CONSTRUCTIVE PACKINGS OF TRIPLE SYSTEMS

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Abstract. Let $\mathcal{F}_0$ and $\mathcal{H}$ be a pair of k-graphs. An $\mathcal{F}_0$-packing of $\mathcal{H}$ is a family $\mathscr{F}$ of pairwise edge-disjoint copies of $\mathcal{F}_0$ in $\mathcal{H}$. Let $\nu_{\mathcal{F}_0}(\mathcal{H})$ denote the maximum size $|\mathscr{F}|$ of an $\mathcal{F}_0$-packing of $\mathcal{H}$. Already in the case of graphs, computing $\nu_{\mathcal{F}_0}(\mathcal{H})$ is NP-hard whenever $\mathcal{F}_0$ has a component with three or more edges [5]. Rödl et al. [17] (cf. [11, 10]) proved that, for any fixed k-graph $\mathcal{F}_0$, one can approximate $\nu_{\mathcal{F}_0}(\mathcal{H})$ within an error of $o(|V(\mathcal{H})|^k)$ in time polynomial in $|V(\mathcal{H})|$. For $k = 3$, we establish an algorithm which, for all $\zeta > 0$ and 3-graphs $\mathcal{F}_0$ and $\mathcal{H}$, constructs in time polynomial in $|V(\mathcal{H})|$ an $\mathcal{F}_0$-packing $\mathscr{F}$ of $\mathcal{H}$ of size $|\mathscr{F}| \geq \nu_{\mathcal{F}_0}(\mathcal{H}) - \zeta|V(\mathcal{H})|^3$.

Our result is in the same vein as two earlier works of its type. Such an algorithm was originated for graphs by Haxell and Rödl [11], and was recently extended to linear k-graphs $\mathcal{F}_0$ by Dizona and Nagle [4]. Our result follows the original approach of Haxell and Rödl, and uses hypergraph regularity tools of Haxell, Rödl and the author [8, 9], together with some details proven here.

1. Introduction

Let $\mathcal{F}_0$ and $\mathcal{H}$ be k-uniform hypergraphs (k-graphs for short, written F_0 and H when $k = 2$). An $\mathcal{F}_0$-*packing* of $\mathcal{H}$ is a family $\mathscr{F}$ of pairwise edge-disjoint copies of $\mathcal{F}_0$ in $\mathcal{H}$. Let $\nu_{\mathcal{F}_0}(\mathcal{H})$ denote the maximum size $|\mathscr{F}|$ of an $\mathcal{F}_0$-packing in $\mathcal{H}$. Already in the case of graphs, computing $\nu_{F_0}(H)$ is NP-hard for any graph F_0 having a component with 3 or more edges (Dor and Tarsi [5]).

Rödl et al. [17, 11, 10] (see Theorem 1.4 below) proved that, for a fixed k-graph $\mathcal{F}_0$, one can approximate $\nu_{\mathcal{F}_0}(\mathcal{H})$ within an error of $o(n^k)$ in time polynomial in $n = |V(\mathcal{H})|$. For graphs ($k = 2$), more is known, where Haxell and Rödl [11] proved the following constructive counterpart.

Theorem 1.1 (Haxell and Rödl [11]). *For every graph F_0, and for all $\zeta > 0$, there exists $n_0 = n_0(F_0, \zeta)$ so that, for a given graph H on $n > n_0$ vertices, an F_0-packing of size at least $\nu_{F_0}(H) - \zeta n^2$ can be constructed in time polynomial in n.*

(Theorem 1.1 also holds for $n \leq n_0$ by exhaustive search.)

Recently, Theorem 1.1 was extended to linear k-graphs $\mathcal{F}_0$, where a k-graph $\mathcal{F}_0$ is *linear* if every pair of its edges overlap in at most one vertex (which is always true of simple graphs F_0).

Theorem 1.2 (Dizona and Nagle [4]). *For every $k \geq 2$, for every linear k-graph $\mathcal{F}_0$, and for all $\zeta > 0$, there exists $n_0 = n_0(k, \mathcal{F}_0, \zeta)$ so that, for a given k-graph $\mathcal{H}$ on $n > n_0$ vertices, an $\mathcal{F}_0$-packing of size at least $\nu_{\mathcal{F}_0}(\mathcal{H}) - \zeta n^k$ can be constructed in time polynomial in n.*

We shall prove an analogue of Theorems 1.1 and 1.2 for $k = 3$ and arbitrary 3-graphs $\mathcal{F}_0$.

Theorem 1.3 (Main result). *For every 3-graph $\mathcal{F}_0$, and for all $\zeta > 0$, there exists $n_0 = n_0(\mathcal{F}_0, \zeta)$ so that, for a given 3-graph $\mathcal{H}$ on $n > n_0$ vertices, an $\mathcal{F}_0$-packing of size $\nu_{\mathcal{F}_0}(\mathcal{H}) - \zeta n^3$ can be constructed in time polynomial in n.*

Theorem 1.3 was thought to be possible by Haxell, Rödl and the author in [9], in light of the tools proven there, at least if one followed the approach of Haxell and Rödl for Theorem 1.1. (We outline this approach momentarily.) This paper is the corresponding realization (see the Acknowledgment at the end of this Introduction). It was also anticipated in [9] that further details would need to be developed (see Remark 1.6 below), which this paper considers. We believe some of the auxiliary tools proven here could be of use in other contexts, and may be of some independent interest.

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The proofs of Theorems 1.1–1.4 all rely on two main ingredients: *fractional packings* and *regularity methods*. (In particular, Theorems 1.1–1.3 can be proven because graphs, linear hypergraphs, and 3-graphs, resp., are precisely where regularity methods have known algorithms.) We next define fractional packings and present Theorem 1.4 (which is in terms of fractional packings). Afterward, we outline some regularity tools employed in the proof of Theorem 1.1 (to motivate upcoming parallels in this paper).

1.1. **Fractional packings.** For k-graphs $\mathcal{F}_0$ and $\mathcal{H}$, let $\binom{\mathcal{H}}{\mathcal{F}_0}$ denote the family of all copies of $\mathcal{F}_0$ in $\mathcal{H}$. For an edge $e \in \mathcal{H}$, let $\binom{\mathcal{H}}{\mathcal{F}_0}_e$ denote the family of copies $\mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}$ containing the edge e. In this notation, an $\mathcal{F}_0$-packing of $\mathcal{H}$ is a family $\mathscr{F} \subseteq \binom{\mathcal{H}}{\mathcal{F}_0}$ satisfying that, for each fixed $e \in \mathcal{H}$,

$$\left| \mathscr{F} \cap \binom{\mathcal{H}}{\mathcal{F}_0}_e \right| \leq 1. \tag{1}$$

Fractional $\mathcal{F}_0$-packings generalize $\mathcal{F}_0$-packings, and can be defined when $\mathcal{H}$ has edge-weights. For a set V and function $\omega : \binom{V}{k} \to [0,1]$, write $\mathcal{H} = \omega^{-1}(0,1]$ and $\mathcal{H}^\omega = \{(e, \omega(e)) : e \in \mathcal{H}\}$. (In the unweighted case when $\omega : \binom{V}{k} \to \{0,1\}$, we identify $\mathcal{H} = \mathcal{H}^\omega$.) A function $\psi : \binom{\mathcal{H}}{\mathcal{F}_0} \to [0,1]$ is a *fractional $\mathcal{F}_0$-packing* of $\mathcal{H}^\omega$ if, for each edge $e \in \mathcal{H}$,

$$\sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_e \right\} \leq \omega(e). \tag{2}$$

Define $|\psi| = \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0} \right\}$ as the *size* of ψ. Denote by $\nu^*_{\mathcal{F}_0}(\mathcal{H}^\omega)$ the maximum size $|\psi|$ of a fractional $\mathcal{F}_0$-packing of $\mathcal{H}^\omega$.

To motivate Theorem 1.4 below, we relate the parameters $\nu_{\mathcal{F}_0}(\mathcal{H})$ and $\nu^*_{\mathcal{F}_0}(\mathcal{H})$ (for a fixed $\mathcal{F}_0$ and an unweighted $\mathcal{H} = \mathcal{H}^\omega$), in terms of relative size and relative complexity. First, consider an $\mathcal{F}_0$-packing $\mathscr{F}$ of $\mathcal{H}$. Then the characteristic function $\chi_{\mathscr{F}} : \binom{\mathcal{H}}{\mathcal{F}_0} \to \{0,1\}$ of $\mathscr{F}$ is a fractional $\mathcal{F}_0$-packing of $\mathcal{H}$ (cf. (1) and (2)), and so $\nu_{\mathcal{F}_0}(\mathcal{H}) \leq \nu^*_{\mathcal{F}_0}(\mathcal{H})$. Second, while computing $\nu_{\mathcal{F}_0}(\mathcal{H})$ is known to be a difficult problem, building a fractional $\mathcal{F}_0$-packing ψ of $\mathcal{H}$ with optimal size $|\psi| = \nu^*_{\mathcal{F}_0}(\mathcal{H})$ is a linear programming problem, and is constructable in time polynomial in $n = |V(\mathcal{H})|$. The following result is therefore significant.

