BOUNDING THE STRONG CHROMATIC INDEX OF DENSE RANDOM GRAPHS

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Abstract. For a graph G, a strong edge coloring of G is an edge coloring in which every color class is an induced matching. The strong chromatic index of G, $\chi_s(G)$, is the smallest number of colors in a strong edge coloring of G. In [9], Z. Palka proved that if $p = p(n) = \Theta(n^{-1})$, then with high probability, $\chi_s(G(n,p)) = O(\Delta(G(n,p)))$. Recently in [12], V. Vu proved that if $n^{-1}(\ln n)^{1+\delta} \le p = p(n) \le n^{-\varepsilon}$ for any $0 < \varepsilon, \delta < 1$, then with high probability, $\chi_s(G(n,p)) = O((pn)^2/\ln(pn))$. In this note, we prove that if $p = p(n) > n^{-\varepsilon}$ for all $\varepsilon > 0$, then with $b = (1-p)^{-1}$, with high probability, $(1-o(1))\frac{p\binom{n}{2}}{\log_b n} \le \chi_s(G(n,p)) \le (2+o(1))\frac{p\binom{n}{2}}{\log_b n}$.

1. Introduction

For a finite simple graph $G = (V(G), E(G))$, a *strong edge coloring* of G is an edge coloring in which every color class is an induced matching. The *strong chromatic index* of G, $\chi_s(G)$, is the minimum number of colors k in a strong edge coloring of G. It is not difficult to see that every graph G satisfies $\chi_s(G) \le 2\Delta(G)^2 - 2\Delta(G) + 1$. Erdős and Nešetřil conjectured, however, that this bound would never be sharp.

Conjecture 1.1 (Erdős and Nešetřil). *For all graphs G,*

$$\chi_s(G) \le \begin{cases} \frac{5}{4}\Delta(G)^2 & \text{if } \Delta(G) \text{ is even,} \\ \frac{5}{4}\Delta(G)^2 - \frac{1}{2}\Delta(G) + \frac{1}{4} & \text{if } \Delta(G) \text{ is odd.} \end{cases}$$

Note that Conjecture 1.1, if true, would be best possible. Indeed, the "blown-up" pentagon $C_5(t)$ requires $\frac{5}{4}\Delta(C_5(t))^2$ colors in a strong edge coloring. (the graph $C_5(t)$ is obtained by replacing each vertex of the pentagon C_5 with a set of t independent vertices and replacing each edge of C_5 with the complete bipartite graph $K_{t,t}$) Conjecture 1.1 is trivial for $\Delta(G) = 2$ and open for graphs G satisfying $\Delta(G) \ge 4$ (cf. [1]).

Conjecture 1.1 seems very difficult and it was non-trivial to show that the upper bound $2\Delta(G)^2$ is not sharp. A question of Erdős and Nešetřil asked if there exists $\varepsilon > 0$ so that for all graphs G, $\chi_s(G) \le (2-\varepsilon)\Delta(G)^2$. Molloy and Reed [8] used sophisticated probabilistic techniques to affirmatively answer this question with $\varepsilon = .002$.

In this note, we consider strong edge colorings of the *random graph* $G(n,p)$ (cf. [6]). By the random graph $G(n,p)$, $0 \le p \le 1$, we mean the probability space consisting of the set of all $2^{\binom{n}{2}}$ graphs on vertex set $\{1, \dots, n\}$ with the probability of a graph H on $\{1, \dots, n\}$ and m edges being $p^m(1-p)^{\binom{n}{2}-m}$. For a graph property $\mathcal{P}$, we say that $G(n,p)$ has property $\mathcal{P}$ *with high probability* if $\lim_{n\to\infty} \text{Prob}[G(n,p)$ satisfies property $\mathcal{P}] = 1$. In what follows, we always write $b = (1-p)^{-1}$. As well, in the entirety of this paper, we assume $p \le c < 1$ for some constant c.

