.1 gives that $p \leq \alpha \leq p+1$, and that $\alpha = p+1$ implies $h(n,k) = np + 2R(k-1) + r + k - 1$. Thus, we prove that $\alpha = p+1$ necessarily holds when $R = 2^p - 1$ and $r \geq 1$, and in the following strong form.

Proposition 5.1. *Let k, n, p, q, $r \geq 1$, $R = 2^p - 1$, and V be given as above. Then, all optimal covers $\mathcal{C}$ of $\binom{V}{k}$ are $(p+1)$-regular, i.e., $\deg_{\mathcal{C}}(v) = p+1$ for all $v \in V$.*

Proposition 5.1 gives all conclusions of Theorem 1.7 when $R = 2^p - 1$ and $r \geq 1$. Indeed, the first conclusion $h(n,k) = np + 2R(k-1) + r + k - 1$ is guaranteed (via Fact 3.1) by $\alpha = p+1$. The latter conclusion (on the number of $(p+1)$-digits of $\boldsymbol{d}(\mathcal{C})$) is trivial since (3) gives precisely

$$2R(k-1) + r + k - 1 = (2R+1)(k-1) + r = (2^p + R)(k-1) + r \overset{(4)}{=} n$$

many $(p+1)$-digits of $\boldsymbol{D} \in \{p, p+1\}^V$ (when $R = 2^p - 1$ and $r \geq 1$).

Proof of Proposition 5.1. Assume, on the contrary, that there exist optimal covers $\mathcal{C}$ of $\binom{V}{k}$ which are not $(p+1)$-regular. From this hypothesis, we shall derive a contradiction proving Proposition 5.1. For this, observe that we may restrict our attention to strong covers $\mathcal{C}$ of $\binom{V}{k}$. Indeed, if $\mathcal{C}$ is an optimal cover of $\binom{V}{k}$ which is not $(p+1)$-regular, then the strong cover $\hat{\mathcal{C}}$ of $\binom{V}{k}$ guaranteed by Lemma 4.4 is optimal and also not $(p+1)$-regular, because $\mathcal{C}$ and $\hat{\mathcal{C}}$ are degree-equivalent (cf. Remark 4.5). Thus,

$$\textit{we assume that there exist strong covers } \mathcal{C} \textit{ of } \binom{V}{k} \textit{ which are not } (p+1)\textit{-regular.} \tag{24}$$

Below in (28), we choose a particular such strong cover $\mathcal{C}^\diamond$ with which to derive the promised contradiction, but for this we require several preparations.

First, Fact 3.1 ensures that an optimal cover $\mathcal{C}$ has (common) average degree $p < \alpha = \alpha(\mathcal{C}) \leq p+1$, where $\alpha = p$ is forbidden by $r \geq 1$ and by $R = 2^p - 1$. Second, for an optimal cover $\mathcal{C}$ of $\binom{V}{k}$, define

$$V_-(\mathcal{C}) = \{v : \deg_{\mathcal{C}}(v) \leq p\}, \quad V_0(\mathcal{C}) = \{v : \deg_{\mathcal{C}}(v) = p+1\}, \quad V_+(\mathcal{C}) = \{v : \deg_{\mathcal{C}}(v) \geq p+2\}. \tag{25}$$

Note that

$$V_-(\mathcal{C}) \neq \emptyset, \tag{26}$$

since otherwise $\alpha \leq p+1$ then requires $V_+(\mathcal{C}) = \emptyset$, while we focus on optimal covers $\mathcal{C}$ which are not already $(p+1)$-regular. Third, observe that

$$\textit{when } \mathcal{C} \textit{ is strong, every bone } S \in \mathcal{S}(\mathcal{C}) \textit{ of } \mathcal{C} \textit{ with } S \subseteq V_-(\mathcal{C}) \textit{ has size } |S| = k-1. \tag{27}$$

Indeed, when $S \subseteq V_-(\mathcal{C})$ is a bone, then $\deg_{\mathcal{C}}(S) \leq p < \alpha = \alpha(\mathcal{C})$ holds in a strong cover $\mathcal{C}$, and so Lemma 4.6 ensures $|S| = k-1$. (In particular, every $v \in V_-(\mathcal{C}) \neq \emptyset$ (cf. (26)) belongs to a bone $S = S_v$ of size $|S| = k-1$.) Finally,

$$\textit{we choose } \mathcal{C} = \mathcal{C}^\diamond \textit{ to minimize } |V_+(\mathcal{C})| \textit{ among all strong covers } \mathcal{C} \textit{ of } \binom{V}{k} \textit{ which are not } (p+1)\textit{-regular (cf. (24)).} \tag{28}$$

We proceed with the following claim.

Claim 5.2. *The strong cover $\mathcal{C}^\diamond$ chosen in (28) satisfies $|V_+(\mathcal{C}^\diamond)| \leq k-1$.*

Proof of Claim 5.2. Assume, on the contrary, that $|V_+(\mathcal{C}^\diamond)| \geq k$. Fix any subset $U \subseteq V_+(\mathcal{C}^\diamond)$ of size $|U| = k-1$, and fix any additional vertex $v_0 \in V_+(\mathcal{C}^\diamond) \setminus U$. Since $V_-(\mathcal{C}^\diamond) \neq \emptyset$ by (26), fix any bone $S \in \mathcal{S}(\mathcal{C}^\diamond)$ of $\mathcal{C}^\diamond$ satisfying $S \subseteq V_-(\mathcal{C}^\diamond)$. Then (27) gives $|S| = k-1$, and so Statements (a) and (b) of Fact 4.10 say that $\mathcal{C}^\diamond_{U,S}$ covers $\binom{V}{k}$ with weight

$$\omega(\mathcal{C}^\diamond_{U,S}) = \omega(\mathcal{C}^\diamond) + |U| + |S| + \sum_{u \in U} \left(\deg_{\mathcal{C}^\diamond}(S) - \deg_{\mathcal{C}^\diamond}(u)\right) \leq h(n,k) + |U| + |S| - 2|U| = h(n,k),$$

where we used $\deg_{\mathcal{C}^\diamond}(S) \leq p$ (from $S \subseteq V_-(\mathcal{C}^\diamond)$), $\deg_{\mathcal{C}^\diamond}(u) \geq p+2$ for each $u \in U \subseteq V_+(\mathcal{C}^\diamond)$, and $|U| = |S| = k-1$. Then $\mathcal{C}^\diamond_{U,S}$ is an optimal cover of $\binom{V}{k}$, where (23) gives

$$\deg_{\mathcal{C}^\diamond_{U,S}}(U) = \deg_{\mathcal{C}^\diamond_{U,S}}(S) = 1 + \deg_{\mathcal{C}^\diamond}(S) \overset{(25)}{\leq} p+1, \qquad \text{while} \qquad \deg_{\mathcal{C}^\diamond_{U,S}}(v) = \deg_{\mathcal{C}^\diamond}(v) \tag{29}$$

holds for each $v \in V \setminus (U \dot\cup S)$. (In particular, $\deg_{\mathcal{C}^\diamond_{U,S}}(v_0) = \deg_{\mathcal{C}^\diamond}(v_0) \geq p+2$ holds for the fixed vertex $v_0 \in V_+(\mathcal{C}^\diamond) \setminus U$, which will be important in a moment.) Thus,

$$V_+(\mathcal{C}^\diamond_{U,S}) = V_+(\mathcal{C}^\diamond) \setminus U \qquad \Longrightarrow \qquad |V_+(\mathcal{C}^\diamond_{U,S})| = |V_+(\mathcal{C}^\diamond)| - (k-1) < |V_+(\mathcal{C}^\diamond)|. \tag{30}$$

To the optimal cover $\mathcal{C}^\diamond_{U,S}$ of $\binom{V}{k}$, we apply Lemma 4.4 to obtain the strong cover $\hat{\mathcal{C}}^\diamond_{U,S}$ of $\binom{V}{k}$. Remark 4.5 says that $\hat{\mathcal{C}}^\diamond_{U,S}$ and $\mathcal{C}^\diamond_{U,S}$ are degree-equivalent, and so

$$V_+(\hat{\mathcal{C}}^\diamond_{U,S}) = V_+(\mathcal{C}^\diamond_{U,S}) \qquad \Longrightarrow \qquad |V_+(\hat{\mathcal{C}}^\diamond_{U,S})| = |V_+(\mathcal{C}^\diamond_{U,S})| \overset{(30)}{<} |V_+(\mathcal{C}^\diamond)|. \tag{31}$$