Theorem 1.4 (Rödl, Schacht, Siggers, Tokushige [17]). *For every $k \geq 2$, for every k-graph $\mathcal{F}_0$, and for all $\zeta > 0$, there exists $n_0 = n_0(k, \mathcal{F}_0, \zeta)$ so that, for every k-graph $\mathcal{H}$ on $n > n_0$ vertices,*

$$\nu^*_{\mathcal{F}_0}(\mathcal{H}) \leq \nu_{\mathcal{F}_0}(\mathcal{H}) + \zeta n^k.$$

Thus, $\nu_{\mathcal{F}_0}(\mathcal{H})$ can be approximated within an error of ζn^k in time polynomial in n.

Various cases of Theorem 1.4 had been earlier considered. Haxell and Rödl initiated Theorem 1.4 when, for graphs ($k = 2$), they proved Theorem 1.4 in the stronger form of Theorem 1.1. Yuster [21] gave an alternative proof of Theorem 1.4 for graphs ($k = 2$) which allowed F_0 to be replaced by a fixed family of graphs. For $k = 3$, Theorem 1.4 was proven by Haxell, Nagle and Rödl [9].

1.2. **Regularity and the proof of Theorem 1.1.** We outline the regularity tools in the proof of Theorem 1.1, and outline the proof of Theorem 1.1 in the case that $F_0 = K_3$ is the triangle.

The main tool from [11] is the Szemerédi Regularity Lemma. Let G be a bipartite graph with vertex bipartition $V(G) = X \cup Y$. For non-empty $X' \subseteq X$ and $Y' \subseteq Y$, the *density* of G w.r.t. X' and Y' is $d_G(X', Y') = |G[X', Y']|/(|X'||Y'|)$, where $G[X', Y']$ is the subgraph of G induced on $X' \cup Y'$. For $d \geq 0$ and $\varepsilon > 0$, we say that G is *(d, ε)-regular* if, for every $X' \subseteq X$, where $|X'| > \varepsilon|X|$, and for every $Y' \subseteq Y$, where $|Y'| > \varepsilon|Y|$, we have $|d_G(X', Y') - d| < \varepsilon$. For $\varepsilon > 0$, we say that G is *ε-regular* if it is (d, ε)-regular for some $d \geq 0$.

Theorem 1.5 (Szemerédi's Regularity Lemma [18, 19]). *For all $\varepsilon > 0$, and for every integer $t_0 \geq 1$, there exist integers $T_0 = T_0(\varepsilon, t_0)$ and $N_0 = N_0(\varepsilon, t_0)$ so that every graph H on $n > N_0$ vertices admits a vertex partition $\Pi : V(H) = V_0 \cup V_1 \cup \cdots \cup V_t$, for some $t_0 \leq t \leq T_0$, satisfying the following conditions:*

(1) Π *is* t-equitable*, meaning* $|V_1| = \cdots = |V_t| \stackrel{\text{def}}{=} m \geq \lfloor n/t \rfloor$*;*
(2) Π *is* ε-regular*, meaning for all but* $\varepsilon\binom{t}{2}$ *pairs* V_i, V_j*,* $1 \leq i < j \leq t$*,* $H[V_i, V_j]$ *is ε-regular.*

In the proof of Theorem 1.1, one needs a *constructive* version of Szemerédi's Regularity Lemma, which was established by Alon, Duke, Lefmann, Rödl and Yuster [1]. Their result guarantees that the partition Π in Theorem 1.5 can be constructed in time $O(M(n))$, where $M(n) = O(n^{2.3727})$ (cf. [20]) is the time required to multiply two $n \times n$ binary matrices over the integers. Kohayakawa, Rödl and Thoma [13] later improved the constructive Regularity Lemma to run in time $O(n^2)$. (Either constructive version of the Regularity Lemma serves our purposes here.)

Proof-sketch of Theorem 1.1 for triangles. Fix $k = 2$ and $F_0 = K_3$.

Input. Let $\zeta > 0$ be given. It is possible to fix $\zeta', \beta, \zeta'', \varepsilon', \varepsilon > 0$ and $t_0, T_0, n \in \mathbb{N}$ satisfying

$$\zeta > \zeta' = \frac{\zeta}{4} \gg \beta \gg \zeta'' \gg \varepsilon' \gg \varepsilon = \frac{1}{t_0} \gg \frac{1}{T_0} \ggg \frac{1}{n}, \tag{3}$$

in such a way as to support all details in the sketch below. (In particular, the notation $x \ll y, z, \ldots$ appearing in (3) means that $x > 0$ can be chosen as a sufficiently small function of $y, z, \ldots > 0$ to satisfy an upcoming list of computations involving $x, y, z, \ldots$. The notation $1/n \lll \ldots$ appearing in (3) means the integer n will be chosen sufficiently large with respect to all constants in (3).) Let H be a graph on n vertices. We build, in time polynomial in n, a triangle packing $\mathscr{T}$ of H of size $|\mathscr{T}| \geq \nu^*_{K_3}(H) - \zeta n^2 \geq \nu_{K_3}(H) - \zeta n^2$.

Step 1: Applying the Regularity Lemma. With $\varepsilon = 1/t_0$ from (3), use the algorithm of Kohayakawa et al. [13] (cf. [1]) to construct, in time $O(n^2)$, an ε-regular, t-equitable partition $\Pi : V(H) = V_0 \cup V_1 \cup \cdots \cup V_t$, where $t_0 \leq t \leq T_0 = O(1)$. (Record also, in time $O(n^2)$, the densities $d(V_i, V_j)$, $1 \leq i < j \leq t$, and the pairs (V_i, V_j), $1 \leq i < j \leq t$, which are[1] ε-regular.) Construct, in time $O(t^2) = O(1)$, the corresponding (weighted) cluster graph H_0^ω, as follows. For $\{h, i\} \in \binom{[t]}{2}$, define $\omega(\{h, i\}) = d_H(V_h, V_i)$ if $H[V_h, V_i]$ is ε-regular, and $\omega(\{h, i\}) = 0$ otherwise. Set $H_0 = \omega^{-1}(0, 1]$ and $H_0^\omega = \{(e, \omega(e)) : e \in H_0\}$. A standard argument shows that

$$\nu^*_{K_3}(H_0^\omega) \geq \frac{\nu^*_{K_3}(H)}{m^2} - \zeta' t^2, \tag{4}$$

where $m = |V_1| = \cdots = |V_t|$. We now pause to reveal a few points of strategy.

Pause (strategy). A main tool for building $\mathscr{T}$ is the following so-called *Packing Lemma* (see Lemma 5 in [11]). Suppose that for some $1 \leq h < i < j \leq t$, each of $H[V_h, V_i]$, $H[V_i, V_j]$, $H[V_h, V_j]$ is (d, ε)-regular with $d \geq \beta$ (cf. (3)). In time polynomial in m, the Packing Lemma constructs a triangle packing $\mathscr{T}^{hij}$ of $H[V_h, V_i, V_j]$ covering all but $\zeta'' m^2$ edges of $H[V_h, V_i, V_j]$ (cf. (3)), in which case

$$\left|\mathscr{T}^{hij}\right| \geq (d - \varepsilon - \zeta'')m^2 \geq (d - 2\zeta'')m^2 \overset{(3)}{\geq} \left(1 - \sqrt{\zeta''}\right) dm^2 \quad (\text{recall } d \geq \beta). \tag{5}$$

Unfortunately, it is unlikely that we find $H[V_h, V_i]$, $H[V_i, V_j]$, $H[V_h, V_j]$ which are all (d, ε)-regular with the same d. The so-called *Slicing Lemma* (see Lemma 6 in [11]) allows us to overcome this problem.

Suppose $H[V_h, V_i]$, $H[V_i, V_j]$, $H[V_h, V_j]$ are, resp., (d_{hi}, ε), (d_{ij}, ε), (d_{hj}, ε)-regular. Write $G^{hi} = H[V_h, V_i]$, and suppose we choose (in a careful way (see Step 2 momentarily)) numbers $\sigma_1^{hi}, \ldots, \sigma_{s_{hi}}^{hi} \geq \beta$ (cf. (3)) where $\sum_{a=1}^{s_{hi}} \sigma_a^{hi} \leq d_{hi}$. The Slicing Lemma constructs, in time $O(m^2)$, a partition $G^{hi} = G_0^{hi} \cup G_1^{hi} \cup \cdots \cup G_{s_{hi}}^{hi}$, where each G_a^{hi}, $1 \leq a \leq s_{hi}$, is $(\sigma_a^{hi}, \varepsilon')$-regular (cf. (3)). Now, if there are slices $G_a^{hi} \cup G_b^{ij} \cup G_c^{hj}$ with

$$\sigma_a^{hi} = \sigma_b^{ij} = \sigma_c^{hj} \overset{\text{def}}{=} \sigma_{abc}^{hij}, \tag{6}$$

[1] Recognizing when $H[V_i, V_j]$, $1 \leq i < j \leq t$, is ε-regular is co-NP-complete (see Theorem 2.1 in [1]). However, the algorithms of Alon et al. [1] and Kohayakawa et al. [13] will have already recognized 'most' of the $H[V_i, V_j]$ which are ε-regular (see [1, 13] or Section 1.1 of [9] for details). For the concept of hypergraph regularity that we use in this paper, we will always be able to recognize all 'regular parts'.

then the Packing Lemma builds a triangle packing $\mathscr{T}^{hij}_{abc}$ of $G^{hi}_a \cup G^{ij}_b \cup G^{hj}_c$ of size

$$\left|\mathscr{T}^{hij}_{abc}\right| \stackrel{(5)}{\geq} \left(1-\sqrt{\zeta''}\right)\sigma^{hij}_{abc}m^2. \tag{7}$$

Now, to choose the numbers $\sigma^{hi}_1, \dots, \sigma^{hi}_{s_{hi}} \geq \beta$ above, we appeal to a so-called *Bounding Lemma* (see Lemma 7 in [11]), which returns us to Step 2 of the algorithm.