Z. Palka was the first to formally consider the strong chromatic index of the random graph $G(n,p)$.

Theorem 1.2 (Palka [9]). *Suppose $p = p(n) = \Theta(n^{-1})$. Then with high probability,*

$$\chi_s(G(n,p)) = O(\Delta(G(n,p))) = O(\ln n / \ln\ln n). \tag{1}$$

Up to the constant, (1) is easily seen to be best possible.

Recently, V. Vu proved the following theorem for a denser and broader range of $p = p(n)$.

Theorem 1.3 (Vu [12]). *Suppose $p = p(n)$ satisfies $n^{-1}(\ln n)^{1+\delta} \le p \le n^{-\varepsilon}$ for any constants $0 < \varepsilon, \delta < 1$. Then with high probability,*

$$\chi_s(G(n,p)) = O\left(\frac{(pn)^2}{\ln(pn)}\right). \tag{2}$$

We prove the following accompaniment to Theorems 1.2 and 1.3.

Theorem 1.4. *Suppose $p = p(n) > n^{-\varepsilon}$ for all $\varepsilon > 0$. Then with high probability,*

$$\chi_s(G(n,p)) \le (2+o(1))\frac{p\binom{n}{2}}{\log_b n}. \tag{3}$$

We continue with the following remark.

Remark 1.5. We mention that one can show from our proof that Theorem 1.4 holds also if $p > n^{-\varepsilon}$ for a suitably small constant $\varepsilon > 0$. As this extension is very slight and is already handled by Vu's result, we choose not to focus much to this case.

Observe that, up to the constants, both (2) and (3) are best possible. Indeed, for an arbitrary graph F, let $m_{\max}(F)$ denote the size of a largest induced matching in F. Then $|E(F)|/m_{\max}(F)$ is an easy lower bound for $\chi_s(F)$. The following fact quickly follows from the first moment method.

Fact 1.6. *Let $p = p(n) = \Omega(n^{-1})$.*

(*i*) *If $p = o(1)$, then with high probability,*

$$m_{\max}(G(n,p)) \le \frac{\ln np}{p}.$$

(*ii*) *If $p > n^{-\varepsilon}$ for all $\varepsilon > 0$, then with high probability,*

$$m_{\max}(G(n,p)) \le \log_b n.$$

Using Fact 1.6 and recalling that with high probability $|E(G(n,p))| = (1-o(1))p\binom{n}{2}$, we see that, up to the constants, both (2) and (3) are best possible.

Particular to (3), we see from Fact 1.6 (*ii*) for $p > n^{-\varepsilon}$ for all $\varepsilon > 0$ that with high probability,

$$\chi_s(G(n,p)) \geq (1-o(1))\frac{p\binom{n}{2}}{\log_b n}.$$

We mention that Vu's proof of Theorem 1.3 follows from more general and technical work using the *nibble method* (cf. [10]). Our proof of Theorem 1.4 is short and elementary and only uses a celebrated result of B. Bollobás [2] concerning the chromatic number of $G(n,p)$.

In the following section, we give our proof of Theorem 1.4. In the final section, we briefly close with a few concluding remarks.

2. Proof of Theorem 1.4

Our proof of Theorem 1.4 uses Bollobás' well known theorem concerning the chromatic number of $G(n,p)$. The following statement is a version of Bollobás' result which we slightly paraphrase from Theorem 7.14 of [6] (pp. 192). This version focuses to Bollobás' result in the case of constant edge probability.

Theorem 2.1 ([6]). *Let $0 < q < 1$ be a constant and set $d = (1-q)^{-1}$. Let $\mu > 0$ be given. Then, with probability at least*

$$1 - 2^m \exp\left\{-\frac{m^2}{\log^{10} m}\right\}, \tag{4}$$

the random graph $G(m,q)$ satisfies

$$\chi(G(m,q)) \leq \frac{m}{2\log_d m - 8\log_d \log_d m} \leq \frac{m}{2\log_d m}(1+\mu).$$

As we show below, Theorem 2.1 implies our Theorem 1.4 in the case when p is constant. To be formal, Theorem 2.1 implies the following lemma.