Now, $\hat{\mathcal{C}}^\diamond_{U,S}$ is a strong cover of $\binom{V}{k}$ satisfying (31), which is also not $(p+1)$-regular, since the fixed vertex $v_0 \in V_+(\mathcal{C}^\diamond) \setminus U$ still satisfies (cf. Remark 4.5 and (29))

$$\deg_{\hat{\mathcal{C}}^\diamond_{U,S}}(v_0) = \deg_{\mathcal{C}^\diamond_{U,S}}(v_0) = \deg_{\mathcal{C}^\diamond}(v_0) \geq p+2.$$

Now, strict inequality in (31) contradicts our choice of $\mathcal{C}^\diamond$ in (28). ☐

For the remainder of the proof, we consider no further alterations to the strong cover $\mathcal{C}^\diamond$ of $\binom{V}{k}$ chosen in (28), so we relax the notation $\mathcal{C}^\diamond$ to $\mathcal{C}$. We then relax the notation in (25) to $V_- = V_-(\mathcal{C})$, $V_0 = V_0(\mathcal{C})$, and $V_+ = V_+(\mathcal{C})$, and we write $\mathcal{S} = \mathcal{S}(\mathcal{C})$ for the skeleton of $\mathcal{C}$. Since each of V_-, V_0, and V_+ is defined in terms of $\mathcal{C}$-degrees, each of these sets is a union of bones $S \in \mathcal{S}$. Analogously to (25), define

$$\mathcal{S}_- = \{S \in \mathcal{S} : S \subseteq V_-\}, \qquad \mathcal{S}_0 = \{S \in \mathcal{S} : S \subseteq V_0\}, \qquad \mathcal{S}_+ = \{S \in \mathcal{S} : S \subseteq V_+\}, \tag{32}$$

where we claim the following inequality.

Claim 5.3. $|\mathcal{S}_0| \geq 2^{p+1} - |\mathcal{S}_-| - 1$.

Proof of Claim 5.3. Indeed, (27) gives $|V_-| = (k-1) \cdot |\mathcal{S}_-|$, and since every bone $S \in \mathcal{S}$ has size $|S| \leq k-1$, we similarly have $|V_0| \leq (k-1) \cdot |\mathcal{S}_0|$. As such, and to $n = |V_-| + |V_0| + |V_+|$, Claim 5.2 adds

$$\begin{aligned}(k-1) \cdot |\mathcal{S}_0| \geq |V_0| = n - |V_-| - |V_+| &\geq n - |V_-| - (k-1) \\ &= n - (k-1)|\mathcal{S}_-| - (k-1) > n - r - (k-1)|\mathcal{S}_-| - (k-1),\end{aligned}$$

where the strict inequality holds from $r \geq 1$. Thus, with $R = 2^p - 1$ in (4), the inequality above gives

$$|\mathcal{S}_0| > \frac{n-r}{k-1} - |\mathcal{S}_-| - 1 \overset{(4)}{=} 2^p + R - |\mathcal{S}_-| - 1 = 2^{p+1} - |\mathcal{S}_-| - 2,$$

and Claim 5.3 follows from the strict inequality above. ☐

We now conclude the proof of Proposition 5.1. Since $\mathcal{C}$ is a strong cover of $\binom{V}{k}$, its surviving family $\mathcal{Z} = \mathcal{Z}(\mathcal{C})$ consists of the skeleton $\mathcal{S}$, together with possibly the empty set. Now, consider the random surviving set $Z = Z_\psi \in \mathcal{Z}$ obtained by selecting $\psi \in \{a,b\}^{\mathcal{C}}$ uniformly (cf. Remark 4.2). Then

$$\begin{aligned}1 = \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}] &= \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}_-] + \mathbb{P}[Z \in \mathcal{S}_0] + \mathbb{P}[Z \in \mathcal{S}_+] \\ &= \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}_+] + \sum_{S \in \mathcal{S}_-} \mathbb{P}[Z = S] + \sum_{S \in \mathcal{S}_0} \mathbb{P}[Z = S]. \end{aligned} \tag{33}$$

For each bone $S \in \mathcal{S} = \mathcal{Z}^*$, we have that $\mathbb{P}[Z = S] = 1/2^{\deg_{\mathcal{C}}(S)}$ (cf. (21)), and so we infer from (25), (32), (33), and Claim 5.3 that

$$\begin{aligned}1 &\geq \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}_+] + |\mathcal{S}_-| \left(\frac{1}{2}\right)^p + |\mathcal{S}_0| \left(\frac{1}{2}\right)^{p+1} \\ &\geq \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}_+] + |\mathcal{S}_-| \left(\frac{1}{2}\right)^p + \left(2^{p+1} - |S_-| - 1\right) \left(\frac{1}{2}\right)^{p+1} \\ &= \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}_+] + 1 + \frac{1}{2^{p+1}} \left(|\mathcal{S}_-| - 1\right). \end{aligned} \tag{34}$$

Now, it is necessarily the case $|\mathcal{S}_-| \leq 1$, where $|\mathcal{S}_-| = 0$ isn't possible by (26). Thus, $|\mathcal{S}_-| = 1$ and so $\mathbb{P}[Z = \emptyset] = \mathbb{P}[Z \in \mathcal{S}_+] = 0$, in which case $\mathcal{S}_+ = \emptyset$. Now, $|\mathcal{S}_-| = 1$ implies that $\mathcal{S}_-$ consists of a single $(k-1)$-tuple (cf. (27)) of vertices of (common) degree at most p, and all other vertices have degree precisely $p+1$. As such,

$$h(n,k) = \omega(\mathcal{C}) \leq (k-1)p + (n-(k-1))(p+1) = n(p+1) - (k-1) = np + 2R(k-1) + r,$$

because $n(p+1) = np + 2R(k-1) + r + k - 1$ when $R = 2^p - 1$ (cf. (4)). Since $r \geq 1$, the bound $h(n,k) \leq np + 2R(k-1) + r$ contradicts the bound $h(n,k) \geq np + 2R(k-1) + 2r$ of Theorem 1.6.

5.2. **inductive step: $2^p - R > 1$.** For the inductive step, we verify the former and latter conclusions of Theorem 1.7 separately, and begin with the former.

Former conclusion of Theorem 1.7. To prove $h(n,k) \geq np + 2R(k-1) + r + k - 1$, we use the following recurrence, which holds when $0 \leq R < 2^p - 1$, and whose proof we give in a moment:

$$h(n,k) \geq h(n+k-1,k) - (k-1)(2+p). \tag{35}$$

Now, recall from (4) that $n = q(k-1) + r$, where $1 \leq r < k-1$, $q = 2^p + R$, and $0 \leq R < 2^p - 1$. Thus, $n + k - 1 = (q+1)(k-1) + r$ has the same modular remainder r, and $q + 1 = 2^p + (R+1)$ has the same exponent p, but $q+1$ has remainder $1 \leq R+1 \leq 2^p - 1$ w.r.t. base 2 expansion. Thus, $1 \leq 2^p - (R+1) < 2^p - R$, and we may apply induction to $h(n+k-1,k)$ to conclude from (35) that

$$h(n,k) \geq (n+k-1)p + 2(R+1)(k-1) + r + k - 1 - (k-1)(2+p) = np + 2R(k-1) + r + k - 1.$$

To prove (35), let $\mathcal{C}$ be a strong cover of $\binom{V}{k}$ (cf. Lemma 4.4), and let $\mathcal{C}$ have average degree $\alpha = \alpha(\mathcal{C})$. Fact 3.1 ensures that $p < \alpha < p+1$, where $\alpha = p$ is forbidden by $r \geq 1$, and $\alpha = p+1$ is forbidden by $R < 2^p - 1$. Thus, some bone $S \in \mathcal{S} = \mathcal{S}(\mathcal{C})$ satisfies $\deg_{\mathcal{C}}(S) \leq p < \alpha$ in the strong cover $\mathcal{C}$, and so Lemma 4.6 ensures that $|S| = k-1$. Let W be a set of $|W| = k-1$ new vertices, and let $\mathcal{C}^{W,S}$ be the S-immersion of W into $\mathcal{C}$. Then Fact 4.10 (Statement (d)) ensures that $\mathcal{C}^{W,S}$ covers $\binom{V \cup W}{k}$ with weight

$$\omega(\mathcal{C}^{W,S}) = \omega(\mathcal{C}) + (k-1)(2 + \deg_{\mathcal{C}}(S)) \leq h(n,k) + (k-1)(2+p), \tag{36}$$

where we used $\deg_{\mathcal{C}}(S) \leq p$. Since $\omega(\mathcal{C}^{W,S}) \geq h(n+k-1,k)$ holds by definition, (35) follows.