Step 2: Applying the Bounding Lemma. The Bounding Lemma is a tool which constructs, in time depending on $t \leq T_0 = O(1)$, a fractional K_3-packing ψ_0 of H^ω_0 (recall H^ω_0 and H_0 from Step 1) satisfying (cf. (3))

$$|\psi_0| \geq \nu^*_{K_3}(H^\omega_0) - \zeta' t^2, \quad \text{and which is } \beta\text{-}\textit{bounded}\text{, meaning that for each } \{h,i,j\} \in \tbinom{H_0}{K_3},$$
$$\psi_0(\{h,i,j\}) \geq \beta, \quad \text{or else,} \quad \psi_0(\{h,i,j\}) = 0. \tag{8}$$

Set

$$\binom{H_0}{K_3}^+ = \left\{\{h,i,j\} \in \binom{H_0}{K_3} : \psi_0(\{h,i,j\}) \geq \beta\right\}. \tag{9}$$

The function ψ_0 defines the numbers $\sigma^{hi}_j \geq \beta$, as follows. For $\{h,i\} \in H_0$ and $\{h,i,j\} \in \binom{H_0}{K_3}^+$, set

$$\sigma^{hi}_j = \psi_0(\{h,i,j\}) \stackrel{(9)}{\geq} \beta. \tag{10}$$

We now apply the Slicing Lemma.

Step 3: Applying the Slicing Lemma. Fix $\{h,i\} \in H_0$. With $\left\{\sigma^{hi}_j : \{h,i,j\} \in \binom{H_0}{K_3}^+\right\}$ from Step 2, apply the Slicing Lemma to $G^{hi} = H[V_h, V_i]$ to construct, in time $O(n^2)$, a partition $H[V_h, V_i] = G^{hi} = G^{hi}_0 \cup \bigcup\left\{G^{hi}_j : \{h,i,j\} \in \binom{H_0}{K_3}^+\right\}$ satisfying that, for each $\{h,i,j\} \in \binom{H_0}{K_3}^+$, G^{hi}_j is $\left(\sigma^{hi}_j, \varepsilon'\right)$-regular. Repeat over all at most $\binom{t}{2} \leq T_0^2 = O(1)$ many $\{h,i\} \in H_0$ so that Step 3 runs in time $O(n^2)$.

Step 4: Applying the Packing Lemma. Fix $\{h,i,j\} \in \binom{H_0}{K_3}^+$. Apply the Packing Lemma to the slices $G^{hi}_j \cup G^{ij}_h \cup G^{hj}_i$, noting that $\sigma^{hi}_j = \sigma^{ij}_h = \sigma^{hj}_i = \psi_0(\{h,i,j\}) \geq \beta$ (cf. (10)). The Packing Lemma constructs, in time polynomial in m, a triangle-packing $\mathscr{T}^{hij} = \mathscr{T}^{hij}_{jhi}$ of $G^{hi}_j \cup G^{ij}_h \cup G^{hj}_i$ of size

$$\left|\mathscr{T}^{hij}\right| \stackrel{(7)}{\geq} \left(1-\sqrt{\zeta''}\right)\psi_0(\{h,i,j\})m^2. \tag{11}$$

Repeat over all at most $\binom{t}{3} \leq T_0^3 = O(1)$ many $\{h,i,j\} \in \binom{H_0}{K_3}^+$ in time polynomial in n.

Output. Construct the family $\mathscr{T} = \bigcup\left\{\mathscr{T}^{hij} : \{h,i,j\} \in \binom{H_0}{K_3}^+\right\}$ in time $O(n^2)$. (That is, collect $O(m^2)$ triangles over at most $\binom{t}{3} \leq T_0^3 = O(1)$ indices.)

Clearly the algorithm above runs in time polynomial in n. To see that $\mathscr{T}$ is a triangle-packing of H, let $\{x,y,z\}, \{x,y,z'\} \in \mathscr{T}$. Then there exist, w.l.o.g., $1 \leq h < i < j, j' \leq t$ so that $\{x,y,z\} \in \mathscr{T}^{hij}$ and $\{x,y,z'\} \in \mathscr{T}^{hij'}$. Then $\{x,y\} \in G^{hi}_j \cap G^{hi}_{j'}$, which implies $j = j'$ since G^{hi}_j and $G^{hi}_{j'}$ are classes of a partition. Then $\{x,y,z\}, \{x,y,z'\} \in \mathscr{T}^{hij}$, which implies $z = z'$ since $\mathscr{T}^{hij}$ is a family of pairwise edge-disjoint triangles. Finally,

$$|\mathscr{T}| = \sum\left\{\left|\mathscr{T}^{hij}\right| : \{h,i,j\} \in \binom{H_0}{K_3}^+\right\} \stackrel{(11)}{\geq} \left(1-\sqrt{\zeta''}\right)m^2 \sum\left\{\psi_0(\{h,i,j\}) : \{h,i,j\} \in \binom{H_0}{K_3}^+\right\}$$
$$= \left(1-\sqrt{\zeta''}\right)m^2\,|\psi_0| \stackrel{(8)}{\geq} \left(1-\sqrt{\zeta''}\right)m^2\left(\nu^*_{K_3}(H^\omega_0) - \zeta' t^2\right) \stackrel{(4)}{\geq} \left(1-\sqrt{\zeta''}\right)\left(\nu^*_{K_3}(H) - 2\zeta' t^2 m^2\right)$$

which by (3) is at least $\nu^*_{K_3}(H) - 4\zeta' n^2 = \nu^*_{K_3}(H) - \zeta n^2$. □

1.3. **Itinerary of paper.** To prove Theorem 1.3, we shall follow the same approach outlined above for the graph case. As such, we need 3-uniform hypergraph analogues of each of the tools sketched in the previous section. We proceed along the following itinerary. In Section 2, we present algorithmic (3-uniform) tools of the following forms:

- a *Regularity Lemma* (upcoming Theorem 2.12) due to Haxell, Nagle and Rödl [9].
- a *Packing Lemma* (upcoming Lemma 2.7), which we prove in Sections 4–6;
- a *Slicing Lemma* (upcoming Lemma 2.4), which we prove in Section 7;
- a *Bounding Lemma* (upcoming Lemma 2.18), taken from [11, 10].

In Section 3, we use these tools to prove our main result, Theorem 1.3.

Remark 1.6. The most important tools in this paper are the Regularity Lemma and the Packing Lemma. In essence, the Packing Lemma is a consequence of a so-called Counting Lemma from [9] (see Theorem 6.2 in this paper). Since the Regularity Lemma and the Counting Lemma were developed in [9], Theorem 1.3 seemed possible if one followed the approach of Haxell and Rödl [11] outlined above.

In the hypergraph setting, deriving the Packing Lemma from the Counting Lemma is somewhat technical, despite following standard lines. We derive the Packing Lemma from a so-called *Extension Lemma* (see Lemma 4.2 in this paper), which we in turn derive from the Counting Lemma. These tools could be of potential use in other settings.

The algorithmic aspects of the Slicing Lemma are of a less standard nature. The Slicing Lemma could be of use in other contexts, and it may be of independent interest. □

1.4. **A minor technicality.** In our outline, we took $F_0 = K_3$ to be the triangle, which illustrates all but one detail in the work of Haxell and Rödl [11]. In particular, whenever F_0 is not complete, one also needs a so-called *Crossing Lemma* from [11] (see Lemma 4 there), which we now state for 3-graphs. For 3-graphs $\mathcal{F}_0$ and $\mathcal{H}$, and for a partition $\Pi_0 : V(\mathcal{H}) = U_1 \cup \cdots \cup U_k$, we say that $\mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}$ *crosses* Π_0 if $|V(\mathcal{F}) \cap U_i| \le 1$ for all $1 \le i \le k$. We write $\binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0}$ for the family of all crossing copies of $\mathcal{F}$ in $\mathcal{H}$.

Lemma 1.7 (Crossing Lemma [11])**.** *For every 3-graph $\mathcal{F}_0$ on f vertices, and for all $\xi > 0$, there exists $K_0 = K_0(\xi, \mathcal{F}_0)$ so that the following holds.*

Let $\mathcal{H}$ be a 3-graph on n vertices, and let ψ be a fractional $\mathcal{F}_0$-packing of $\mathcal{H}$. There exists an algorithm which constructs, in time $O(n^f)$, a vertex partition $\Pi_0 : V(\mathcal{H}) = U_1 \cup \cdots \cup U_k$, for some $k \le K_0$, where $|U_1| \le \cdots \le |U_k| \le |U_1| + 1$, satisfying that $|\psi_{\Pi_0}| \stackrel{\text{def}}{=} \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0} \right\} \ge (1 - \xi)|\psi|$.

Haxell and Rödl [11] proved Lemma 1.7 (see Lemma 11 there) in a setting more general than that of (k-uniform) hypergraphs and fractional packings. (See Remark 2.10 in [4] for some related comments.)

