A NOTE ON CODEGREE PROBLEMS FOR HYPERGRAPHS

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ABSTRACT. A recent question of S. Abbasi provides a degree condition on a hypergraph and asks if every 3-uniform hypergraph $\mathcal{G}$ satisfying this condition admits a vertex covering by copies of $K_4^{(3)}$, the complete clique on 4 vertices. In this note, we answer this question negatively, and in turn, ask whether every $\mathcal{G}$ satisfying the hypothesis of Abbasi's original question need even contain one copy of $K_4^{(3)}$ as a subhypergraph. While our question remains open, we prove that if true, Abbasi's degree condition would be best possible.

1. Introduction

Packing and covering problems for graphs belong to some of the most interesting yet difficult questions considered in graph theory. In general, a packing problem can be formulated as follows: *Given a fixed graph H on k vertices and a graph G on n vertices, what conditions on G guarantee a collection within $\mathcal{G}$ of $\lfloor \frac{n}{k} \rfloor$ vertex-disjoint copies of H?* An early result, due to Hajnal and Szemerédi [HS], relates the packing of copies of $H = K_k$, the clique on k vertices, to the minimum degree $\delta(G)$ of G. Specifically, their result states that any graph G on $n \geq n_0(k)$ vertices satisfying $\delta(G) \geq \left(1 - \frac{1}{k}\right) n$ contains $\lfloor \frac{n}{k} \rfloor$ vertex-disjoint copies of K_k. Using the well-known Regularity Lemma of Szemerédi (cf. [KS], [Sz]), Alon and Yuster [AY1] and [AY2] gave an "approximate" extension of the result of Hajnal and Szemerédi for a general fixed graph H. Using the Blow-up Lemma (cf. [KSS]), Komlós, Sárközy and Szemerédi [KSS1] further extended [AY2] in the following theorem.

Theorem 1.1. *For every graph H, there is a constant K so that any graph G on $n \geq n_0(H)$ vertices satisfying $\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n$ contains a collection of vertex-disjoint copies of H covering all but at most K vertices.*

One can consider similar packing problems for l-uniform hypergraphs. One such question is to ask for a degree condition which will guarantee a covering of a hypergraph $\mathcal{G}$ with vertex-disjoint copies of a fixed hypergraph $\mathcal{H}$. In this note, we are interested in the case when $l = 3$ (triple systems) and $\mathcal{H} = K_4^{(3)}$, the complete triple system on four vertices. Note, however, that one can consider two types of vertex degree in a triple system. For a vertex v, we can define the *degree* of v, $\deg(v)$, as

$$\deg(v) = |\{e \in \mathcal{G} | v \in e\}|,$$

and for vertices u, v, $u \neq v$, we can define the *codegree* of the pair $\{u, v\}$ as

$$\text{codeg}(\{u, v\}) = |\{e \in \mathcal{G} | u, v \in e\}|.$$

Set

$$\delta(\mathcal{G}) = \min_{v} \deg(v),$$

and define the *codegree* of $\mathcal{G}$, $\text{codeg}(\mathcal{G})$, to be

$$\text{codeg}(\mathcal{G}) = \min_{u \neq v} \text{codeg}(\{u, v\}).$$

Although $\delta(\mathcal{G})$ is certainly an interesting parameter to consider, it seems that the codegree of a triple system $\mathcal{G}$ is a parameter which implies more structure on $\mathcal{G}$.

The following problem, recently asked by S. Abbasi [A], relates the codegree of a triple system $\mathcal{G}$ with packing copies of $K_4^{(3)}$.

Problem 1.2. *Does every triple system $\mathcal{G}$ on n vertices, $n \equiv 0 \pmod 4$, satisfying $\text{codeg}(\mathcal{G}) \geq \frac{n}{2}$ admit a covering by a collection of vertex-disjoint copies of $K_4^{(3)}$?*

We will show that the answer to the above problem is negative by constructing, for every $\epsilon > 0$, a triple system $\mathcal{G}$ with $\text{codeg}(\mathcal{G}) \geq \left(\frac{3}{5} - \epsilon\right) n$ which has the property that every vertex-disjoint collection of copies of $K_4^{(3)}$ within $\mathcal{G}$ avoids many vertices.