Latter conclusion of Theorem 1.7. We continue with the considerations above, where $\mathcal{C}$ is a strong cover of $\binom{V}{k}$, $S \in \mathcal{S} = \mathcal{S}(\mathcal{C})$ is a $(k-1)$-bone of $\mathcal{C}$ with degree $\deg_{\mathcal{C}}(S) \leq p$, and $\mathcal{C}^{W,S}$ is the $\mathcal{S}$-immersion of a set of $k-1$ new vertices W into the cover $\mathcal{C}$. In (36), we observed that

$$h(n+k-1,k) \leq \omega(\mathcal{C}^{W,S}) = \omega(\mathcal{C}) + (k-1)(2 + \deg_{\mathcal{C}}(S)) \leq h(n,k) + (k-1)(2+p), \tag{37}$$

where $n + k - 1$ has the same modular remainder r, where $q+1$ has the same exponent p, but where $q+1$ has remainder $1 \leq R+1 \leq 2^p - 1$ w.r.t. base 2 expansion. Since Theorem 1.3 is now proven in full, we apply it to both sides of (37) to obtain

$$(n+k-1)p + 2(R+1)(k-1) + r + k - 1 \leq \omega(\mathcal{C}^{W,S}) \leq np + 2R(k-1) + r + k - 1 + (k-1)(2+p), \tag{38}$$

and so equality holds throughout (37) and (38). Thus, $\mathcal{C}^{W,S}$ is an optimal cover of $\binom{V\cup W}{k}$, and it is necessarily the case that $\deg_{\mathcal{C}}(S) = p$. Since $\mathcal{C}^{W,S}$ is optimal with $2^p - (R+1) < 2^p - R$, induction guarantees that its degree sequence $\boldsymbol{d}(\mathcal{C}^{W,S})$ is the unique $\boldsymbol{D}_{n+k-1,k} \in \{p, p+1\}^{V\cup W}$ with precisely

$$2(R+1)(k-1) + r + k - 1 = 2R(k-1) + r + k - 1 + 2(k-1)$$

many $(p+1)$-digits (cf. (3)). We now compare the sequences $\boldsymbol{d}(\mathcal{C}^{W,S})$ and $\boldsymbol{d}(\mathcal{C})$, which by Definition 4.9 differ only on the $|W \cup S| = 2(k-1)$ many coordinates corresponding to $W \cup S$. First, each $(W \cup S)$-coordinate of $\mathcal{C}^{W,S}$ is a $(p+1)$-digit, since we observed that $\deg_{\mathcal{C}}(S) = p$, where Definition 4.9 gives $\deg_{\mathcal{C}^{W,S}}(W \cup S) = 1 + \deg_{\mathcal{C}}(S)$. Second, the $|W| = k-1$ many W-coordinates of $\boldsymbol{d}(\mathcal{C}^{W,S})$ don't appear in $\boldsymbol{d}(\mathcal{C})$ at all. Third, the $|S| = k-1$ many S-coordinates of $\boldsymbol{d}(\mathcal{C}^{W,S})$ do appear in $\boldsymbol{d}(\mathcal{C})$, but as p-digits (as noted above). Thus, $\boldsymbol{d}(\mathcal{C})$ consists of precisely

$$2R(k-1) + r + k - 1 + 2(k-1) - 2(k-1) = 2R(k-1) + r + k - 1$$

many $(p+1)$-digits, and all remaining coordinates are p-digits, making $\boldsymbol{d}(\mathcal{C}) = \boldsymbol{D}_{n,k} \in \{p, p+1\}^V$ the unique sequence described in (3).

6. Proof of Lemma 4.4

Let V be a finite set, and let $\mathcal{C}$ cover $\binom{V}{k}$ with surviving family $\mathcal{Z}$ and skeleton $\mathcal{S}$, as described in Definitions 4.1 and 4.3. The following two observations will initiate the proof of Lemma 4.4.

Observation 6.1. *Every surviving set $Z_\psi \in \mathcal{Z}$ is a union of bones $S \in \mathcal{S}$.*

Proof. For sake of argument, assume $Z_\psi \neq \emptyset$ and fix $v \in Z_\psi = \bigcap_{\{A,B\}\in\mathcal{C}} Z^\psi_{\{A,B\}}$. Fix $\{A, B\} \in \mathcal{C}$, and let $S_v \in \mathcal{S}$ be the unique bone containing v (cf. (22)). By (19), $Z^\psi_{\{A,B\}}$ is $V \setminus A$ (resp. $V \setminus B$) iff $\psi(\{A, B\})$ is a (resp. b). Definition 4.3 then ensures $S_v \subseteq Z^\psi_{\{A,B\}}$, and hence $S_v \subseteq \bigcap_{\{A,B\}\in\mathcal{C}} Z^\psi_{\{A,B\}} = Z_\psi$. □

Observation 6.2. *If $\mathcal{Z}^* \subseteq \mathcal{S}$, then in fact $\mathcal{Z}^* = \mathcal{S}$.*

Proof. Fix $S \in \mathcal{S}$, and define $\psi_S \in \{a, b\}^{\mathcal{C}}$ by $\psi_S(\{A, B\}) = a$ if, and only if, $S \cap A = \emptyset$. We will show

$$S \subseteq Z_{\psi_S}. \tag{39}$$

If true, S is a bone and $Z_{\psi_S} \in \mathcal{Z}^* \subseteq \mathcal{S}$ is also a bone, so as overlapping equivalence classes $S = Z_{\psi_S}$. To see (39), fix $\{A, B\} \in \mathcal{C}$. If $\psi_S(\{A, B\}) = a$, then $S \cap A = \emptyset$ and (19) gives $Z^{\psi_S}_{\{A,B\}} = V \setminus A \supseteq S$. If $\psi_S(\{A, B\}) = b$, then $S \cap A \neq \emptyset$ and so (22) gives $S \subseteq A$ and $S \cap B = \emptyset$, and (19) gives $Z^{\psi_S}_{\{A,B\}} = V \setminus B \supseteq S$. Either way, $S \subseteq Z^{\psi_S}_{\{A,B\}}$ and hence $S \subseteq \bigcap_{\{A,B\}\in\mathcal{C}} Z^{\psi_S}_{\{A,B\}} = Z_{\psi_S}$. □

Observations 6.1 and 6.2 allow a sketch of the main idea for proving Lemma 4.4. Indeed, if $\mathcal{Z}^* \subseteq \mathcal{S}$, then we set $\hat{\mathcal{C}} = \mathcal{C}$ and Observation 6.2 says we are done. For sake of argument, let $Z_0 = Z_{\psi_0} \in \mathcal{Z}^* \setminus \mathcal{S}$ be a surviving set which is itself not a bone. Then Observation 6.1 says that Z_0 is a union of at least two bones $S \in \mathcal{S}$, so we choose $S_0 \in \mathcal{S}$ to satisfy

$$S_0 \subsetneq Z_0, \quad \text{where} \quad \deg_{\mathcal{C}}(S_0) = \min_{S\in\mathcal{S}} \{\deg_{\mathcal{C}}(S) : S \subseteq Z_0\}, \quad \text{and we set} \quad U_0 = Z_0 \setminus S_0 \neq \emptyset. \tag{40}$$

Let $\mathcal{C}_0 = \mathcal{C}^*_{U_0,S_0} = \mathcal{C}_{U_0,S_0} \setminus \{U_0, S_0\}$ be the S_0-shift of U_0 in $\mathcal{C}$ (cf. Definition 4.7). We claim the following.