In our previous outline, if F_0 were not complete, the Crossing Lemma would appear as Step 0. We would construct a fractional F_0-packing ψ of H of size $|\psi| = \nu^*_{F_0}(H)$ (via linear programming, running in time polynomial in n). We would then apply the Crossing Lemma to H and ψ to construct Π_0 in time $O(n^f)$. In Step 1, we would require Π to *refine* Π_0 in the usual way (which is always possible with a regularity lemma). All remaining details would proceed as we described.

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2. Algorithmic 3-graph Regularity Tools

In this section, we present (1) the Slicing Lemma (Lemma 2.4), (2) the Packing Lemma (Lemma 2.7), (3) the Regularity Lemma (Theorem 2.12), and (4) the Bounding Lemma (Lemma 2.18), where this order is determined by inclusion of the concepts needed to present each statement.

2.1. **(α,δ)-minimality.** For graphs, *ε-regularity* may be viewed as the central concept of the Introduction. For 3-graphs, we shall consider the corresponding concept of *(α,δ)-minimality* (see upcoming Definition 2.3). For that concept, triples of a 3-graph $\mathcal{G}$ will be defined on pairs from an underlying graph P. To make this precise, for a graph P, write $\mathcal{K}_3(P) = \{\{v_1, v_2, v_3\} : \{v_i, v_j\} \in P \text{ for all } 1 \le i < j \le 3\}$ for the family of all *triangles* K_3 of P. We then say P *underlies* $\mathcal{G}$ if $\mathcal{G} \subseteq \mathcal{K}_3(P)$. Whenever $P' \subseteq P$ satisfies $\mathcal{K}_3(P') \neq \emptyset$, we define the *density $d_{\mathcal{G}}(P')$ of $\mathcal{G}$ w.r.t. P'* by $d_{\mathcal{G}}(P') = |\mathcal{G} \cap \mathcal{K}_3(P')|/|\mathcal{K}_3(P')|$. Unless otherwise indicated, we reserve the symbol α for $\alpha = d_{\mathcal{G}}(P) = |\mathcal{G}|/|\mathcal{K}_3(P)|$.

In the context above, the underlying graphs P will be 3-partite, balanced, and well-behaved. We call the following environment a *triad*.

Definition 2.1 (triad). Let $d \ge 0$, $\varepsilon > 0$, and $m \in \mathbb{N}$ be given. We call a graph P a *triad* if $P = P^{12} \cup P^{23} \cup P^{13}$ is 3-partite with vertex partition $V(P) = V_1 \cup V_2 \cup V_3$, where $|V_1| = |V_2| = |V_3| = m$, and where each $P^{ij} = P[V_i, V_j]$ is (d,ε)-regular, $1 \le i < j \le 3$.

Throughout this paper, we use the following well-known fact (cf. upcoming Lemma 8.2).

Fact 2.2 (triangle counting lemma). *For all $d, \tau > 0$, there exists $\varepsilon = \varepsilon_{\text{Fact 2.2}}(d,\tau) > 0$ so that whenever P is a triad with the parameters $d, \varepsilon > 0$ and m sufficiently large, then $|\mathcal{K}_3(P)| = (1 \pm \tau) d^3 m^3$.*

Now, let P be a triad and let $\mathcal{G} \subseteq \mathcal{K}_3(P)$ be given with $\alpha = d_{\mathcal{G}}(P)$. As usual, let $K^{(3)}_{2,2,2}$ denote the complete 3-partite 3-graph with 2 vertices in each vertex class. We then define the family $\mathcal{K}_{2,2,2}(\mathcal{G}) = \left\{ J \in \binom{V(\mathcal{G})}{6} : J \text{ induces a copy of } K^{(3)}_{2,2,2} \text{ in } \mathcal{G} \right\}$. In the context of Definition 2.1, if $\varepsilon = \varepsilon(\alpha, d) > 0$ is sufficiently small and $m = m(\alpha, d, \varepsilon)$ is sufficiently large, it is not difficult to prove (see [9] for a proof) that $|\mathcal{K}_{2,2,2}(\mathcal{G})| \ge \alpha^8 d^{12} \binom{m}{2}^3 \left(1 - \varepsilon^{1/10}\right)$. The following concept is therefore motivated.

Definition 2.3 ((α,δ)-minimality). Let P and $\mathcal{G} \subseteq \mathcal{K}_3(P)$ be given as in Definition 2.1 with $\alpha = d_{\mathcal{G}}(P)$. For $\delta > 0$, we say $\mathcal{G}$ is *(α,δ)-minimal w.r.t. P* if $|\mathcal{K}_{2,2,2}(\mathcal{G})| \le \alpha^8 d^{12} \binom{m}{2}^3 (1 + \delta)$.

2.2. **The Slicing Lemma.** With the definitions above, we can already present the Slicing Lemma. (In what follows, $x = y \pm z$ denotes $y - z \le x \le y + z$.)

Lemma 2.4 (Slicing Lemma). *For all $\alpha_0, \delta' > 0$, there exists $\delta = \delta_{\text{Lem.2.4}}(\alpha_0, \delta') > 0$ so that, for all $d > 0$, there exists $\varepsilon = \varepsilon_{\text{Lem.2.4}}(\alpha_0, \delta', \delta, d) > 0$ so that the following statement holds.*

Let P be a triad with parameters d, ε and a sufficiently large integer m. Let $\mathcal{G} \subseteq \mathcal{K}_3(P)$ be (α,δ)-minimal w.r.t. P, for some $\alpha \ge \alpha_0$. Suppose $\sigma_1, \ldots, \sigma_s \ge \alpha_0$ are given with $\sum_{i=1}^s \sigma_i \le \alpha$. Then, in time $O(m^3)$, one can construct a partition $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_s$ so that, for each $1 \le i \le s$, $\mathcal{G}_i$ is (α_i, δ')-minimal w.r.t. P, where $\alpha_i = d_{\mathcal{G}_i}(P) = \sigma_i \pm \delta'$.

We prove Lemma 2.4 in Section 7.

2.3. **The Packing Lemma.** To present the Packing Lemma (Lemma 2.7), we require some additional considerations. We summarize these conditions in the following environment and subsequent definition.

Setup 2.5 (Packing Setup). Suppose $\alpha, \delta, d, \varepsilon > 0$ and $f, m \in \mathbb{N}$ are given together with a fixed 3-graph $\mathcal{F}_0$ on the vertex set $[f] = \{1, \ldots, f\}$. (This will be the same $\mathcal{F}_0$ appearing in Theorem 1.3.) Suppose

(1) $P = \bigcup \{P^{ij} : 1 \le i < j \le f\}$ is an f-partite graph with vertex partition $V_1 \cup \cdots \cup V_f$, where $|V_1| = \cdots = |V_f| = m$, and where each $P^{ij} = P[V_i, V_j]$ is (d,ε)-regular, $1 \le i < j \le f$;
(2) $\mathcal{G} = \bigcup \{\mathcal{G}^{hij} : 1 \le h < i < j \le f\} \subseteq \mathcal{K}_3(P)$ is a 3-graph satisfying that, for each $1 \le h < i < j \le f$, $\mathcal{G}^{hij} = \mathcal{G}[V_h, V_i, V_j]$ is (α_{hij}, δ)-minimal w.r.t. $P^{hi} \cup P^{ij} \cup P^{hj}$, where $\alpha_{hij} = \alpha \pm \delta$ if $\{h, i, j\} \in \mathcal{F}_0$, and $\alpha_{hij} = 0$ otherwise (i.e., $\mathcal{G}^{hij} = \emptyset$ otherwise).

Definition 2.6 (partite-isomorphic). Let $\mathcal{F}_0$, P and $\mathcal{G}$ be given as in Setup 2.5. Let $\mathcal{F} \subset \mathcal{G}$ be a subhypergraph of $\mathcal{G}$ on vertices $v_1, \ldots, v_f$, where $v_1 \in V_1, \ldots, v_f \in V_f$. We say that $\mathcal{F}$ is a *partite-isomorphic copy* of $\mathcal{F}_0$ if for every $1 \le i < j \le f$, $\{v_i, v_j\} \in P^{ij}$, and if $v_i \mapsto i$ defines an isomorphism from $\mathcal{F}$ to $\mathcal{F}_0$.

Lemma 2.7 (Packing Lemma)**.** *Let $\mathcal{F}_0$ be a fixed 3-graph with $V(\mathcal{F}_0) = [f]$. For all $\alpha_0, \rho > 0$, there exists $\delta = \delta_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho) > 0$ so that, for all $d^{-1} \in \mathbb{N}$, there exists $\varepsilon = \varepsilon_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho, \delta, d) > 0$ so that the following holds.*

Let P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5 with $\mathcal{F}_0$ and some $\alpha \geq \alpha_0$, with $\delta, d, \varepsilon > 0$ above, and with a sufficiently large integer m. Then, one may construct, in time polynomial in m, an $\mathcal{F}_0$-packing $\mathscr{F}_{\mathcal{G}}$ of $\mathcal{G}$ covering all but $\rho|\mathcal{G}|$ edges of $\mathcal{G}$, where $\mathscr{F}_{\mathcal{G}}$ consists entirely of partite-isomorphic copies of $\mathcal{F}_0$ in $\mathcal{G}$. One has, in particular, $|\mathscr{F}_{\mathcal{G}}| \geq (1 - 2\rho)\alpha d^3 m^3$.