Lemma 2.2. *Let $0 < p < 1$ be a constant and write $b = (1-p)^{-1}$. For every $\gamma > 0$, with high probability,*

$$\chi_s(G(n,p)) \leq \frac{2p\binom{n}{2}}{\log_b n}(1+\gamma).$$

We then see that Lemma 2.2 establishes Theorem 1.4 in the case when p is constant.

Our proof of Lemma 2.2 below is easily extended to establish Theorem 1.4 in the range $p \geq n^{-\varepsilon}$ for all $\varepsilon > 0$. The proof is, in fact, essentially the same except that instead of appealing to Theorem 2.1 above, one appeals to Theorem 5 and Corollary 6 in [2].

Proof of Lemma 2.2.

We begin with some notation. Let j be an integer satisfying $1 \leq j \leq n/2$. Define $\mathcal{M}_j$ as the set of all matchings in K_n with j edges. Clearly,

$$|\mathcal{M}_j| \leq n^{2j}.$$

Define

$$\mathcal{M}^{\mathrm{big}} = \bigcup_{j=n/\log_b^2 n}^{n/2} \mathcal{M}_j.$$

Clearly,

$$|\mathcal{M}^{\mathrm{big}}| \leq n^{n+1}. \tag{5}$$

We continue with some notation. Let G be a fixed graph. Let $M_j = \{e_1, \dots, e_j\} \in \mathcal{M}_j$ be fixed. Define the graph $H_{M_j}(G)$ on vertex set M_j as

$$H_{M_j}(G) = \{\{e_k, e_l\} : \text{ there exists } e \in G \text{ with } e_k \cap e \neq \emptyset \neq e_l \cap e\}.$$

We now define an event important to our argument. Let $G \in G(n,p)$. Let $\mu > 0$ and $M_j \in \mathcal{M}^{\mathrm{big}}$ be fixed. Let $\mathcal{Q}_{M_j}(\mu)$ denote the event that

$$\chi(H_{M_j}(G(n,p))) > (1+\mu)\frac{2j}{\log_b j}.$$

Let

$$\mathcal{Q}(\mu) = \bigcup_{M_j \in \mathcal{M}^{\mathrm{big}}} \mathcal{Q}_{M_j}(\mu).$$

We proceed with the following proposition.

Proposition 2.3. *For each $\mu > 0$,*

$$\mathrm{Prob}[\mathcal{Q}(\mu)] = o(1).$$

Before continuing with our argument for Lemma 2.2, we prove Proposition 2.3.

Proof of Proposition 2.3.

Let $\mu > 0$ be given. Fix $M_j \in \mathcal{M}^{\mathrm{big}}$ and write $M_j = \{e_1, \dots, e_j\}$. Set $p' = 1 - (1-p)^4$ and $b' = (1-p')^{-1} = b^4$. Clearly, $0 < p' < 1$ is a constant. Observe that for all $x > 0$, $\log_b x = 4 \log_{b'} x$.

For fixed $1 \leq k < l \leq j$, observe $\{e_k, e_l\} \in H_{M_j}(G(n,p))$ with probability p'. We therefore identify $H_{M_j}(G(n,p))$ with the random graph $G(j,p')$ where with $j \geq n/\log_b^2 n$, $j \to \infty$ as $n \to \infty$. The event $\mathcal{Q}_{M_j}(\mu)$ is identified with the event that

$$\chi(G(j,p')) > (1+\mu)\frac{2j}{\log_b j} = (1+\mu)\frac{j}{2\log_{b'} j}.$$

By Theorem 2.1, we see

$$\mathrm{Prob}[\mathcal{Q}_{M_j}(\mu)] < 2^j \exp\left\{-\frac{j^2}{\ln^{10} j}\right\} < 2^n \exp\left\{-\frac{n^2}{\log_b^4 n \ln^{10}(n/\log_b^2 n)}\right\}. \tag{6}$$

Using (5) and (6), we see

$$\mathrm{Prob}[\mathcal{Q}(\mu)] \leq n^{n+1} 2^n \exp\left\{-\frac{n^2}{\log_b^4 n \ln^{10}(n/\log_b^2 n)}\right\} = o(1).$$