Proposition 6.3. *The family $\mathcal{C}_0$ covers $\binom{V}{k}$. Moreover, the skeleton $\mathcal{S}_0$ of $\mathcal{C}_0$ satisfies $|\mathcal{S}_0| < |\mathcal{S}|$.*

Proposition 6.3, upon possible iteration, will give Lemma 4.4. To see this, we first note that $\mathcal{C}_0$ from Proposition 6.3 satisfies $\deg_{\mathcal{C}_0}(v) \leq \deg_{\mathcal{C}}(v)$ for each $v \in V$. Indeed, fix $u \in U_0$ and $v \in V \setminus U_0$. By (23), $\mathcal{C}_0 = \mathcal{C}^*_{U_0,S_0}$ admits the identities $\deg_{\mathcal{C}_0}(v) = \deg_{\mathcal{C}}(v)$ and $\deg_{\mathcal{C}_0}(u) = \deg_{\mathcal{C}_0}(S_0) = \deg_{\mathcal{C}}(S_0)$, where (40) adds that $\deg_{\mathcal{C}_0}(u) = \deg_{\mathcal{C}}(S_0) \leq \deg_{\mathcal{C}}(u)$. Second, we consider the surviving family $\mathcal{Z}_0$ of $\mathcal{C}_0$. If $\mathcal{Z}_0^* \subseteq \mathcal{S}_0$, then set $\hat{\mathcal{C}} = \mathcal{C}_0$ and Observation 6.2 says we are done. Otherwise, $\mathcal{Z}_0^* \setminus \mathcal{S}_0 \neq \emptyset$, and we repeat (40). By Proposition 6.3, we can't repeat (40) indefinitely, and so Lemma 4.4 follows.

6.1. **Proof of Proposition 6.3: first assertion.** To verify the first assertion of Proposition 6.3, we fix $K \in \binom{V}{k}$ and consider three cases. (Recall S_0 and U_0 from (40).)

Case 1 ($K \cap U_0 = \emptyset$). Let $\{A, B\} \in \mathcal{C}$ cover K. Then

$$\{A_{U_0,S_0}, B_{U_0,S_0}\} \in \mathcal{C}_0 \text{ also covers } K, \tag{41}$$

because Definition 4.7 gives $K \cap A_{U_0,S_0} = K \cap A$ and $K \cap B_{U_0,S_0} = K \cap B$.

Case 2 ($K \cap S_0 \neq \emptyset$). For our most delicate case, let $\{A, B\} \in \mathcal{C}$ cover K. Since K meets S_0, then S_0 meets $A \dot\cup B$, and so by (22) we take, w.l.o.g., $S_0 \subseteq A$. Now, Definition 4.7 ensures $A_{U_0,S_0} = A \cup U_0 \supseteq A$ and $B_{U_0,S_0} = B \setminus U_0$, and so we will infer (41) if we can prove that

$$B_{U_0,S_0} = B, \quad \text{which holds if} \quad B \cap U_0 = \emptyset. \tag{42}$$

To see (42), recall that $Z_0 = Z_{\psi_0} \in \mathcal{Z}^* \setminus \mathcal{S}$ is a surviving set, where $\psi_0 \in \{a, b\}^{\mathcal{C}}$ (recall (19)) denotes that function for which $Z_0 = Z_{\psi_0} = \bigcap_{\{C,D\} \in \mathcal{C}} Z^{\psi_0}_{\{C,D\}}$. Thus, for the element $\{A, B\} \in \mathcal{C}$ fixed above, we have that $Z_0 = S_0 \cup U_0$ satisfies $S_0 \subset Z_0 \subseteq Z^{\psi_0}_{\{A,B\}}$. Since we know $S_0 \subseteq A$, it can only be (recall (19)) that $Z^{\psi_0}_{\{A,B\}} = V \setminus B$, and so $U_0 \subset Z_0 \subseteq Z^{\psi_0}_{\{A,B\}} = V \setminus B$ is disjoint from B, as desired in (42).

Case 3 ($K \cap U_0 \neq \emptyset$ and $K \cap S_0 = \emptyset$). Fix $u \in K \cap U_0$ and fix $v \in S_0$, where we necessarily have $v \notin K$. Define $K_{u,v} = (K \setminus \{u\}) \cup \{v\}$, which as a k-tuple of V is covered by some fixed $\{A, B\} \in \mathcal{C}$. Applying Case 2 to $K_{u,v}$, we infer that $K_{u,v}$ is also covered by $\{A_{U_0,S_0}, B_{U_0,S_0}\} \in \mathcal{C}_0$, and moreover and w.l.o.g. that $S_0 \subseteq A$, $A_{U_0,S_0} = A \cup U_0$ and $B_{U_0,S_0} = B$. As such, to see (41), we simply note that

$$K \triangle K_{u,v} = \{u, v\} \subseteq A_{U_0,S_0}, \tag{43}$$

and so $K \cap B_{U_0,S_0} = K_{u,v} \cap B_{U_0,S_0} \neq \emptyset$, and $u \in K \cap A_{U_0,S_0} \neq \emptyset$.

6.2. **Proof of Proposition 6.3: second assertion.** It remains to verify the second assertion of Proposition 6.3. For that, we make a natural but important observation.

Observation 6.4. *$Z_0 = U_0 \dot\cup S_0$ is a $\mathcal{C}_0$-bone, i.e., a bone of $\mathcal{C}_0 = \mathcal{C}^*_{U_0,S_0}$.*

Proof. Definition 4.7 forces all vertices of $Z_0 = U_0 \dot\cup S_0$ to be $\mathcal{C}_0$-equivalent in $\mathcal{C}_0 = \mathcal{C}^*_{U_0,S_0}$, and so a unique $\mathcal{C}_0$-bone ($\mathcal{C}_0$-equivalence class) $T \in \mathcal{S}_0$ of $\mathcal{C}_0$ contains Z_0. If $Z_0 \subsetneq T$ isn't already that bone, then fix any $v \in T \setminus Z_0$. Now, recall that $Z_0 \in \mathcal{Z}^*$ is a surviving set of $\mathcal{C}$, where $\psi_0 \in \{a, b\}^{\mathcal{C}}$ (recall (19)) satisfies $Z_0 = Z_{\psi_0} = \bigcap_{\{C,D\} \in \mathcal{C}} Z^{\psi_0}_{\{C,D\}}$. Since $v \notin Z_0$, there exists $\{A, B\} \in \mathcal{C}$ for which $v \notin Z^{\psi_0}_{\{A,B\}}$, and here we take $Z^{\psi_0}_{\{A,B\}} = V \setminus A$ (w.l.o.g. (cf. (19))). Thus, $v \in A$ while $S_0 \subset Z_0 = Z_{\psi_0} \subseteq Z^{\psi_0}_{\{A,B\}} = V \setminus A$ entirely misses A. Since $S_0 \cap A = \emptyset$, Definition 4.7 gives $A_{U_0,S_0} = A \setminus U_0$, which still contains v because $v \in A$ but $v \notin Z_0 \supset U_0$. At the same time, S_0 entirely misses $A_{U_0,S_0} = A \setminus U_0$, and so v and S_0 do not agree on $\{A_{U_0,S_0}, B_{U_0,S_0}\} \in \mathcal{C}_0 = \mathcal{C}^*_{U_0,S_0}$. Thus, v can't be $\mathcal{C}_0$-equivalent to S_0, and so $v \in T \setminus Z_0$ can't be part of the $\mathcal{C}_0$-bone $T \in \mathcal{S}_0$ of $\mathcal{C}_0$ which contains Z_0. □

The remaining assertion of Proposition 6.3 is a formal corollary of Observation 6.4. Indeed, write $U_0 = S_1 \dot\cup \dots \dot\cup S_t$ as a union of $t \geq 1$ many $\mathcal{C}$-bones, in which case $Z_0 = S_0 \dot\cup U_0 = S_0 \dot\cup S_1 \dot\cup \dots \dot\cup S_t$ is the union of $t + 1 \geq 2$ many $\mathcal{C}$-bones. Define the relation $f : \mathcal{S} \setminus \{S_0, S_1, \dots, S_t\} \to \mathcal{S}_0 \setminus \{Z_0\}$ by $f(S) = T$ if, and only if, $S \subseteq T$. We claim that f is a well-defined surjection, which would conclude Proposition 6.3:

$$|\mathcal{S}| - (t+1) = |\mathcal{S} \setminus \{S_0, S_1, \dots, S_t\}| \geq |\mathcal{S}_0 \setminus \{Z_0\}| = |\mathcal{S}_0| - 1 \quad \Longrightarrow \quad |\mathcal{S}_0| \leq |\mathcal{S}| - t \leq |\mathcal{S}| - 1 < |\mathcal{S}|.$$

To see that f is well-defined, fix $S \in \mathcal{S} \setminus \{S_0, S_1, \dots, S_t\}$. Since $S, S_0, S_1, \dots, S_t$ are all $\mathcal{C}$-bones, it follows that $S \cap U_0 = S \cap (S_1 \dot\cup \dots \dot\cup S_t) = \emptyset$, in which case no part of S *moves*[1] upon shifting U_0 to S_0 in $\mathcal{C}_0 = \mathcal{C}^*_{U_0,S_0}$. As such, the vertices of S are $\mathcal{C}_0$-equivalent, and so a unique $\mathcal{C}_0$-bone $T \in \mathcal{S}_0$ contains

[1]That is, for each $\{A, B\} \in \mathcal{C}$, we have, e.g., $S \subseteq A$ ($S \cap A = \emptyset$) if, and only if, $S \subseteq A_{U_0,S_0}$ ($S \cap A_{U_0,S_0} = \emptyset$).