We prove Lemma 2.7 in Section 4.

Remark 2.8. Lemma 2.7 holds whether $d > 0$ satisfies $d^{-1} \in \mathbb{N}$ or not. We make the assumption here because results we use from [9] to prove Lemma 2.7 made the same assumption. (Moreover, it suffices to take $d^{-1} \in \mathbb{N}$ because the Regularity Lemma always provides this condition.) $\square$

Remark 2.9. The last assertion of Lemma 2.7 is an easy consequence of its predecessor. To see this, we may assume, w.l.o.g., that $2\delta \leq \rho \times \alpha_0^2$ and that, with $\tau = \delta$, we have $\varepsilon \leq \varepsilon_{\text{Fact 2.2}}(d, \tau)$. Now, let $\mathcal{G}' \subseteq \mathcal{G}$ denote the set of edges covered by $\mathscr{F}_{\mathcal{G}}$. Then every element $\mathcal{F} \in \mathscr{F}_{\mathcal{G}}$ covers precisely $|\mathcal{F}_0|$ edges of $\mathcal{G}'$, and every edge of $\mathcal{G}'$ is covered by precisely one element $\mathcal{F} \in \mathscr{F}_{\mathcal{G}}$. Thus,

$$\begin{aligned}
|\mathscr{F}_{\mathcal{G}}| \times |\mathcal{F}_0| = |\mathcal{G}'| \geq (1-\rho)|\mathcal{G}| &= (1-\rho) \sum \left\{ |\mathcal{G}^{hij}| : \{h, i, k\} \in \mathcal{F}_0 \right\} \\
&= (1-\rho) \sum \left\{ \alpha_{hij} |\mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj})| : \{h, i, j\} \in \mathcal{F}_0 \right\} \\
&\geq (1-\rho)(\alpha - \delta) \sum \left\{ |\mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj})| : \{h, i, j\} \in \mathcal{F}_0 \right\} \\
&\overset{\text{Fact 2.2}}{\geq} |\mathcal{F}_0| \times (1-\rho)(\alpha - \delta)(1-\delta) d^3 m^3 \geq |\mathcal{F}_0| \times (1-\rho) \left(1 - \frac{\delta}{\alpha_0}\right)(1-\delta)\alpha d^3 m^3,
\end{aligned}$$

and so now the result follows by a few routine calculations. $\square$

2.4. **The Regularity Lemma.** We now present the Regularity Lemma from [9] (see Theorem 2.5 there). For a 3-graph $\mathcal{H}$, the Regularity Lemma from [9] will partition the vertices $V = V(\mathcal{H})$ and partition the pairs $\binom{V}{2}$, in such a way that the following holds.

Definition 2.10 ((ℓ, t, ε)-partition)**.** Let $\ell, t \in \mathbb{N}$ and $\varepsilon > 0$ be given, and suppose V is a set with size $|V| = n$. An *(equitable) (ℓ, t, ε)-partition* of V is a pair $(\Pi, \mathscr{P})$ of partitions of the following form:

(1) $\Pi : V = V_0 \cup V_1 \cup \cdots \cup V_t$ is a t-equitable partition of V, i.e, $|V_1| = \cdots = |V_t| \overset{\text{def}}{=} m \geq \lfloor n/t \rfloor$;
(2) $\mathscr{P}$ is a partition of $K[V_1, \ldots, V_t]$ with classes, for each $1 \leq i < j \leq t$, $K[V_i, V_j] = P_1^{ij} \cup \cdots \cup P_\ell^{ij}$, where every $P_a^{ij} \in \mathscr{P}$ is (ℓ^{-1}, ε)-regular.

For a 3-graph $\mathcal{H}$, the Regularity Lemma from [9] will construct an (ℓ, t, ε)-partition of $V(\mathcal{H})$ and a 'large' subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ with the following property.

Definition 2.11 ((α_0, δ)-minimal partition)**.** Let $\ell, t \in \mathbb{N}$ and $\alpha_0, \delta, \varepsilon > 0$ be given, and suppose $\mathcal{H}$ is a 3-graph and $(\Pi, \mathscr{P})$ is an (ℓ, t, ε)-partition of $V(\mathcal{H})$. For a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$, we say $(\Pi, \mathscr{P})$ is *(α_0, δ)-minimal w.r.t. $\mathcal{H}'$* if for every $\{x, y, z\} \in \mathcal{H}'$, there exist $1 \leq h < i < j \leq t$ and $1 \leq a, b, c \leq \ell$ so that $\{x, y, z\} \in \mathcal{K}_3(P_a^{hi} \cup P_b^{ij} \cup P_c^{hj})$, where $\mathcal{H} \cap \mathcal{K}_3(P_a^{hi} \cup P_b^{ij} \cup P_c^{hj})$ is $(\alpha_{abc}^{hij}, \delta)$-minimal w.r.t. $P_a^{hi} \cup P_b^{ij} \cup P_c^{hj}$ for some $\alpha_{abc}^{hij} \geq \alpha_0$.

We now give the regularity lemma from [9] (see Theorem 2.5 there).

Theorem 2.12 (Regularity Lemma [9])**.** *For all $\alpha_0, \delta > 0$, and for all functions $\boldsymbol{\varepsilon} : \mathbb{N} \to (0, 1)$, there exist positive integers $T_0 = T_0(\alpha_0, \delta, \boldsymbol{\varepsilon})$, $L_0 = L_0(\alpha_0, \delta, \boldsymbol{\varepsilon})$, and $N_0 = N_0(\alpha_0, \delta, \boldsymbol{\varepsilon})$ so that the following holds.*

Let $\mathcal{H}$ be a 3-graph with vertex set $V = V(\mathcal{H})$, where $|V| = n > N_0$. Then, in time $O(n^6)$, one can construct an $(\ell, t, \boldsymbol{\varepsilon}(\ell))$-partition $(\Pi, \mathscr{P})$ of V, for some $\ell \leq L_0$ and some $t \leq T_0$, and a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$, where $|\mathcal{H}'| > |\mathcal{H}| - (\alpha_0 + \delta)n^3 - tn^2$, and with respect to which $(\Pi, \mathscr{P})$ is (α_0, δ)-minimal.

We make the following Remark for future reference.

Remark 2.13. In Theorem 2.12, suppose a fixed integer $k \geq 1$ is given with α_0, δ, and $\boldsymbol{\varepsilon} : \mathbb{N} \to (0,1)$. (The constants T_0, L_0 and N_0 will now depend also on k.) Suppose $\mathcal{H}$ is given with a pre-partition $\Pi_0 : V = U_1 \cup \cdots \cup U_k$, where $|U_1| \leq \cdots \leq |U_k| \leq |U_1| + 1$. If we allow $|V_0|$ to be as large as $t + k$, i.e.,

$$|\mathcal{H}'| > |\mathcal{H}| - (\alpha_0 + \delta)n^3 - (t+k)n^2, \tag{12}$$

then the proof of Theorem 2.12 allows that Π can be taken to refine Π_0, in the sense that for each $1 \leq i \leq t$, there exists $1 \leq i' \leq k$ so that $V_i \subseteq U_{i'}$. In this context, we shall call a triple $1 \leq h < i < j \leq t$ *transversal* if $V_h \subseteq U_{h'}$, $V_i \subseteq U_{i'}$, and $V_j \subseteq U_{j'}$, where h', i', j' are distinct. Since the integers $t \leq T_0$ and $k \leq K_0$ (where K_0 will be given by Lemma 1.7) will always be constants in this paper, while $n \to \infty$ whenever needed, we abbreviate (12) to say

$$|\mathcal{H}'| > |\mathcal{H}| - (\alpha_0 + \delta)n^3 - O(n^2). \tag{13}$$

□

2.5. **The Bounding Lemma.** We now present the Bounding Lemma from [10], which appeared as Lemma 3.6 there. The Bounding Lemma concerns fractional packings in weighted multi-hypergraphs related to (ℓ, t, ε)-partitions (cf. Definition 2.10). To make this precise, we require several definitions, taken mostly from [10].

Definition 2.14 ((ℓ,t)-augmented (weighted) 3-graph)**.** On vertex set $[t] = \{1, 2, \ldots, t\}$, let $M = M^{(2)}(\ell, t) = \left\{p_a^{ij} : 1 \leq i < j \leq t,\ 1 \leq a \leq \ell\right\}$ be the complete multigraph with edge-multiplicity ℓ, where the set of multiedges connecting $1 \leq i < j \leq t$ is $\{p_1^{ij}, \ldots, p_\ell^{ij}\}$. We call M the *complete (ℓ,t)-multigraph.* Define $\mathcal{M} = \mathcal{M}^{(3)}(\ell, t) = \left\{\left\{p_a^{hi}, p_b^{ij}, p_c^{hi}\right\} : 1 \leq h < i < j \leq t,\ 1 \leq a, b, c \leq \ell\right\}$ to be the *complete (ℓ,t)-augmented 3-graph.* Any subset $\mathcal{A} \subseteq \mathcal{M}$ is called an *(ℓ,t)-augmented 3-graph.* If $\omega : \mathcal{M} \to [0,1]$ is a weight function, then $\mathcal{A} = \omega^{-1}(0,1]$ is an (ℓ,t)-augmented 3-graph, and we define $\mathcal{A}^\omega = \{(A, \omega(A)) : A \in \mathcal{A}\}$ to be the *(ℓ,t)-augmented ω-weighted 3-graph.*

Clearly, (ℓ,t)-augmented and ω-weighted 3-graphs $\mathcal{A}^\omega$ provide the 'cluster objects' of the Regularity Lemma (Theorem 2.12). We make this precise in the following remark.

