Strong Edge Colorings of Uniform Graphs

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Abstract

For a graph $G = (V(G), E(G))$, a strong edge coloring of G is an edge coloring in which every color class is an induced matching. The strong chromatic index of G, $\chi_s(G)$, is the smallest number of colors in a strong edge coloring of G. The strong chromatic index of the random graph $G(n, p)$ was considered in [3], [4], [12], and [16]. In this paper, we consider $\chi_s(G)$ for a related class of graphs G known as uniform or ϵ-regular graphs. In particular, we prove that for $0 < \epsilon \ll d < 1$, all (d, ϵ)-regular bipartite graphs $G = (U \cup V, E)$ with $|U| = |V| \geq n_0(d, \epsilon)$ satisfy $\chi_s(G) \leq \zeta(\epsilon)\Delta(G)^2$, where $\zeta(\epsilon) \to 0$ as $\epsilon \to 0$ (this order of magnitude is easily seen to be best possible). Our main tool in proving this statement is a powerful packing result of Pippenger and Spencer [11].

Key words: Strong chromatic index, the regularity lemma

1 Introduction

For a finite simple graph $G = (V(G), E(G))$, a *strong edge coloring* of G is an edge coloring in which every color class is an induced matching. The *strong chromatic index* of G, $\chi_s(G)$, is the minimum number of colors k in a strong edge coloring of G.

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The following conjecture of Erdős and Nešetřil (cf. [5]) is central to the area of strong chromatic index problems: $\chi_s(G) \leq \frac{5}{4}\Delta(G)^2$ *holds for every graph* G. This conjecture has proven to be difficult and only very few partial results are known.

Studying the strong chromatic index for special classes of graphs has yielded interesting results. We focus our discussion to classes of random and pseudo-random graphs and begin with the former.

In [3], [4], [12], and [16], the strong chromatic index of the random graph $G(n,p)$ was studied (cf. [7]). In particular, Z. Palka [12] showed that if $p = \Theta(n^{-1})$, then *asymptotically almost surely* (cf. [7]), $\chi_s(G(n,p)) = \Theta(\Delta(G(n,p))$. V. Vu [16] more recently showed that for positive $\delta, \epsilon < 1$, if $n^{-1}\log^{1+\delta} n \leq p \leq n^{-\epsilon}$, then asymptotically almost surely, $\chi_s(G(n,p)) = \Theta\left(\frac{\Delta(G(n,p))^2}{\ln \Delta(G(n,p))}\right)$. In [3], the current authors recently extended Vu's result to the range $p \geq n^{\epsilon_0}$ for a suitable $\epsilon_0 > 0$.

In this paper, we consider an analogous problem of estimating $\chi_s(G)$ for so-called *pseudo-random* or *uniform* graphs G. As we define them below, these are graphs obtained from and identified with the well-known Szemerédi Regularity Lemma (cf [9], [14]). Uniform graphs were studied by Alon, Rödl, and Rucinski [1] who estimated the number of perfect matchings of a super-regular pair (see below for definitions) and by Frieze [6] who estimated the number of hamiltonian cycles and perfect matchings in uniform graphs.

For a bipartite graph $G = (U \cup V, E)$, let $U' \subseteq U$ and $V' \subseteq V$ be two nonempty sets, and let $E_G(U', V') = \{\{u, v\} \in E(G) : u \in U',\ v \in V'\}$ and $e_G(U', V') = |E_G(U', V')|$. Define the *density* of the graph $(U' \cup V', E_G)$ by $d_G(U', V') = e_G(U', V')|U'|^{-1}|V'|^{-1}$. For constant d and $\epsilon > 0$, we say that a bipartite graph $G = (U \cup V, E)$ is (d, ϵ)-*regular* if for all $U' \subseteq U$, $|U'| > \epsilon|U|$, and all $V' \subseteq V$, $|V'| > \epsilon|V|$, we have $|d - d_G(U', V')| < \epsilon$. If $G = (U \cup V, E)$ is (d, ϵ)-regular for some $0 \leq d \leq 1$, then G is called ϵ-*regular*. Bipartite graphs which are (d, ϵ)-regular, $0 < \epsilon \ll d$, have uniform edge distributions and therefore behave, in some senses, in a "random-like" manner.

Our theorem is stated as follows.

Theorem 1.1 (Main Theorem) *For every* $0 < d < 1$ *and* $\mu > 0$, *there exist* $\epsilon > 0$ *and integer* n_0 *such that if* $G = (U \cup V, E)$ *is a* (d, ϵ)*-regular bipartite graph with* $|U| = |V| \geq n_0$, *then* $\chi_s(G) \leq \mu\Delta(G)^2$.