S. Since S was disjoint from $Z_0 = S_0 \dot\cup S_1 \dot\cup \ldots \dot\cup S_t$, this unique $\mathcal{C}_0$-bone T is not the $\mathcal{C}_0$-bone Z_0 of Observation 6.4. Thus, the unique containment $S \subseteq T \in \mathcal{S}_0 \setminus \{Z_0\}$ verifies that $f(S) = T$ is well-defined.

To see that f is surjective, fix $T \in \mathcal{S}_0 \setminus \{Z_0\}$. Moreover, fix any $v \in T$, and let $S_v \in \mathcal{S}$ be the unique $\mathcal{C}$-bone containing v. Since $T, Z_0 \in \mathcal{S}_0$ are distinct $\mathcal{C}_0$-bones, it must be the case that $T \cap Z_0 = \emptyset$, and so $v \in T$ satisfies $v \notin Z_0 = S_0 \dot\cup S_1 \dot\cup \ldots \dot\cup S_t$. As such, the $\mathcal{C}$-bone S_v can't overlap any of the $\mathcal{C}$-bones $S_0, S_1, \ldots, S_t$, which places S_v in the domain of f. In particular, S_v can't overlap $U_0 = S_1 \dot\cup \ldots \dot\cup S_t$, and so S_v doesn't move upon shifting U_0 to S_0 in $\mathcal{C}_0$. Thus, vertices of S_v are $\mathcal{C}_0$-equivalent, where $v \in S_v \cap T$ belongs to the $\mathcal{C}_0$-bone T. Thus, $S_v \subseteq T$, and so S_v satisfies $f(S_v) = T$.

7. A Special Case of Lemma 4.6, and a Tool for the General Case

We prove a special case of Lemma 4.6, and we also establish a tool critical for the general case. Fix a set V of size n and fix a strong cover $\mathcal{C}$ of $\binom{V}{k}$ with skeleton $\mathcal{S} = \mathcal{Z}^*$, where p, q, r, and R satisfy (4). Lemma 4.6 asserts that every bone $S \in \mathcal{S} = \mathcal{Z}^*$ with $\deg_{\mathcal{C}}(S) < \alpha = \alpha(\mathcal{C})$ below the average has maximum size $|S| = k-1$. For $r = 0$, we use Theorem 1.6 to prove this assertion in strong form.

Fact 7.1. *When $r = 0$, every bone $S \in \mathcal{S} = \mathcal{Z}^*$ of the strong cover $\mathcal{C}$ satisfies $|S| = k-1$.*

Theorem 1.6 is valid to apply because we established it in Section 3. Moreover, Theorem 1.6 says that, since $\mathcal{C}$ is optimal and $r = 0$, the degree sequence $\boldsymbol{d}(\mathcal{C})$ is given uniquely by $\boldsymbol{D} = \boldsymbol{D}_{n,k} \in \{p, p+1\}^V$ from (3), with precisely $2R(k-1)$ many coordinates of $p+1$.

Proof of Fact 7.1. As in Section 3, consider again the random surviving set Z of $\mathcal{C}$ from (8), and recall from (9) that $\mathbb{E}[|Z|] = \sum_{v \in V} 2^{-\deg_{\mathcal{C}}(v)}$. Applying Theorem 1.6 according to the discussion above,

$$k - 1 \overset{(9)}{\geq} \mathbb{E}[|Z|] \overset{(9)}{=} 2R(k-1)\left(\frac{1}{2}\right)^{p+1} + (n - 2R(k-1))\left(\frac{1}{2}\right)^{p} = \left(\frac{1}{2}\right)^{p}(k-1)\left(\frac{n}{k-1} - R\right) \overset{(4)}{=} k-1.$$

Thus, the random surviving set averages the maximum size of $k-1$, and so all surviving sets of $Z \in \mathcal{Z} = \mathcal{Z}^*$ achieve $|Z| = k-1$. Since $\mathcal{C}$ is strong, i.e., $\mathcal{S} = \mathcal{Z}^*$, all bones $S \in \mathcal{S}$ satisfy $|S| = k-1$. ☐

The proof above used Theorem 1.6, which established (in Section 3) the case $r = 0$ of Theorem 1.4. For $r \geq 1$, we must proceed more carefully because Theorem 1.4 for $r \geq 1$ depends on Theorem 1.7, which depends on Lemma 4.6 for $r \geq 1$, whose establishment won't be complete until Section 9. Nonetheless, we can still apply some ideas from Section 3 to the strong cover $\mathcal{C}$ for general $r \geq 0$, and this will in fact make a critical step in the desired direction.

Proposition 7.2. *For $r \geq 0$, every bone $S \in \mathcal{S} = \mathcal{Z}^*$ of the strong cover $\mathcal{C}$ with $\deg_{\mathcal{C}}(S) < \alpha = \alpha(\mathcal{C})$ below the average has size $|S| \geq (r + k - 1)/2$ (which is at least half of what Lemma 4.6 promises).*

Proof of Proposition 7.2. Fix a bone $S_0 \in \mathcal{S} = \mathcal{Z}^*$ of the strong cover $\mathcal{C}$ with $\deg_{\mathcal{C}}(S_0) < \alpha = \alpha(\mathcal{C})$ below the average. Since $\mathcal{C}$ is optimal, Fact 3.1 gives $\alpha \leq p+1$, and so

$$\deg_{\mathcal{C}}(S_0) \leq p. \tag{44}$$

Since $\mathcal{C}$ is a strong cover, the bone $S_0 \in \mathcal{S} = \mathcal{Z}^*$ is a surviving set, and therefore has the form $S_0 = Z_{\psi_0}$ for some function $\psi_0 \in \{a, b\}^{\mathcal{C}}$ (cf. Definition 4.1). Thus, $S_0 = Z_{\psi_0}$ is a possible outcome of the random surviving set Z from (8), which is obtained when $\psi \in \{a, b\}^{\mathcal{C}}$ is chosen uniformly. As such, we pivot the size $|S_0| = |Z_{\psi_0}|$ against the expected size $\mathbb{E}[|Z|]$ given in (20):

$$\mathbb{E}[|Z|] = \sum_{Z_\Psi \in \mathcal{Z}} \big(|Z_\Psi| \cdot \mathbb{P}[Z = Z_\Psi]\big). \tag{45}$$

Since $S_0 \in \mathcal{S} = \mathcal{Z}^* \subseteq \mathcal{Z}$ appears in (45), and all surviving sets $Z_\Psi \in \mathcal{Z}$ satisfy $|Z_\Psi| \leq k-1$, we infer

$$\begin{aligned}\mathbb{E}[|Z|] &= |S_0| \cdot \mathbb{P}[Z = S_0] + \sum_{S_0 \neq Z_\Psi \in \mathcal{Z}} \big(|Z_\Psi| \cdot \mathbb{P}[Z = Z_\Psi]\big) \leq |S_0| \cdot \mathbb{P}[Z = S_0] + (k-1) \sum_{S_0 \neq Z_\Psi \in \mathcal{Z}} \mathbb{P}[Z = Z_\Psi] \\ &= |S_0| \cdot \mathbb{P}[Z = S_0] + (k-1)\big(1 - \mathbb{P}[Z = S_0]\big) = k - 1 - \mathbb{P}[Z = S_0]\big(k - 1 - |S_0|\big). \end{aligned} \tag{46}$$

Again, since $S_0 \in \mathcal{S} = \mathcal{Z}^*$ is a surviving set, we recall from (21) that

$$\mathbb{P}[Z = S_0] = \left(\frac{1}{2}\right)^{\deg_{\mathcal{C}}(S_0)} \overset{(44)}{\geq} \frac{1}{2^p},$$

$$\overset{(46)}{\Longrightarrow} \quad \mathbb{E}[|Z|] \leq k-1-\frac{1}{2^p}\big(k-1-|S_0|\big) = (k-1)\left(1-\frac{1}{2^p}\right)+\frac{1}{2^p}|S_0|. \quad (47)$$