Although the answer to Problem 1.2 is negative, it is interesting to ask whether $\text{codeg}(\mathcal{G}) \geq \frac{n}{2}$ ensures that $\mathcal{G}$ contains even one copy of $K_4^{(3)}$. While the answer is unknown and appears to be difficult, we conjecture that the affirmative statement is indeed the case, and prove that, if true, this would be best possible. This problem relates to the famous Turán problem for triple systems where one asks for the minimum number of triples on n vertices which will force a copy of $K_4^{(3)}$. P. Turán (cf. [Si]) conjectured that this number is $\frac{5}{9}\binom{n}{3}(1 + o(1))$, and there are many constructions known ([B], [K]) which show that the number of such triples on n vertices must be at least this large. However, in all these constructions for the Turán problem, the codegree is much smaller than $n/2$ (asymptotically $n/3$). On the other hand, in the case when $\text{codeg}(\mathcal{G}) \geq \frac{n}{2}$, the number of triples is only guaranteed to be at least $\frac{1}{2}\binom{n}{3}(1 + o(1))$, so we hope that the additional structural assumption will beat the general case.

The rest of this note is organized as follows. In Section 2, we present a constructive counterexample to Problem 1.2. In Section 3, we present a probabilistic construction which shows that there exists a triple system $\mathcal{G}$ with $\text{codeg}(\mathcal{G}) = \frac{n}{2}(1 - o(1))$ which does not contain a copy of $K_4^{(3)}$.

2. A negative answer to Problem 1.2

We prove the following theorem, providing a negative answer to Problem 1.2.

Theorem 2.1. *For all $\epsilon \in (0, 1/8)$, there exists an infinite family $\{\mathcal{G}_i\}_{i \in I}$ of triple systems so that each $\mathcal{G} \in \{\mathcal{G}_i\}_{i \in I}$ satisfies that $\text{codeg}(\mathcal{G}) \geq \left(\frac{3}{5} - \epsilon\right) |V(\mathcal{G})|$ and that every collection of vertex-disjoint copies of $K_4^{(3)}$ within $\mathcal{G}$ leaves at least $\epsilon |V(\mathcal{G})|$ vertices uncovered.*

Proof of Theorem 2.1. Pick $\epsilon \in (0, 1/8)$. We construct an infinite family $\{\mathcal{G}_i\}_{i\in I}$ of triple systems so that every $\mathcal{G} \in \{\mathcal{G}_i\}_{i\in I}$ satisfies the two properties stated in the conclusion of Theorem 2.1. Consider all ordered triples (n, α, β) of integers satisfying

$$n > \frac{4}{\epsilon}, \tag{1}$$

$$\alpha = \lfloor \left(\frac{2}{5} + \frac{\epsilon}{2}\right) n \rfloor, \tag{2}$$

$$2\alpha + \beta = n. \tag{3}$$

For each ordered triple (n, α, β) defined above, we define a triple system $\mathcal{G} = \mathcal{G}_{(n,\alpha,\beta)}$. Let A, B and C be pairwise disjoint sets satisfying $|A| = |C| = \alpha$ and $|B| = \beta$. Define

$$\mathcal{F} = \{\{c, c', a\} : c, c' \in C, a \in A\}. \tag{4}$$

Define triple system $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ by

$$V(\mathcal{G}) = A \cup B \cup C, \tag{5}$$

$$E(\mathcal{G}) = [A \cup B \cup C]^3 \setminus ([A]^3 \cup [B]^3 \cup \mathcal{F}). \tag{6}$$

Note that $\mathcal{G}$ is a triple system on n vertices. We show that $\mathcal{G}$ satisfies the two properties in the conclusion of Theorem 2.1, and begin by showing that codeg$(\mathcal{G})$ is as large as promised. Note that

$$[V(\mathcal{G})]^2 = \left([A]^2 \cup [B]^2 \cup [C]^2 \cup P(\{A, B\}) \cup P(\{A, C\}) \cup P(\{B, C\})\right),$$

where

$$P(\{A, B\}) = \{\{a, b\} : a \in A, b \in B\},$$
$$P(\{A, C\}) = \{\{a, c\} : a \in A, c \in C\},$$
$$P(\{B, C\}) = \{\{b, c\} : b \in B, c \in C\}.$$