As any (d, ϵ)-regular bipartite graph $G = (U \cup V, E)$, $|U| = |V| = n$, satisfies $\Delta(G) \geq (d - \epsilon)n$, it suffices to prove Theorem 1.1 in the following form.

Theorem 1.2 *For every* $0 < d < 1$ *and* $\mu > 0$, *there exist* $\epsilon > 0$ *and integer* n_0 *such that if* $G = (U \cup V, E)$ *is a* (d, ϵ)*-regular bipartite graph with* $|U| =$

$|V| = n \geq n_0$, *then* $\chi_s(G) \leq \mu n^2$.

The following observation shows that the order of magnitude for the upper bound in Theorem 1.1 is best possible.

Fact 1 *Let* $0 < d < 1$ *be fixed. For all* $\epsilon > 0$ *and integers* n*, there exists a* (d, ϵ)*-regular bipartite graph* $G_0 = (U \cup V, E)$*,* $|U| = |V| = N \geq n$*, satisfying* $\chi_s(G_0) \geq \frac{\epsilon^2}{2}\Delta(G_0)^2$.

The observation in Fact 1, in various forms, has been noted by various researchers (e.g. Prof. T. Łuczak [8] and also an anonymous referee). The proof of Fact 1 is easy and we present it at the end of Section 3.

The rest of the paper is organized as follows. In Section 2, we state some background material we use to prove Theorem 1.2. In Section 3, we prove Theorem 1.2 and verify Fact 1.

1.1 Acknowledgement

We wish to thank the referees for suggestions which lead to simplified details in this paper.

2 Definitions and Facts

In this section, we give some background material we use to prove Theorem 1.1. We begin our discussion with basic notation and considerations. For a graph $G = (V(G), E(G))$ and a vertex $v \in V(G)$, let $N(v) = \{x \in V(G) : \{v, x\} \in G\}$ and set $\deg(v) = |N(v)|$. In all that follows, graphs $G = (V, E)$ are identified with their edge sets. For convenience of calculations, we use the convention $s = (a \pm b)t$ to mean $t(a - b) \leq s \leq t(a + b)$.

2.1 (d, ϵ)-regular bipartite graphs

We begin with the following well-known fact which may be found as Fact 1.3 in [9].

Fact 2 *Let* $G = (U \cup V, E)$ *be a* (d, ϵ)*-regular bipartite graph. Then, all but* $2\epsilon|U|$ *vertices* $u \in U$ *and all but* $2\epsilon|V|$ *vertices* $v \in V$ *satisfy, respectively,*

$$\deg(u) = (d \pm \epsilon)|V|, \quad \deg(v) = (d \pm \epsilon)|U|. \tag{1}$$

We use following definition (cf. Definition 1.6 of [9]).

Definition 3 ((d,ϵ)-super regularity) Let $G = (U \cup V, E)$ be a (d,ϵ)-regular bipartite graph. We say that G is (d,ϵ)-*super-regular* if all vertices $u \in U$ and all vertices $v \in V$ satisfy (1).

We continue with the following fact.

Fact 4 Let $G = (U \cup V, E)$ be a (d,ϵ)-regular bipartite graph where, say, $3\epsilon < 1 - d$, and $|U| = |V| = n$. Then G has a (d,ϵ')-super-regular induced bipartite subgraph $G_0 = G_0[U_0 \cup V_0]$, where $\epsilon' = \frac{6\epsilon}{d}$ and $|U_0| = |V_0| > (1-2\epsilon)n$.

The precise details of the proof of Fact 4 are extremely standard and so we omit them. The idea behind proving Fact 4, however, is to delete the vertices $u \in U$ and $v \in V$ not satisfying (1). Then, appealing to the *Slicing Lemma*, Fact 1.5 of [9], Fact 4 immediately follows.

2.2 Hypergraph Packings

At the heart of our argument for Theorem 1.1 lies an application of the following strong theorem of Pippenger and Spencer (cf. [11]). Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be an l-uniform hypergraph. For a vertex $u \in V(\mathcal{H})$, define the *degree* of the vertex u, $\deg(u)$, as $\deg(u) = |\{h \in E(\mathcal{H}) : u \in h\}|$. Set $\delta(\mathcal{H})$ to be the minimum degree of any vertex in $\mathcal{H}$ and set $\Delta(\mathcal{H})$ to the maximum degree of any vertex in $\mathcal{H}$. For a pair of distinct vertices $u, v \in V(\mathcal{H})$, set $\text{codeg}(\{u,v\}) = |\{h \in E(\mathcal{H}) : u, v \in h\}|$ and let $\text{codeg}(\mathcal{H}) = \max_{u,v \in V(\mathcal{H}), u \neq v} \text{codeg}(\{u,v\})$. Then the theorem of [11] is stated as follows.