On the other hand, Fact 3.2 yields

$$\mathbb{E}[|Z|] \overset{(9)}{=} \sum_{v \in V}\left(\frac{1}{2}\right)^{d_v} \geq r\left(\frac{1}{2}\right)^{p+1} + (k-1)\left(1-\left(\frac{1}{2}\right)^{p+1}\right). \quad (48)$$

Comparing (47) and (48) yields $2|S_0| \geq k-1+r$, which gives Proposition 7.2. □

8. Shifting Considerations for Proving Lemma 4.6

It remains to prove Lemma 4.6, which we do in Section 9 using shifting. This section develops a few helpful considerations on shifting which are motivated by the following basic questions. *Fix an arbitrary cover $\mathcal{C}$ of $\binom{V}{k}$, bone $S \in \mathcal{S}$, and subset $U \subseteq V \setminus S$ of size $1 \leq |U| \leq k-1$:*

$$\textit{Does } \mathcal{C}_{U,S} \textit{ cover } \tbinom{V}{k}? \qquad \textit{If so, does } \mathcal{C}^*_{U,S} \textit{ cover } \tbinom{V}{k}? \quad (49)$$

Sections 5 and 6 featured the following conditions sufficient for confirming parts of (49). First, Section 5 used the condition $|S| = k-1$, which Fact 4.10 proved is sufficient[2] for $\mathcal{C}_{U,S}$ to cover $\binom{V}{k}$. However, this condition matches the conclusion of Lemma 4.6. Second, Section 6 used the condition that $U \cup S$ was a surviving set of $\mathcal{C}$ which was specifically not a bone, which Proposition 6.3 proved is sufficient for $\mathcal{C}^*_{U,S}$ to cover $\binom{V}{k}$. However, Lemma 4.6 assumes $\mathcal{C}$ is strong, where (non-empty) surviving sets and bones are indistinguishable concepts. We can't use these earlier conditions in Section 9.

We now initiate further insights on (49) that we use in Section 9. In fact, it will be enough for our purposes to resolve (49) (see Proposition 8.4 below) under the following restrictions:

- (i) we always assume $|S| < k-1$;
- (ii) we only consider when $U = \{u\} \subseteq V \setminus S$ is a singleton;
- (iii) we only consider $\mathcal{C}^*_{\{u\},S}$.

By the discussion above, (i) is necessary for further investigation on (49). From (ii), we abbreviate $\mathcal{C}_{\{u\},S}$ to $\mathcal{C}_{u,S}$ and $\mathcal{C}^*_{\{u\},S}$ to $\mathcal{C}^*_{u,S}$, and we abbreviate each $\{A_{\{u\},S}, B_{\{u\},S}\} \in \mathcal{C}_{u,S}$ to $\{A_{u,S}, B_{u,S}\}$. Finally, (i) and (ii) warrant (iii), because $\{u, S\} \in \mathcal{C}_{u,S}$ covers none of $\binom{V}{k}$.

8.1. Observations on (49) under $(i)-(iii)$.

We start with the following very easy observation.

Observation 8.1. *With $|S| < k-1$, the family $\mathcal{C}^*_{u,S}$ covers all $K \in \binom{V}{k}$ for which $u \notin K$ or $S \setminus K \neq \emptyset$.*

Proof. Indeed, if $u \notin K$, then $\{A_{u,S}, B_{u,S}\} \in \mathcal{C}^*_{u,S}$ covers K whenever $\{A, B\} \in \mathcal{C}$ does, which happens at least once in the cover $\mathcal{C}$. If $u \in K$ but $v \in S \setminus K$, then $\{A_{u,S}, B_{u,S}\} \in \mathcal{C}^*_{u,S}$ covers K whenever it covers $K_{u,v} = (K \setminus \{u\}) \cup \{v\}$, which happens at least once by the previous case. □

Observation 8.1, prompts that we investigate the coverage of a fixed element K satisfying

$$\{u\} \dot\cup S \subsetneq K \in \tbinom{V}{k}, \quad (50)$$

where $\subsetneq$ holds by $|S| < k-1$. Note that $L = K \setminus \{u\}$ is not a bone of $\mathcal{C}$, because it properly contains the bone S. The following curious concept will characterize all K in (50) not covered by $\mathcal{C}^*_{u,S}$.

Definition 8.2 (limb)**.** A $(k-1)$-set $L \subset V$ is a *limb* of $\mathcal{C}$ if $L \notin \mathcal{S}$ is not a bone, but $\forall \{A, B\} \in \mathcal{C}$,

$$L \subseteq A \cup B \qquad \Longrightarrow \qquad L \subseteq A \quad \text{or} \quad L \subseteq B. \quad (51)$$

[2] The same condition prevents $\mathcal{C}^*_{U,S}$ from covering $\binom{V}{k}$, since then $\binom{U \cup S}{k} \neq \emptyset$ requires $\{U, S\} \in \mathcal{C}_{U,S}$ for coverage.

Remark 8.3. Every bone $S \in \mathcal{S}$ satisfies (51) as an equivalence class of $\sim_{\mathcal{C}}$ (cf. (22)). The condition (51) is weaker than that of (22), since for a fixed limb L and a given $\{A, B\} \in \mathcal{C}$, condition (51) allows for

$$L \cap (A \cup B) \neq \emptyset \neq L \setminus (A \cup B). \tag{52}$$

In fact, (52) must hold for some $\{A, B\} \in \mathcal{C}$. Indeed, Definition 8.2 insists that $L \notin \mathcal{S}$ is not a bone of $\mathcal{C}$, whence $u \not\sim_{\mathcal{C}} v$ holds for some $u, v \in L$. By (22), there exists $\{A, B\} \in \mathcal{C}$ so that, w.l.o.g.,

(i) $u \in A \cup B$ but $v \notin A \cup B$, or
(ii) $u \in A$ but $v \in B$.

In (i), $u \in L \cap (A \cup B)$ and $v \in L \setminus (A \cup B)$, so $\{A, B\}$ satisfies (52). In (ii), $u, v \in L \cap (A \cup B)$, but if $L \subseteq A \cup B$, then $u \in A$ and $v \in B$ contradict (51). Let us also note at this time that a limb L is of maximum size w.r.t. (51), since $|L| = k$ would require some $\{A, B\} \in \mathcal{C}$ to cover it. □

We proceed with the promised characterization, which is crucial in Section 9.

Proposition 8.4. *With $|S| < k - 1$ and $u \in V \setminus S$ fixed, every $K \in \binom{V}{k}$ satisfies*

$$K \text{ is not covered by } \mathcal{C}^*_{u,S} \quad \Longleftrightarrow \quad K \text{ satisfies (50) and } L = K \setminus \{u\} \text{ is a limb of } \mathcal{C}.$$

Proof. Let K satisfy (50) where $L = K \setminus \{u\}$ is a limb of $\mathcal{C}$. To see that K is not covered in $\mathcal{C}^*_{u,S}$, fix $\{A_{u,S}, B_{u,S}\} \in \mathcal{C}^*_{u,S}$ with $K \subseteq A_{u,S} \dot\cup B_{u,S}$. Since $A_{u,S} \subseteq A \cup \{u\}$ and $B_{u,S} \subseteq B \cup \{u\}$ (cf. Definition 4.7),

$$L = K \setminus \{u\} \subseteq (A_{u,S} \dot\cup B_{u,S}) \setminus \{u\} = (A_{u,S} \setminus \{u\}) \dot\cup (B_{u,S} \setminus \{u\}) \subseteq A \dot\cup B.$$

Since L is a limb of $\mathcal{C}$, $L \subseteq A$ or $L \subseteq B$, and w.l.o.g. we assume the former. Now, $S \subset K \setminus \{u\} = L \subseteq A$, so Definition 4.7 guarantees $A_{u,S} = A \cup \{u\}$. Now, $K \setminus \{u\} = L \subseteq A$ implies $K \subseteq A \cup \{u\} = A_{u,S}$.

Conversely, let K be uncovered in $\mathcal{C}^*_{u,S}$. Observation 8.1 guarantees K satisfies (50), so $L = K \setminus \{u\}$ is a $(k-1)$-set properly containing S and can't be a bone of $\mathcal{C}$. To see that L is a limb of $\mathcal{C}$, let $\{A, B\} \in \mathcal{C}$ satisfy $L \subseteq A \dot\cup B$. Now, $S \subset L \subseteq A \dot\cup B$, so we take w.l.o.g. $S \subseteq A$. Definition 4.7 guarantees $A_{u,S} = A \cup \{u\}$ and $B_{u,S} = B \setminus \{u\}$, so $K = L \cup \{u\} \subseteq A_{u,S} \dot\cup B_{u,S}$. Now, K meets $A_{u,S}$ in u and S, but $\{A_{u,S}, B_{u,S}\} \in \mathcal{C}^*_{u,S}$ does not cover K, so $K \subseteq A_{u,S} = A \cup \{u\}$ and $L = K \setminus \{u\} \subseteq A$. □

Proposition 8.4 shows that limbs resolve (49) under the restrictions (i)–(iii). However, Section 9 will need to understand limbs beyond just this context, and for this we collect a few more observations.