Thus, Proposition 2.3 is proved. $\square$

We now conclude our proof of Lemma 2.2. Let $\mathcal{A}$ be the event that

$$\Delta(G(n,p)) \leq pn(1+n^{-1/4}). \tag{7}$$

As is well known, it easily follows from the Chernoff inequality that $\mathrm{Prob}(\mathcal{A}) = 1-o(1)$. Consequently, from Proposition 2.3 we thus infer that for all $\gamma > 0$,

$$\mathrm{Prob}[\mathcal{A} \cap \neg\mathcal{Q}(\gamma/2)] = 1 - o(1). \tag{8}$$

The following claim, together with (8), implies Lemma 2.2.

Claim 2.4. *Let $\gamma > 0$ be given. Let $G \in G(n,p)$ satisfy $G \in \mathcal{A} \cap \neg\mathcal{Q}(\gamma/2)$. Then*

$$\chi_s(G) \leq \frac{2p\binom{n}{2}}{\log_b n}(1+\gamma).$$

We prove Claim 2.4 below.
Proof of Claim 2.4.
Let $\gamma > 0$ be given. Let $G \in G(n,p)$ satisfy $G \in \mathcal{A} \cap \neg\mathcal{Q}(\gamma/2)$. Let $G = M^{(j_1)} \cup \ldots \cup M^{(j_t)}$ be any proper edge coloring of G with t as small as possible. By Vizing's Theorem, $t \leq \Delta(G)+1$. With $G \in \mathcal{A}$, we see by (7) that

$$t \leq pn(1+2n^{-1/4}). \tag{9}$$

Let $\mathcal{M}^{\mathrm{big}}(G) = \mathcal{M}^{\mathrm{big}} \cap \{M^{(j_1)}, \ldots, M^{(j_t)}\}$ and let $\mathcal{M}^{\mathrm{small}}(G) = \{M^{(j_1)}, \ldots, M^{(j_t)}\} \setminus \mathcal{M}^{\mathrm{big}}(G)$. Since $G \in \neg\mathcal{Q}(\gamma/2)$, every $M^{(j)} \in \mathcal{M}^{\mathrm{big}}(G)$ satisfies

$$\chi(H_{M^{(j)}}(G)) \leq \left(1+\frac{\gamma}{2}\right)\frac{2|M^{(j)}|}{\log_b |M^{(j)}|}.$$

Fix $M^{(j)} \in \mathcal{M}^{\mathrm{big}}(G)$. Observe that each independent set in $H_{M^{(j)}}(G)$ corresponds to an induced matching in G. Using the least number of colors, we color each independent set of $H_{M^{(j)}}(G)$ with its own color; we need only $(1+\gamma/2)\frac{2|M^{(j)}|}{\log_b |M^{(j)}|}$ colors.

Over all $M^{(j)} \in \mathcal{M}^{\mathrm{big}}(G)$, color each independent set of $H_{M^{(j)}}(G)$ with its own color (where we choose pairwise disjoint palettes over different $M^{(j)} \in \mathcal{M}^{\mathrm{big}}(G)$). Note that, at the most, we use

$$\sum_{M^{(j)} \in \mathcal{M}^{\mathrm{big}}(G)} \left(1+\frac{\gamma}{2}\right)\frac{2|M^{(j)}|}{\log_b |M^{(j)}|} \leq t\left(1+\frac{\gamma}{2}\right)\frac{n}{\log_b(n/\log_b^2 n)}$$

colors. Using (9), we see we use at most

$$\left(1+\frac{2\gamma}{3}\right)\frac{pn^2}{\log_b n} \tag{10}$$

colors.

Now, for each $M^{(j)} \in \mathcal{M}^{\mathrm{small}}(G)$, simply color each edge of $M^{(j)}$ with a new and unique color. As $|M^{(j)}|$ is small, we need few colors. Doing this over all $M^{(j)} \in \mathcal{M}^{\mathrm{small}}(G)$ (and using pairwise disjoint palettes each time) requires at most

$$t\frac{n}{\log_b^2 n} \leq \frac{2pn^2}{\log_b^2 n} \tag{11}$$

colors.