Theorem 2.1 (Pippenger, Spencer, [11]) *For all positive integers l and positive constants γ, there exists $\epsilon = \epsilon(l,\gamma)$ so that if $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is an l-uniform hypergraph with minimum degree $\delta(\mathcal{H})$ satisfying $\delta(\mathcal{H}) > (1-\epsilon)\Delta(\mathcal{H})$ and $\text{codeg}(\mathcal{H}) \leq \epsilon\Delta(\mathcal{H})$, then there exists a set $M \subseteq E(\mathcal{H})$, $h \cap h' = \emptyset$ for every $h \neq h'$ in M, which covers all but $\gamma|V(\mathcal{H})|$ vertices of $\mathcal{H}$.*

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. The following theorem, combined with Fact 4, almost immediately implies Theorem 1.2.

Theorem 3.1 *For every $0 < d < 1$ and every $\zeta > 0$, there exist $\epsilon > 0$ and integer n_0 such that if $G = (U \cup V, E)$ is a (d,ϵ)-super-regular bipartite graph* *with* $|U| = |V| = n \geq n_0$*, then* $\chi_s(G) \leq \zeta n^2$*.*

In view of Theorem 3.1 and Fact 4, we may prove Theorem 1.2 by producing a promised strong edge coloring in "two rounds". Indeed, Fact 4 guarantees a large (d, ϵ')-super-regular induced subgraph G_0 of G. Theorem 3.1 guarantees G_0 admits strong edge colorings using few colors. Fix one such coloring. As $G \setminus G_0$ is small, we may greedily color the remaining edges. As the subgraph G_0 of G is *induced*, the greedy coloring of $G \setminus G_0$ does not disturb the strong edge coloring of G_0 guaranteed by Theorem 3.1.

It remains to prove Theorem 3.1. We make preparations to that end in what follows.

3.1 Setting up the argument of Theorem 3.1

For an integer $k \geq 1$, graph G and edge $e \in G$, define $M_k(e, G)$ to be the set of all induced matchings of size k containing edge e. Set $m_k(e, G) = |M_k(e, G)|$.

To prove Theorem 3.1, we use the following well-known fact.

Lemma 3.2 *Let* $0 < d < 1$ *be given. For every integer* $k \geq 1$*, for every* $\rho > 0$*, there exists* $\epsilon > 0$ *so that if* $G = (U \cup V, E)$ *is a* (d, ϵ)*-super regular bipartite graph,* $|U| = |V| = n \geq n_0(d, k, \rho, \epsilon)$*, then for all* $e \in G$*,*

$$m_k(e, G) = d^{k-1}(1-d)^{(k-1)^2-(k-1)} \frac{n^{2(k-1)}}{(k-1)!}(1 \pm \rho).$$

"Counting results" for ϵ-regular graphs are well-studied by many researchers. We refer the reader to one of the first papers in this area, [13]. The proof of Lemma 3.2, while not trivial, is extremely standard and we therefore omit it. The details of this proof are given in full in [2].

3.2 Proof of Theorem 3.1

Let $0 < d < 1$ and $\zeta > 0$ be given. Set $\gamma = \zeta/2$ and $l = \lceil \frac{1}{\gamma} \rceil$. Let $\epsilon_{2.1} = \epsilon_{2.1}(l, \gamma)$ be that constant guaranteed by Theorem 2.1 for the parameters l and γ. For $k = l$ and $\rho = \frac{\epsilon_{2.1}}{2}$, let $\epsilon = \epsilon_{3.2}(d, l, \frac{\epsilon_{2.1}}{2})$ be that constant guaranteed by Corollary 3.2. Let $G = (U \cup V, E)$ be a (d, ϵ)-super-regular bipartite graph where $|U| = |V| = n$. We show that $\chi_s(G) \leq \zeta n^2$.

To that end, with $l = \lceil \frac{1}{\gamma} \rceil$, define auxiliary l-uniform hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$

to have vertex set $V(\mathcal{H}) = G$, the edge set of G, and $E(\mathcal{H}) = M_l(G)$, the set of all induced matchings in G of size l. For $e \in V(\mathcal{H})$, note that $\deg_{\mathcal{H}}(e) = m_l(e, G)$. With $\epsilon = \epsilon_{3.2}(d, l, \frac{\epsilon_{2.1}}{2})$, we infer from Corollary 3.2 that for every $e \in V(\mathcal{H})$,

$$\deg_{\mathcal{H}}(e) = d^{l-1}(1-d)^{(l-1)^2-(l-1)} \left(\frac{n^{2(l-1)}}{(l-1)!}\right)(1 \pm \frac{\epsilon_{2.1}}{2}).$$