8.2. **Observations on limbs.** Our remaining observations relate the limbs of a cover $\mathcal{C}$ to its bones. To maintain neutrality from (49) (and $S \in \mathcal{S}$ in particular), we write an arbitrary bone of $\mathcal{C}$ as $T \in \mathcal{S}$.

Observation 8.5. *Every limb L of $\mathcal{C}$ is a union of at least two bones of $\mathcal{C}$. In particular, if L is a limb of $\mathcal{C}$ and $T \in \mathcal{S}$ is any bone of $\mathcal{C}$,*

$$T \cap L \neq \emptyset \quad \Longrightarrow \quad T \subseteq L. \tag{53}$$

Proof. Indeed, let $v \in T \cap L$, but suppose $w \in T \setminus L$. Then $K_{w,L} = \{w\} \cup L$ is a k-tuple of V covered by some $\{A, B\} \in \mathcal{C}$. Now, $L = K_{w,L} \setminus \{w\} \subseteq A \dot\cup B$, and so by definition L satisfies, w.l.o.g., $L \subseteq A$. This forces $v \in K_{w,L} \cap A = L \cap A$ and $K_{w,L} \cap B = \{w\}$, contradicting that $v, w \in T$ are $\sim_{\mathcal{C}}$-equivalent. Now, L is a union of bones, and by definition necessarily more than one. □

Our final observation is critical in Section 9, and will relate closely to our earlier Proposition 7.2.

Proposition 8.6. *Every bone $T \in \mathcal{S}$ of $\mathcal{C}$ of size $|T| \geq k/2$ is contained within at most one limb L of $\mathcal{C}$.*

Proof. Suppose, on the contrary, that $T_0 \in \mathcal{S}$ is a bone of $\mathcal{C}$ of size $|T_0| \geq k/2$, and suppose $L_1 \neq L_2$ are distinct limbs of $\mathcal{C}$ for which $T_0 \subseteq L_1 \cap L_2$. By (53), the union $L_1 \cup L_2 = T_0 \dot\cup T_1 \dot\cup \ldots \dot\cup T_t$ is partitioned into bones, which necessarily includes T_0, and where necessarily $t \geq 1$. Observe that $t \leq k - 2$, since

$$t \leq \big|(L_1 \cup L_2) \setminus T_0\big| \leq |L_1 \setminus T_0| + |L_2 \setminus T_0| = |L_1| + |L_2| - 2|T_0| = 2(k-1) - 2|T_0| \leq k - 2.$$

Now, choose any k-tuple $K \subseteq L_1 \cup L_2$ (noting $L_1 \neq L_2$ implies $|L_1 \cup L_2| \geq k$) meeting each of $T_0, T_1, \ldots, T_t$. Let $\{A, B\} \in \mathcal{C}$ cover K. Since K meets each bone $T_0, T_1, \ldots, T_t$ of $L_1 \cup L_2$, and since $K \subseteq A \dot\cup B$, we have from (22) that $L_1 \cup L_2 \subseteq A \dot\cup B$. Since $L_1 \subset A \dot\cup B$ is a limb, take (w.l.o.g.) $L_1 \subseteq A$

so that $T_0 \subseteq A$. Then $\emptyset \neq T_0 \subseteq L_2 \cap A$, and since $L_2 \subset A \dot\cup B$ is a limb, it must be that $L_2 \subset A$. Now, $K \subseteq L_1 \cup L_2 \subseteq A$, contradicting that $\{A, B\}$ covered K. □

9. Proof of Lemma 4.6

Let V be an n-set, let $\mathcal{C}$ be a strong cover of $\binom{V}{k}$ with skeleton $\mathcal{S} = \mathcal{Z}^*$, and let p, q, r, and R satisfy (4). We prove that every bone $S \in \mathcal{S}$ with $\deg_{\mathcal{C}}(S) < \alpha = \alpha(\mathcal{C})$ below the average has maximum size $|S| = k-1$. When $r = 0$, Fact 7.1 already proved this assertion, so it suffices to take $r \geq 1$. We proceed indirectly: *suppose that there exists a bone $S_0 \in \mathcal{S}$ with*

$$\deg_{\mathcal{C}}(S_0) < \alpha = \alpha(\mathcal{C}) \qquad \textit{but} \qquad |S_0| < k-1. \tag{54}$$

Proposition 7.2 and $r \geq 1$ guarantee that S_0 is large:

$$|S_0| \geq (r+k-1)/2 \geq k/2. \tag{55}$$

Using (54) and (55), we will find a vertex $u_0 \in V \setminus S_0$ satisfying both

(I) $\deg_{\mathcal{C}}(u_0) \geq \alpha$, *and*
(II) $\mathcal{C}^*_{u_0,S_0}$ *covers* $\binom{V}{k}$.

When so, (I) and (II) immediately contradict the optimality of $\mathcal{C}$, because Fact 4.10 guarantees

$$\omega(\mathcal{C}^*_{u_0,S_0}) \overset{\text{(c)}}{=} \omega(\mathcal{C}_{u_0,S_0}) - 1 - |S_0| \overset{\text{(b)}}{=} \omega(\mathcal{C}) + \deg_{\mathcal{C}}(S_0) - \deg_{\mathcal{C}}(u_0) \overset{(54),\,(\text{I})}{<} \omega(\mathcal{C}). \tag{56}$$

In other words, (54) is incorrect, which would prove Lemma 4.6. It remains only to guarantee (I) and (II).

With (II), we are in precisely the context of Section 8 with restrictions (i)–(iii). Indeed, to the bone S_0 of (54) satisfying $|S_0| < k-1$ (as in (i)), we seek to shift a single vertex $u = u_0 \in V \setminus S_0$ (as in (ii)), and we want $\mathcal{C}^*_{u,S_0}$ to cover $\binom{V}{k}$ (as in (iii)). For the moment, fix an arbitary $u \in V \setminus S_0$. Observation 8.1 guarantees $\mathcal{C}^*_{u,S_0}$ covers all $K \in \binom{V}{k}$ for which $u \notin K$ or $K \setminus S_0 \neq \emptyset$, and Proposition 8.4 guarantees that a remaining K is not covered in $\mathcal{C}^*_{u,S_0}$ if, and only if, $L = K \setminus \{u\}$ is a limb of $\mathcal{C}$. In summary,

$$\textit{every element } K \in \binom{V}{k} \textit{ which is uncovered in } \mathcal{C}^*_{u,S_0} \textit{ contains the vertex } u$$
$$\textit{and bears a limb } L = K \setminus \{u\} \textit{ necessarily containing the bone } S_0 \textit{ from (54).} \tag{57}$$

Since S_0 satisfies $|S_0| \geq k/2$ from (55), Proposition 8.6 guarantees that S_0 is contained within at most one limb. We therefore consider the following two cases:

Case 1 (*S_0 is contained within no limbs L of $\mathcal{C}$*)**.** Choose any $u_0 \in V \setminus S_0$ with $\deg_{\mathcal{C}}(u_0) \geq \alpha$, which exists by $\deg_{\mathcal{C}}(S_0) < \alpha$ from (54). Then (I) is satisfied by u_0, and (II) is satisfied by Case 1 and (57).