Remark 2.15. For a set V, an (ℓ,t,ε)-partition $(\Pi, \mathscr{P})$ of V corresponds to the complete (ℓ,t)-multigraph M defined above, where $P_a^{hi} \in \mathscr{P}$ corresponds to $p_a^{hi} \in M$. The family of all triads of $(\Pi, \mathscr{P})$ corresponds to the complete (ℓ,t)-augmented 3-graph $\mathcal{M}$ defined above, where

$$A = \left\{p_a^{hi}, p_b^{ij}, p_c^{hi}\right\} \in \mathcal{M} \quad \text{corresponds to the triad} \quad P^A \overset{\text{def}}{=} P_a^{hi} \cup P_b^{ij} \cup P_c^{hj} \subset \mathscr{P}. \tag{14}$$

If $V = V(\mathcal{H})$ and $(\Pi, \mathscr{P})$ is (α_0, δ)-minimal w.r.t. $\mathcal{H}' \subseteq \mathcal{H}$, then $\mathcal{H}'$ and $(\Pi, \mathscr{P})$ will correspond to an (ℓ,t)-augmented 3-graph $\mathcal{A}$. Indeed, for $A \in \mathcal{M}$, write

$$\mathcal{H}^A \overset{\text{def}}{=} \mathcal{H} \cap \mathcal{K}_3\left(P^A\right) \quad \text{and} \quad \alpha^A \overset{\text{def}}{=} d_{\mathcal{H}}\left(P^A\right), \tag{15}$$

and so one would take

$$A \in \mathcal{A} \quad \Longleftrightarrow \quad \mathcal{H}^A \text{ is } (\alpha^A, \delta)\text{-minimal w.r.t. } P^A \text{ for some } \alpha^A \geq \alpha_0. \tag{16}$$

More generally, we can define weight function $\omega : \mathcal{M} \to [0,1]$ by

$$\omega(A) = \begin{cases} \alpha^A & \text{if } \alpha^A \geq \alpha_0 \text{ and } \mathcal{H}^A \text{ is } (\alpha^A, \delta)\text{-minimal w.r.t. } P^A, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Then $\mathcal{A} = \omega^{-1}(0,1]$ is the (ℓ,t)-augmented 3-graph defined in (16), and $\mathcal{A}^\omega = \{(A, \omega(A)) : A \in \mathcal{A}\}$ is an (ℓ,t)-augmented ω-weighted 3-graph. When Π has refined a pre-partition Π_0 of V (cf. Remark 2.13), we alter (17) to say

$$\omega(A) = \begin{cases} \alpha^A & \text{if } A \text{ is transversal, } \alpha^A \geq \alpha_0, \text{ and } \mathcal{H}^A \text{ is } (\alpha^A, \delta)\text{-minimal w.r.t. } P^A, \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

and define $\mathcal{A}$ and $\mathcal{A}^\omega$ identically to before. □

For an (ℓ, t)-augmented 3-graph $\mathcal{A}$, we next define *copies* $\mathcal{F} \subset \mathcal{A}$ of a fixed 3-graph $\mathcal{F}_0$ in $\mathcal{A}$, and copies *containing* a fixed edge $A \in \mathcal{A}$.

Definition 2.16 (copy, edge containment)**.** Let $\mathcal{F}_0$ be a 3-graph and let $\mathcal{A}$ be an (ℓ, t)-augmented 3-graph. A *copy* $\mathcal{F}$ of $\mathcal{F}_0$ in $\mathcal{A}$ is a pair (ϕ_1, ϕ_2) of functions, where the first function $\phi_1 : V(\mathcal{F}_0) \to [t]$ is an injection and where we write $\Phi_1 \overset{\text{def}}{=} \phi_1(V(\mathcal{F}_0))$, and where the second function $\phi_2 : \binom{\Phi_1}{2} \to [\ell]$ satisfies

$$\begin{aligned} \{u, v, w\} \in \mathcal{F}_0 \quad \Longrightarrow \quad & \left\{p_a^{hi}, p_b^{ij}, p_c^{hj}\right\} \in \mathcal{A} \quad \text{where} \\ & \{h, i, j\} = \phi_1(\{u, v, w\}) \quad \text{and} \quad (a, b, c) = (\phi_2(\{h, i\}), \phi_2(\{i, j\}), \phi_2(\{h, j\})). \end{aligned}$$

In reverse, suppose $A = \{p_a^{hi}, p_b^{ij}, p_c^{hj}\} \in \mathcal{A}$ and $\mathcal{F} = (\phi_1, \phi_2)$ is a copy of $\mathcal{F}_0$ in $\mathcal{A}$. We say $\mathcal{F}$ *contains* A, and write $A \in \mathcal{F}$, if $\phi_1^{-1}(\{h, i, j\}) \in \mathcal{F}_0$ and if $(a, b, c) = (\phi_2(\{h, i\}), \phi_2(\{i, j\}), \phi_2(\{h, j\})$. Finally, we write $\binom{\mathcal{A}}{\mathcal{F}_0}$ to denote the family of all copies of $\mathcal{F}_0$ in $\mathcal{A}$, and $\binom{\mathcal{A}}{\mathcal{F}_0}_A = \left\{\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0} : A \in \mathcal{F}\right\}$.

In what follows, we define a fractional $\mathcal{F}_0$-packing of an (ℓ, t)-augmented weighted 3-graph $\mathcal{A}^\omega$ (the definition is identical to (2)), and we also define a concept of boundedness.

Definition 2.17 ((β-bounded) fractional $\mathcal{F}_0$-packing of $\mathcal{A}^\omega$)**.** Let $\mathcal{A}^\omega$ be an (ℓ, t)-augmented ω-weighted 3-graph. A *fractional* $\mathcal{F}_0$*-packing* of $\mathcal{A}^\omega$ is a function $\psi : \binom{\mathcal{A}}{\mathcal{F}_0} \to [0, 1]$ satisfying that, for each $A \in \mathcal{A}$, $\sum \left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A\right\} \leq \omega(A)$. For a fractional $\mathcal{F}_0$-packing ψ of $\mathcal{A}^\omega$, we write $|\psi| = \sum \left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}\right\}$ for the *size* of ψ, and we write $\nu^*_{\mathcal{F}_0}(\mathcal{A})$ for the maximum size of a fractional $\mathcal{F}_0$-packing of $\mathcal{A}$. For $\beta > 0$, we say that ψ is β*-bounded* if for every $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}$, $\psi(\mathcal{F}) \geq \beta$, or else, $\psi(\mathcal{F}) = 0$.

We may finally state the Bounding Lemma from [10] (see Lemma 3.6 there).

Lemma 2.18 (Bounding Lemma)**.** *For every 3-graph $\mathcal{F}_0$ and for all positive η, there exists $\beta = \beta_{\text{Lem.2.18}}(\mathcal{F}_0, \eta) > 0$ so that, for every (ℓ, t)-augmented ω-weighted 3-graph $\mathcal{A}^\omega$, there exists a β-bounded fractional $\mathcal{F}_0$-packing ψ of $\mathcal{A}^\omega$ so that $|\psi| \geq \nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) - \eta\ell^3 t^3$. Moreover, ψ may be constructed in time depending on ℓ and t.*

Remark 2.19. Lemma 2.18 was proven in [10], but without regard to the constructive assertion. However, this assertion follows easily from the proof in [10], which consists of an application of a statement of Haxell and Rödl appearing as Theorem 18 in [11]. Theorem 18, in turn, is proven by a standard probabilistic argument on a family of sets on $\ell^3 t^3 = O(1)$ vertices, which one exhaustively derandomizes. Algorithmic aspects are briefly discussed by Haxell and Rödl in [11] (see Section 3). □

3. Proof of Main Result

In this section, we prove Theorem 1.3 in Steps 0–4. In particular, Step 0 will apply the Crossing Lemma (Lemma 1.7), and Steps 1–4 will align with those in the Introduction. We begin by discussing our input, and by defining some auxiliary constants and parameters. The Reader not interested in these details may refer to the hierarchies provided below in (22), (26) and (28).