In particular, we see

$$d^{l-1}(1-d)^{(l-1)^2-(l-1)} \left(\frac{n^{2(l-1)}}{(l-1)!}\right)(1 - \frac{\epsilon_{2.1}}{2}) \leq \delta(\mathcal{H}),$$

$$\Delta(\mathcal{H}) \leq d^{l-1}(1-d)^{(l-1)^2-(l-1)} \left(\frac{n^{2(l-1)}}{(l-1)!}\right)(1 + \frac{\epsilon_{2.1}}{2}),$$

and consequently,

$$\delta(\mathcal{H}) \geq \frac{1 - \frac{\epsilon_{2.1}}{2}}{1 + \frac{\epsilon_{2.1}}{2}} \Delta(\mathcal{H}) > (1 - \epsilon_{2.1})\Delta(\mathcal{H}).$$

Clearly, $\operatorname{codeg}(\mathcal{H}) \leq n^{2(l-2)}$, which with n sufficiently large satisfies $\operatorname{codeg}(\mathcal{H}) < \epsilon_{2.1}\Delta(\mathcal{H})$. With $\epsilon_{2.1} = \epsilon_{2.1}(l, \gamma)$, we apply Theorem 2.1 to $\mathcal{H}$ to conclude that there exists a set $\{h_1, \ldots, h_t\} \subset E(\mathcal{H})$, $h_i \cap h_j = \emptyset$ for all $1 \leq i < j \leq t$, which covers all but $\gamma|V(\mathcal{H})|$ vertices $e \in V(\mathcal{H})$. Note that $tl \leq |V(\mathcal{H})| = |E(G)|$ trivially follows.

We now give the strong edge coloring of G using no more than the maximum number of colors required by Lemma 3.1. The edge classes $\{h_1, \ldots, h_t\}$ constitue t color classes in our coloring. Let $X = \bigcup_{1 \leq i \leq t} \{e | e \in h_i\}$. Then the singelton classes $\{\{e\} | e \in E(G) \setminus X\}$ constitue the remaining coloring classes in our coloring. Since there are at most $\gamma|V(\mathcal{H})| = \gamma|E(G)|$ edges in $E(G) \setminus X$ the number of colors used in the above colorings is at most

$$t + \gamma|E(G)| \leq \frac{|E(G)|}{l} + \gamma|E(G)| \leq 2\gamma n^2,$$

where the last inequality follows from the fact that $l = \lceil \frac{1}{\gamma} \rceil$. With $\gamma = \zeta/2$, we see that at most ζn^2 colors have been used. It is easy to see that the obtained coloring is a strong edge coloring of $E(G)$. $\square$

3.3 Proof of Fact 1

Let $0 < d < 1$ be given along with ϵ and integer n. We produce a graph G_0 satisfying the conclusion of Fact 1. Indeed, fix disjoint sets U and V with

$|U| = |V| = N$ where $N \geq n$ is a sufficiently large integer. Take any $(d, \epsilon/2)$-regular bipartite graph G on $U \cup V$. (the existence of such a graph is easily established by the probabilistic method provided N is sufficiently large) Now, fix any $U_0 \subset U$ where $|U_0| = \frac{\epsilon^2}{2}|U|$. Define the graph G_0 on $U \cup V$ by $G_0 = G \cup K[U_0, V]$. In other words, G_0 is obtained from G by replacing the edges $G[U_0, V]$ with the complete bipartite graph $K[U_0, V]$. Clearly, $\Delta(G_0) = |V| = N$ and

$$\chi_s(G_0) \geq \frac{\epsilon^2}{2} N^2 = \frac{\epsilon^2}{2} \Delta(G_0)^2.$$

What remains to be shown is that G_0 is (d, ϵ)-regular. Indeed, let $U' \subseteq U$ and $V' \subseteq V$ be given, $|U'| > \epsilon|U|$ and $|V'| > \epsilon|V|$. Set $U_0' = U' \cap U_0$. Then

$$d_G(U', V') \leq d_{G_0}(U', V') \leq d_G(U', V') + \frac{|U_0'|}{|U'|}.$$

Since $|U_0'| \leq (\epsilon^2/2)|U|$,

$$d_G(U', V') \leq d_{G_0}(U', V') \leq d_G(U_1', V') + \frac{\epsilon}{2}. \tag{2}$$

As $|U'| > \frac{\epsilon}{2}|U|$ and $|V'| > \frac{\epsilon}{2}|V|$, we see from the $(d, \epsilon/2)$-regularity of G that $|d_G(U', V') - d| < \frac{\epsilon}{2}$. Consequently, in (2), we see $|d_{G_0}(U', V') - d| < \epsilon$. This proves Fact 1. □

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