Case 2 (*S_0 is contained within precisely one limb L_0 of $\mathcal{C}$*)**.** Observation 8.5 guarantees that $L_0 = S_0 \dot\cup S_1 \dot\cup \dots \dot\cup S_t$ is a union of $t+1 \geq 2$ bones, one of which is S_0. Observe that $\deg_{\mathcal{C}}(S_1), \dots, \deg_{\mathcal{C}}(S_t) \geq \alpha$, for if $S_i \in \{S_1, \dots, S_t\}$ would be otherwise, Proposition 7.2 would apply to S_i precisely as it did with S_0 in (55), yielding $k - 1 = |L_0| \geq |S_0| + |S_i| \geq (k/2) + (k/2) = k$. Now, choose $u_0 \in L_0 \setminus S_0$ arbitrarily so that (I) is satisfied with $\deg_{\mathcal{C}}(u_0) \geq \alpha$. With u_0 now chosen, (II) is necessarily satisfied: if $K \in \binom{V}{k}$ is uncovered in $\mathcal{C}^*_{u_0,S_0}$, then (57) guarantees $L_1 = K \setminus \{u_0\}$ is a limb of $\mathcal{C}$ containing S_0, but $u_0 \in L_0$ and $u_0 \notin L_1$ ensure that $L_1 \neq L_0$ are distinct limbs containing S_0, contradicting Case 2. □

Appendix: Proof of Fact 4.10

We prove Fact 4.10 by elementary arguments using Definitions 4.7 and 4.9. Throughout this section, V, $\mathcal{C}$, $\mathcal{S}$, S, U, W, $\mathcal{C}_{U,S}$, $\mathcal{C}^*_{U,S}$, and $\mathcal{C}^{W,S}$ are given as in Definitions 4.7 and 4.9.

9.1. **Statement (a) of Fact 4.10.** For $1 \le |U| \le k-1 = |S|$, we show that $\mathcal{C}_{U,S}$ covers an arbitrary $K \in \binom{V}{k}$. For sake of argument, assume that $\{U,S\} \in \mathcal{C}_{U,S}$ does not already cover K, and proceed by induction on $|K \cap U|$. For $K \cap U = \emptyset$, let $\{A,B\} \in \mathcal{C}$ cover K. Then $\{A_{U,S}, B_{U,S}\} \in \mathcal{C}_{U,S}$ also covers K, because Definition 4.7 ensures $K \cap A_{U,S} = K \cap A$ and $K \cap B_{U,S} = K \cap B$. Inductively, let $u \in K \cap U$ and let $v \in S \setminus K$, where v exists by $|S| = k-1 = |K|-1$ and our assumption that $\{U,S\}$ does not cover K. Set $K_{u,v} = (K \setminus \{u\}) \cup \{v\}$, which like (43) satisfies $K \triangle K_{u,v} = \{u,v\}$. Observe that $K_{u,v}$ is not covered by $\{U,S\} \in \mathcal{C}_{U,S}$, since otherwise either K would be too (contrary to our assumption), or $K \subseteq U$ (contrary to $|U| \le k-1 < |K|$). Thus, by induction, some $\{A_{U,S}, B_{U,S}\} \in \mathcal{C}_{U,S}$ covers $K_{u,v}$, and we claim the same $\{A_{U,S}, B_{U,S}\} \in \mathcal{C}_{U,S}$ covers K. Indeed, since $v \in K_{u,v}$, which is covered by $\{A_{U,S}, B_{U,S}\} \in \mathcal{C}_{U,S}$, we take w.l.o.g. $v \in A_{U,S}$. Now, $v \in A_{U,S} \cap S$, and so Definition 4.7 ensures that $A_{U,S} = A \cup U \supseteq U \dot\cup S$. Now, $K \triangle K_{u,v} = \{u,v\} \subseteq U \dot\cup S \subseteq A_{U,S}$ gives that $K \subseteq A_{U,S} \dot\cup B_{U,S}$, where $u \in K \cap A_{U,S} \ne \emptyset$ and $K \cap B_{U,S} = K_{u,v} \cap B_{U,S} \ne \emptyset$.

9.2. **Statements (b) and (c) of Fact 4.10.** For each $v \in V$, recall from (23) that $\deg_{\mathcal{C}_{U,S}}(v) = 1 + \deg_{\mathcal{C}}(S)$ if $v \in U \dot\cup S$, and $\deg_{\mathcal{C}_{U,S}}(v) = \deg_{\mathcal{C}}(v)$ otherwise. Thus,

$$\begin{aligned}\omega(\mathcal{C}_{U,S}) &= \sum_{v\in V} \deg_{\mathcal{C}_{U,S}}(v) = \sum_{v \in U} \deg_{\mathcal{C}_{U,S}}(v) + \sum_{v\in S} \deg_{\mathcal{C}_{U,S}}(v) + \sum_{v \in V \setminus (S \dot\cup U)} \deg_{\mathcal{C}_{U,S}}(v) \\ &= |U|\big(1+\deg_{\mathcal{C}}(S)\big) + \sum_{v\in S}\big(1+\deg_{\mathcal{C}}(v)\big) + \sum_{v \in V\setminus (S\dot\cup U)} \deg_{\mathcal{C}}(v) \\ &= |U| + |S| + \sum_{v\in U} \deg_{\mathcal{C}}(S) + \sum_{v\in V\setminus U} \deg_{\mathcal{C}}(v) = \omega(\mathcal{C}) + |U| + |S| + \sum_{v \in U}\big(\deg_{\mathcal{C}}(S) - \deg_{\mathcal{C}}(u)\big).\end{aligned}$$

Now, Statement (c) of Fact 4.10 is trivial, since if $\mathcal{C}^*_{U,S}$ covers $\binom{V}{k}$, then so does $\mathcal{C}_{U,S}$, and whatever its weight, we have $\omega(\mathcal{C}^*_{U,S}) = \omega(\mathcal{C}_{U,S}) - |U| - |S|$ by construction.

9.3. **Statement (d) of Fact 4.10.** Statement (4) can be similarly established directly from Definition 4.9, but we infer it from Statements (a) and (b). For that, we construct $\mathcal{C}^{W,S}$ (the S-immersion of W in $\mathcal{C}$) indirectly as follows. Set $X = W \cup V$ and $\mathcal{C}^X = \{\{W,V\}\} \cup \mathcal{C}$ so that $\mathcal{C}^X$ covers $\binom{X}{k}$ by construction, and $\mathcal{C}^X_{W,S}$ (the S-shift of W in $\mathcal{C}^X$) covers $\binom{X}{k}$ by Statement (a) of Fact 4.10. We claim

$$\mathcal{C}^X_{W,S} = \{\{\emptyset, X\}\} \cup \mathcal{C}^{W,S}, \qquad \text{or equivalently,} \qquad \mathcal{C}^{W,S} = \mathcal{C}^X_{W,S} \setminus \{\{\emptyset, X\}\}, \tag{58}$$

which would imply that $\mathcal{C}^{W,S}$ covers $\binom{X}{k}$, because $\mathcal{C}^X_{W,S}$ covers $\binom{X}{k}$ while $\{\emptyset, X\} \in \mathcal{C}^X_{W,S}$ covers nothing. To see (58), note first that $\{W,V\} \in \mathcal{C}^X$ corresponds to $\{\emptyset, X\} \in \mathcal{C}^X_{W,S}$, since $S \cap W = \emptyset$, $S \subseteq V$, and Definition 4.7 give

$$W_{W,S} = W \setminus W = \emptyset \qquad \text{and} \qquad V_{W,S} = V \cup W = X.$$

Otherwise, for each $\{A,B\} \in \mathcal{C}$, we have $A \cap W = \emptyset$ and so Definitions 4.7 and 4.9 agree that either

$$A_{W,S} = A \cup W = A^{W,S} \qquad \text{or} \qquad A_{W,S} = A \setminus W = A = A^{W,S},$$

and similarly, $B_{W,S} = B^{W,S}$. Now, Statement (b) of Fact 4.10 gives

$$\omega(\mathcal{C}^X_{W,S}) = \omega(\mathcal{C}^X) + |W| + |S| + \sum_{w \in W} \left(\deg_{\mathcal{C}^X}(S) - \deg_{\mathcal{C}^X}(w)\right), \tag{59}$$

where by construction $\omega(\mathcal{C}^X) = \omega(\mathcal{C}) + |W| + |V| = \omega(\mathcal{C}) + |X|$, $\deg_{\mathcal{C}^X}(S) = \deg_{\mathcal{C}}(S) + 1$, and where $\deg_{\mathcal{C}^X}(w) = 1$ holds for each $w \in W$. Returning to (59), we infer

$$\omega(\mathcal{C}^X_{W,S}) = \omega(\mathcal{C}) + |X| + |S| + |W|\left(1 + \deg_{\mathcal{C}}(S)\right), \tag{60}$$

and applying (58) to (60), we infer

$$\omega(\mathcal{C}^{W,S}) + |X| = \omega(\mathcal{C}^X_{W,S}) = \omega(\mathcal{C}) + |X| + |S| + |W|\left(1 + \deg_{\mathcal{C}}(S)\right),$$

which implies the desired formula for $\omega(\mathcal{C}^{W,S})$.

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