We have the following equalities:

(i) If $a, a' \in A$, codeg$(\{a, a'\}) = \alpha + \beta$.
(ii) If $b, b' \in B$, codeg$(\{b, b'\}) = 2\alpha$.
(iii) If $c, c' \in C$, codeg$(\{c, c'\}) = \alpha + \beta - 2$.
(iv) If $a \in A, b \in B$, codeg$(\{a, b\}) = 2\alpha + \beta - 2$.
(v) If $a \in A, c \in C$, codeg$(\{a, c\}) = \alpha + \beta - 1$.
(vi) If $b \in B, c \in C$, codeg$(\{b, c\}) = 2\alpha + \beta - 2$.

Thus, (i)-(vi) imply

$$\begin{aligned}\text{codeg}(\mathcal{G}) &= \min_{\{u,v\}\in[V(\mathcal{H})]^2} \text{codeg}(\{u, v\}),\\ &= \min\{2\alpha, \alpha + \beta - 2\},\\ &= \alpha + \beta - 2,\end{aligned}$$

where the last equality follows from the fact that $\beta \leq \alpha$. Due to (2) and (3), we have that

$$\text{codeg}(\mathcal{G}) = n - \alpha - 2 \geq \left(\frac{3}{5} - \frac{\epsilon}{2}\right) n - 2,$$

and since $n \geq 4/\epsilon$,

$$\text{codeg}(\mathcal{G}) \geq \left(\frac{3}{5} - \epsilon\right) n.$$

We now show that any vertex-disjoint collection of copies of $K_4^{(3)}$ within $\mathcal{G}$ leaves at least ϵn vertices uncovered. Indeed, let $\{X_1, \dots, X_l\}$ be a set of pairwise disjoint subsets of $V(\mathcal{G})$, each of which spans a copy of $K_4^{(3)}$ in $\mathcal{G}$. We show that $|V(\mathcal{H}) \setminus \cup_{i=1}^{l} X_i| \geq \epsilon n$. We begin by observing that for each $i \in [l]$,

$$|X_i \cap A| \leq 2.$$

Indeed, if $|X_i \cap A| \geq 3$, then $[X_i]^3 \cap [A]^3 \neq \emptyset$ and since $[A]^3 \cap E(\mathcal{G}) = \emptyset$ from (6), $[X_i]^3 \not\subset E(\mathcal{G})$, contradicting that $X_i \in \{X_1, \dots, X_l\}$. Next, we observe that for each $i \in [l]$ such that $X_i \cap A \neq \emptyset$, $X_i \cap B \neq \emptyset$. Indeed, suppose $X_i \cap A \neq \emptyset$ but $X_i \cap B = \emptyset$. Since $|X_i \cap A| \leq 2$, then $|X_i \cap C| \geq 2$. However, for $c_1, c_2 \in X_i \cap C$, $a \in X_i \cap A$, $\{c_1, c_2, a\} \in \mathcal{F}$ from (4). Since $\mathcal{F} \cap E(\mathcal{G}) = \emptyset$ from (6), $[X_i]^3 \not\subset E(\mathcal{G})$, again contradicting that $X_i \in \{X_1, \dots X_l\}$. As a result of our observations above and the fact that the X_i's are chosen to be pairwise disjoint,

$$|V(\mathcal{G}) \setminus \cup_{i=1}^{l} X_i| \geq |A \setminus \cup_{i=1}^{l} X_i| \geq |A| - 2|B|.$$

Since $|A| = \alpha$, $|B| = \beta$, we infer from (3) that

$$|A| - 2|B| = 5\alpha - 2n$$

which by (2) and (1) is at least

$$\frac{5\epsilon n}{2} - 5 > \epsilon n.$$

Consequently,

$$|V(\mathcal{G}) \setminus \cup_{i=1}^{l} X_i| > \epsilon n.$$

Thus, the proof of Theorem 2.1 is complete. □

3. Discussion

Although the answer to the original problem of Abbasi is negative, it is possible that if the codegree of a triple system $\mathcal{G}$ is at least $n/2$, then $\mathcal{G}$ must contain at least one copy of $K_4^{(3)}$. We conjecture that this is indeed the case.