Input and auxiliary constants. Let $\mathcal{F}_0$ be a fixed 3-graph on f vertices, and let $\zeta > 0$ be given. Set

$$\xi = \rho = \eta = \tau = \frac{\zeta}{8}. \tag{19}$$

We now define some constants related to the Crossing and Bounding Lemmas. With ξ given above, let

$$K_0 = K_0(\xi, \mathcal{F}_0) \tag{20}$$

be the constant guaranteed by the Crossing Lemma (Lemma 1.7). With $\eta > 0$ given above, let

$$\beta = \beta_{\text{Lem.2.18}}(\mathcal{F}_0, \eta) > 0 \tag{21}$$

be the constant guaranteed by the Bounding Lemma (Lemma 2.18). We have the first hierarchy

$$\zeta > \xi = \rho = \eta = \tau \gg \frac{1}{K_0}, \beta. \tag{22}$$

We next define some constants related to the Packing and Slicing Lemmas. With β given above, set

$$\alpha_0 = \beta. \tag{23}$$

With ρ in (19) and α_0 above, let $\delta_{\text{Lem.2.7}} = \delta_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho) > 0$ be the constant guaranteed by the Packing Lemma (Lemma 2.7). With α_0 in (23) and $\delta' = \delta_{\text{Lem.2.7}}$, let $\delta_{\text{Lem.2.4}} = \delta_{\text{Lem.2.4}}(\alpha_0, \delta_{\text{Lem.2.7}}) > 0$ be the constant guaranteed by the Slicing Lemma (Lemma 2.4). It is the case that $\delta_{\text{Lem.2.4}} < \delta_{\text{Lem.2.7}}$ (as will be seen in the proof of Lemma 2.4), and so we set

$$\delta = \delta_{\text{Lem.2.4}}. \tag{24}$$

Continuing, let $\ell \in \mathbb{N}$ be an integer variable. With ρ in (19), α_0 in (23), $\delta_{\text{Lem.2.7}}$ above and $d = 1/\ell$, let $\boldsymbol{\varepsilon}_{\text{Lem.2.7}}(\ell) = \boldsymbol{\varepsilon}_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho, \delta_{\text{Lem.2.7}}, 1/\ell) > 0$ be the function (of integer variable ℓ) guaranteed by the Packing Lemma (Lemma 2.7). With α_0 in (23), $\delta' = \delta_{\text{Lem.2.7}}$ above, $\delta = \delta_{\text{Lem.2.4}}$ in (24) and $d = 1/\ell$, let $\boldsymbol{\varepsilon}_{\text{Lem.2.4}}(\ell) = \boldsymbol{\varepsilon}_{\text{Lem.2.4}}(\alpha_0, \delta_{\text{Lem.2.7}}, \delta, 1/\ell) > 0$ be the function guaranteed by the Slicing Lemma (Lemma 2.4). Finally, with $\tau > 0$ given in (19) and $d = 1/\ell$, let $\boldsymbol{\varepsilon}_{\text{Fact 2.2}}(\ell) = \boldsymbol{\varepsilon}_{\text{Fact 2.2}}(1/\ell, \tau) > 0$ be the function guaranteed by the Triangle Counting Lemma (Fact 2.2). Set

$$\boldsymbol{\varepsilon}(\ell) = \min\left\{\boldsymbol{\varepsilon}_{\text{Lem.2.7}}(\ell), \boldsymbol{\varepsilon}_{\text{Lem.2.4}}(\ell), \boldsymbol{\varepsilon}_{\text{Fact2.2}}(\ell)\right\}. \tag{25}$$

With integer variable ℓ, we have the second hierarchy (cf. (22))

$$\beta \gg \delta_{\text{Lem.2.7}} \gg \delta_{\text{Lem.2.4}} = \delta \geq \min\left\{\frac{1}{\ell}, \delta\right\} \gg \boldsymbol{\varepsilon}_{\text{Lem.2.7}}, \boldsymbol{\varepsilon}_{\text{Lem.2.4}}, \boldsymbol{\varepsilon}_{\text{Fact 2.2}}(\ell) \geq \boldsymbol{\varepsilon}(\ell). \tag{26}$$

Finally, we define some constants related to the Regularity Lemma. Let k be an integer variable. With constants α_0 in (23) and δ in (24), and with function $\boldsymbol{\varepsilon}$ in (25), let $\boldsymbol{T}_0(k) = \boldsymbol{T}_0(k, \alpha_0, \delta, \boldsymbol{\varepsilon})$, $\boldsymbol{L}_0(k) = \boldsymbol{L}_0(k, \alpha_0, \delta, \boldsymbol{\varepsilon})$, and $\boldsymbol{N}_0(k) = \boldsymbol{N}_0(k, \alpha_0, \delta, \boldsymbol{\varepsilon})$ be the functions (of integer variable k) guaranteed by Theorem 2.12 (cf. Remark 2.13). (These are not functions of ℓ.) With K_0 from (20), let

$$T_0 = \max_{1 \leq k \leq K_0} \boldsymbol{T}_0(k), \quad L_0 = \max_{1 \leq k \leq K_0} \boldsymbol{L}_0(k), \quad N_0 = \max_{1 \leq k \leq K_0} \boldsymbol{N}_0(k). \tag{27}$$

We take integer n_0 so that, in the final hierarchy (cf. (22), (26))

$$\min_{1 \leq \ell \leq L_0} \boldsymbol{\varepsilon}(\ell) \gg \frac{1}{T_0}, \frac{1}{L_0}, \frac{1}{N_0} \ggg \frac{1}{n_0}. \tag{28}$$

Now, let $\mathcal{H}$ be a given 3-graph on $n \geq n_0$ vertices. We construct, in time polynomial in n, an $\mathcal{F}_0$-packing $\mathscr{F}_{\mathcal{H}}$ of $\mathcal{H}$ of size

$$|\mathscr{F}_{\mathcal{H}}| \geq \nu^*_{\mathcal{F}_0}(\mathcal{H}) - \zeta n^3. \tag{29}$$

Since $\nu^*_{\mathcal{F}_0}(\mathcal{H}) \geq \nu_{\mathcal{F}_0}(\mathcal{H})$, this will prove Theorem 1.3. We proceed to the first step of the algorithm.

Step 0: Applying the Crossing Lemma. Our first step is to apply the Crossing Lemma to $\mathcal{H}$. (We do so in order to prove (32) below.) For that purpose, construct a maximum fractional $\mathcal{F}_0$-packing $\psi : \binom{\mathcal{H}}{\mathcal{F}_0} \to [0,1]$, i.e., one for which $|\psi| = \nu^*_{\mathcal{F}_0}(\mathcal{H})$ (which is a linear programming problem running in time polynomial in n). With $\xi > 0$ in (19), we apply the Crossing Lemma (Lemma 1.7) to $\mathcal{H}$ and ψ to construct, in time $O(n^f)$, a partition $\Pi_0 : V(\mathcal{H}) = U_1 \cup \cdots \cup U_k$, where $k \leq K_0$ (see (20)) and $|U_1| \leq \cdots \leq |U_k| \leq |U_1| + 1$, and where

$$|\psi_{\Pi_0}| = \sum\left\{\psi(F) : F \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0}\right\} \geq (1 - \xi)|\psi| = (1 - \xi)\nu^*_{\mathcal{F}_0}(\mathcal{H}). \tag{30}$$

(recall the notation $\binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0}$ from Lemma 1.7).

Step 1: Applying the Regularity Lemma. Our next step is to apply the Regularity Lemma (Theorem 2.12) to $\mathcal{H}$ and its vertex partition $V(\mathcal{H}) = U_1 \cup \cdots \cup U_k$ from Step 1. To that end, recall the constants α_0 in (23) and δ in (24), the integer k above (where $k \leq K_0$ (cf. (20))), and the function $\boldsymbol{\varepsilon}$ in (25). With these parameters, Theorem 2.12 constructs, in time $O(n^6)$, an $(\ell, t, \boldsymbol{\varepsilon}(\ell))$-partition $(\Pi, \mathscr{P})$ of $V(\mathcal{H})$, for some $\ell \leq L_0$ and some $t \leq T_0$ (cf. (27)), which refines Π_0 (cf. Remark 2.13), and constructs a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$, where $|\mathcal{H}'| > |\mathcal{H}| - (\alpha_0 + \delta)n^3 - O(n^2)$, and with respect to which $(\Pi, \mathscr{P})$ is (α_0, δ)-minimal. To simplify notation slightly, now that Theorem 2.12 has been applied, the integers $\ell \leq L_0$ and $t \leq T_0$ are *fixed* (they are no longer variables), and so we shall write (cf. (25))

$$\varepsilon = \boldsymbol{\varepsilon}(\ell), \quad \varepsilon_{\text{Lem.2.7}} = \boldsymbol{\varepsilon}_{\text{Lem.2.7}}(\ell), \quad \varepsilon_{\text{Lem.2.4}} = \boldsymbol{\varepsilon}_{\text{Lem.2.4}}(\ell), \quad \varepsilon_{\text{Fact 2.2}} = \boldsymbol{\varepsilon}_{\text{Fact 2.2}}(\ell). \tag{31}$$

We construct the corresponding (ℓ, t)-augmented (ω-weighted) 3-graph $\mathcal{A}$ ($\mathcal{A}^\omega$) for $\mathcal{H}'$ and $(\Pi, \mathscr{P})$ above using (16) ((18)) from Remark 2.15. Clearly, $\mathcal{A}$ and $\mathcal{A}^\omega$ are constructed in time $O(n^6)$. Indeed, for fixed $A \in \mathcal{M}$ (of which there are $|\mathcal{M}| = t^3\ell^3 \leq T_0^3 L_0^3 = O(1)$ many), testing $\alpha^A \geq \alpha_0$ takes time $O(n^3)$, and verifying Definition 2.3 (by greedy count) takes time $O(n^6)$. In Section 3.1, we shall prove that (cf. (4))

$$\frac{m^3}{\ell^3} \nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) \geq |\psi_{\Pi_0}| - (\alpha_0 + \delta + \tau + o(1))\, n^3 \overset{(30)}{\geq} (1 - \xi)\nu^*_{\mathcal{F}_0}(\mathcal{H}) - (\alpha_0 + \delta + \tau + o(1))\, n^3, \tag{32}$$

where $o(1) \to 0$ as $n \to \infty$.