Combining (10) and (11), we have a strong edge coloring of G using no more than

$$\left(1+\frac{2\gamma}{3}\right)\frac{pn^2}{\log_b n} + \frac{2pn^2}{\log_b^2 n} \leq \frac{2p\binom{n}{2}}{\log_b n}(1+\gamma)$$

colors. This proves Claim 2.4 and hence, Lemma 2.2. □

3. Concluding Remarks

In this section, we discuss some issues related to Theorem 1.4.

A *graph property* $\mathcal{P}$ is an infinite class of graphs closed under isomorphism. For a graph property $\mathcal{P}$, let $\mathcal{P}_n$ denote the set of all graphs from $\mathcal{P}$ which are on vertex set $\{1, \dots, n\}$. Let $\mathcal{P}_{\leq n} = \bigcup_{i=1}^n \mathcal{P}_i$. Let $\mathcal{U}$ denote the trival property consisting of all graphs. We say *almost all graphs belong to* $\mathcal{P}$ if

$$\frac{|\mathcal{P}_{\leq n}|}{|\mathcal{U}_{\leq n}|} = 1 - o(1).$$

Setting $p = 1/2$ and applying Theorem 1.4, we obtain the following easy corollary.

Corollary 3.1. *Almost all graphs satisfy Conjecture 1.1.*

In [4], the problem of estimating $\chi_s(G)$ was studied for a class of so-called *pseudo-random* bipartite graphs. These graphs are obtained from and identified with the well-known Szemerédi Regularity Lemma (cf. [7], [11]). We define these graphs precisely below.

For a graph G, let X and Y be two nonempty disjoint subsets of $V(G)$ and let $E_G(X,Y) = \{\{x,y\} \in E(G) : x \in X,\ y \in Y\}$ and $e_G(X,Y) = |E_G(X,Y)|$. Define the *density* of the pair X, Y by

$$d_G(X,Y) = \frac{e_G(X,Y)}{|X||Y|}.$$

For constant d and $\varepsilon > 0$, we say that a bipartite graph $G = (U \cup V, E)$ is (d,ε)*-regular* if for all $U' \subseteq U$, $|U'| > \varepsilon|U|$, and all $V' \subseteq V$, $|V'| > \varepsilon|V|$, the following holds,

$$|d - d_G(U',V')| < \varepsilon. \tag{12}$$

If $G = (U \cup V, E)$ is (d,ε)-regular for some $0 \leq d \leq 1$, then G is called ε*-regular*.

Bipartite graphs which are (d,ε)-regular, $0 < \varepsilon \ll d$ (i.e. ε is sufficiently smaller than d), have very uniform edge distributions and therefore behave, in some senses, in a "random-like" manner. While stated precisely in, for example, [7] and [11], Szemerédi's Regularity Lemma essentially says that the edge set of any large enough graph may be decomposed into a bounded number of ε-regular bipartite subgraphs.

The following theorem was proved in [4].

Theorem 3.2. *For every $0 < d < 1$ and $\mu > 0$, there exist $\varepsilon > 0$ and integer n_0 such that if $G = (U \cup V, E)$ is a (d, ε)-regular bipartite graph with $|U| = |V| = n \geq n_0$, then*

$$\chi_s(G) \leq \mu \Delta(G)^2.$$

Note that, like the random graph $G(n,p)$, (d, ε)-regular graphs (with appropriately given parameters) easily satisfy Conjecture 1.1. As well, recall that Theorems 1.2, 1.3 and 1.4 for $G(n,p)$ are all, up to the constant, best possible. We mention that an easy probabilistic construction is given in [4] showing that, up to the constant, Theorem 3.2 is also best possible.

As a final topic, we state the following conjecture.

Conjecture 3.3. *For $0 < p < 1$ constant,*

$$\chi_s(G(n,p)) = \frac{p\binom{n}{2}}{\log_b n}(1 + o(1)).$$

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