Conjecture 3.1. *Let $\mathcal{G}$ be a triple system on n vertices. If* $\text{codeg}(\mathcal{G}) \geq \frac{n}{2}$*, then $\mathcal{G}$ contains a copy of $K_4^{(3)}$.*

Modifying an example from [NR], we show that the constant $1/2$ in the lower bound for $\text{codeg}(\mathcal{G})$ in Conjecture 3.1 cannot be improved.

Proposition 3.2. *There exists a triple system $\mathcal{G}$ on n vertices which satisfies* $\text{codeg}(\mathcal{G}) = \frac{n}{2} - o(n)$ *but which does not contain a copy of $K_4^{(3)}$.*

Proof of Proposition 3.2. Consider a random tournament T on $\{1,\dots,n\}$, that is, a complete directed graph on vertices $\{1,\dots,n\}$ where for any $i<j$, $(i,j)\in T$ with probability 1/2. Define a triple system $\mathcal{G}$ on $\{1,\dots,n\}$ as follows: for $\{i,j,k\}$ with $i=\min\{i,j,k\}$, we add $\{i,j,k\}\in E(\mathcal{G})$ if and only if (i,j) and (k,i) are arcs of T. We observe two simple facts.

Fact 3.3. *$\mathcal{G}$ does not contain a copy of $K_4^{(3)}$.*

Indeed, take $S=\{i,j,k,l\}$ and suppose that $i=\min\{i,j,k,l\}$. If S induces a copy of $K_4^{(3)}$, then $\{i,j,k\}$ and $\{i,k,l\}$ are triples of $\mathcal{G}$, implying that $(i,j),(k,i),(i,l)$ are arcs of T, but then $\{i,j,l\}$ is not a triple of $\mathcal{G}$.

Fact 3.4. $\Pr\left[\operatorname{codeg}(\mathcal{G})\geq\left(1-\frac{1}{n^{1/4}}\right)\frac{n-2}{2}\right]>0$*, when n is large enough.*

Indeed, fix $i,j\in\{1,\dots,n\}$, $i<j$, where by symmetry, we assume $(i,j)\in T$. For $k\neq i,j$, let X_{ijk} be an indicator random variable which is equal to one if $\{i,j,k\}\in\mathcal{G}$ and zero otherwise. If $i<k$, then $\Pr[X_{ijk}=1]=1/2$, since with probability 1/2, (k,i) is a arc of T. If $i>k$, then again $\Pr[X_{ijk}=1]=1/2$, since with probability 1/2, we will have either (k,i) and (j,k) as arcs of T or (k,j) and (i,k) as arcs of T. Consequently, for $X_{ij}=\sum_{k\neq i,j}X_{ijk}$, we have

$$E(X_{ij})=\frac{n-2}{2}. \tag{7}$$

Since for fixed i,j, the random variables X_{ijk}, $1\leq k\leq n$, $k\neq i,j$, are independent, we can apply the Chernoff bound (cf. [MR]) to conclude that

$$\Pr\left[X_{ij}<\left(1-\frac{1}{n^{1/4}}\right)E(X_{ij})\right]<\exp\left(-E(X_{ij})/2n^{1/2}\right). \tag{8}$$

Therefore, combining (7) with (8) yields

$$\Pr\left[X_{ij}<\left(1-\frac{1}{n^{1/4}}\right)\frac{n-2}{2}\right]<\exp(-\sqrt{n-2}/5) \tag{9}$$

for n large enough. Thus, the probability that there exist i,j such that $X_{ij}<\left(1-\frac{1}{n^{1/4}}\right)\frac{n-2}{2}$ is less than

$$\binom{n}{2}\exp\left(-\sqrt{n-2}/5\right)$$

which is less than one, when n is sufficiently large. Therefore, there exists a triple system $\mathcal{G}$ on n vertices with $\operatorname{codeg}(\mathcal{G})\geq(1-1/n^{1/4})\frac{n-2}{2}$ without a copy of $K_4^{(3)}$. □

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