Step 2: Applying the Bounding Lemma. We now apply the Bounding Lemma (Lemma 2.18) to $\mathcal{A}^\omega$. To that end, with $\eta > 0$ in (19) and $\beta > 0$ in (21), we apply Lemma 2.18 to $\mathcal{A}^\omega$ to construct a β-bounded (cf. Definition 2.17) fractional $\mathcal{F}_0$-packing ψ_0 of $\mathcal{A}^\omega$ satisfying

$$|\psi_0| \geq \nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) - \eta\ell^3 t^3. \tag{33}$$

Recall from Lemma 2.18 that ψ_0 is constructed in time depending on $\ell \leq L_0 = O(1)$ and $t \leq T_0 = O(1)$.

Let us also define (construct) a few related objects. To begin, for $A \in \mathcal{A}$ (recall Definition 2.16), set

$$\binom{\mathcal{A}}{\mathcal{F}_0}^+ = \left\{ \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0} : \psi_0(\mathcal{F}) \geq \beta \right\} \quad \text{and} \quad \binom{\mathcal{A}}{\mathcal{F}_0}^+_A = \binom{\mathcal{A}}{\mathcal{F}_0}^+ \cap \binom{\mathcal{A}}{\mathcal{F}_0}_A. \tag{34}$$

For $A \in \mathcal{A}$ and $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+_A$, define

$$\sigma^A_{\mathcal{F}} = \psi_0(\mathcal{F}) \overset{(34)}{\geq} \beta. \tag{35}$$

Note that the sets and numbers above are constructed in time $O(1)$, since in an (ℓ, t)-augmented 3-graph $\mathcal{A}$, there are at most (recall $f = |V(\mathcal{F}_0)|$)

$$\left| \binom{\mathcal{A}}{\mathcal{F}_0} \right| \leq t^f \ell^{3|\mathcal{F}_0|} \leq T_0^f L_0^{3f^3} = O(1) \tag{36}$$

copies $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}$.

Step 3: Applying the Slicing Lemma. Fix $A \in \mathcal{A}$. With $\left\{\sigma^A_{\mathcal{F}} : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+_A\right\}$ from Step 2, we wish to apply the Slicing Lemma (Lemma 2.4) to $\mathcal{G} = \mathcal{H}^A$ (cf. (15)), but first check that it is appropriate to do so. Since $A \in \mathcal{A} = \omega^{-1}(0, 1]$ (cf. (18)), we have that $\mathcal{H}^A$ is (α^A, δ)-minimal w.r.t. P^A, where $\alpha^A \geq \alpha_0 = \beta$ (cf. (23). From (35), every $\sigma^A_{\mathcal{F}}$, $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+_A$, satisfies $\sigma^A_{\mathcal{F}} \geq \beta = \alpha_0$, and moreover,

$$\sum \left\{ \sigma^A_{\mathcal{F}} : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+_A \right\} \overset{(35)}{=} \sum \left\{ \psi_0(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+_A \right\} = \sum \left\{ \psi_0(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A \right\} \leq \omega(A) \overset{(18)}{=} \alpha^A,$$

where the second equality holds on account that ψ_0 vanishes outside of $\binom{\mathcal{A}}{\mathcal{F}_0}^+_A$, and the inequality holds since ψ_0 is a fractional $\mathcal{F}_0$-packing of $\mathcal{A}^\omega$. From (24) and (25) (cf. (31)), we have that $\delta =$

$\delta_{\text{Lem.2.4}}(\alpha_0, \delta' = \delta_{\text{Lem.2.7}})$ and $\varepsilon \le \varepsilon_{\text{Lem.2.4}}(\alpha_0, \delta' = \delta_{\text{Lem.2.7}}, \delta, 1/\ell)$ are appropriately chosen for an application of Lemma 2.4. Applying the Slicing Lemma to $\mathcal{G} = \mathcal{H}^A$, we construct, in time $O(m^3)$, a partition

$$\mathcal{H}^A = \mathcal{H}_0^A \cup \bigcup \left\{ \mathcal{H}_{\mathcal{F}}^A : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A^+ \right\} \tag{37}$$

satisfying that, for each $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A^+$, the 3-graph $\mathcal{H}_{\mathcal{F}}^A$ is $(\alpha_{\mathcal{F}}^A, \delta' = \delta_{\text{Lem.2.7}})$-minimal with respect to P^A, where $\alpha_{\mathcal{F}}^A = \sigma_{\mathcal{F}}^A \pm \delta'$. Repeat over all at most $\binom{t}{3}\ell^3 \le T_0^3 L_0^3 = O(1)$ many $A \in \mathcal{A}$ in time $O(n^3)$.

Step 4: Applying the Packing Lemma. Fix $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+$. We apply the Packing Lemma to the following subhypergraph $\mathcal{G} = \mathcal{H}_{\mathcal{F}} \subset \mathcal{H}$:

$$V(\mathcal{H}_{\mathcal{F}}) = \bigcup_{i \in V(\mathcal{F})} V_i \qquad \text{and} \qquad \mathcal{H}_{\mathcal{F}} = \bigcup_{A \in \mathcal{F}} \mathcal{H}_{\mathcal{F}}^A. \tag{38}$$

Said differently, for each $1 \le i \le t$, we include $V_i \subset V(\mathcal{H}_{\mathcal{F}})$ if, and only if, $i \in V(\mathcal{F})$, and for each $A \in \mathcal{A}$, we take (recall (37))

$$\mathcal{H}_{\mathcal{F}} \cap \mathcal{K}_3\left(P^A\right) = \begin{cases} \mathcal{H}_{\mathcal{F}}^A & A \in \mathcal{F}, \\ \emptyset & A \in \mathcal{A} \setminus \mathcal{F}. \end{cases} \tag{39}$$

Let us check that it is appropriate to apply the Packing Lemma (Lemma 2.7) to $\mathcal{G} = \mathcal{H}_{\mathcal{F}}$.

We first confirm that $\mathcal{H}_{\mathcal{F}}$ meets the conditions of Setup 2.5 (the Packing Setup). For simplicity, but w.l.o.g., we assume $V(\mathcal{F}) = [f] \subset [t] = V(\mathcal{A})$. For the function $\phi_2 = \phi_2(\mathcal{F})$ in Definition 2.16, we write $a_{ij} = \phi_2(\{i,j\})$ for each $1 \le i < j \le f$. Then, the graph $P_{\mathcal{F}} = \bigcup \left\{ P_{a_{ij}}^{ij} : 1 \le i < j \le f \right\}$ underlies $\mathcal{H}_{\mathcal{F}}$, i.e., $\mathcal{H}_{\mathcal{F}} \subseteq \mathcal{K}_3(P_{\mathcal{F}})$, since for each $A = \{p_{a_{hi}}^{hi}, p_{b_{ij}}^{ij}, p_{c_{hj}}^{hj}\} \in \mathcal{F}$, we have by (15) and (37) (resp.) that $\mathcal{H}_{\mathcal{F}}^A \subseteq \mathcal{H}^A \subseteq \mathcal{K}_3(P^A)$. Note that $P_{\mathcal{F}}$ is an f-partite graph with vertex partition $V_1 \cup \cdots \cup V_f$, where $|V_1| = \cdots = |V_f| = m$ and where, for each $1 \le i < j \le f$, $P_{a_{ij}}^{ij}$ is (ℓ^{-1}, ε)-regular. Finally, suppose $A = \{p_{a_{hi}}^{hi}, p_{b_{ij}}^{ij}, p_{c_{hj}}^{hj}\} \in \mathcal{F}$. By (39), we see that $\mathcal{H}_{\mathcal{F}} \cap \mathcal{K}_3(P^A) = \mathcal{H}_{\mathcal{F}}^A$, which by Step 3 is $(\alpha_{\mathcal{F}}^A, \delta' = \delta_{\text{Lem.2.7}})$-minimal w.r.t. P^A. Moreover, by the Slicing Lemma, $\alpha_{\mathcal{F}}^A = \sigma_{\mathcal{F}}^A \pm \delta'$, where $\sigma_{\mathcal{F}}^A \overset{(35)}{=} \psi_0(\mathcal{F}) \ge \beta \overset{(23)}{=} \alpha_0$ is constant and bounded over all $A \in \mathcal{F}$. (In other words, $\psi_0(\mathcal{F})$ plays the role of α in Setup 2.5.) Note that our constants are also chosen appropriately for an application of the Packing Lemma (Lemma 2.7). Indeed, for α_0 above and $\rho > 0$ in (19), we defined $\delta' = \delta_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho) > 0$ (cf. (24)) to be the constant guaranteed by the Packing Lemma, and we defined $\varepsilon \le \varepsilon_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho, \delta', 1/\ell)$ in (25) (cf. (31)) to be appropriate for an application of the Packing Lemma.

Applying Lemma 2.7 to $\mathcal{G} = \mathcal{H}_{\mathcal{F}}$ and $P = P_{\mathcal{F}}$ above, we construct, in time polynomial in m, an $\mathcal{F}_0$-packing $\mathscr{F}_{\mathcal{H}_{\mathcal{F}}}$ of $\mathcal{H}_{\mathcal{F}}$ of size

$$|\mathscr{F}_{\mathcal{H}_{\mathcal{F}}}| \ge (1 - 2\rho)\psi_0(\mathcal{F}) \frac{m^3}{\ell^3}, \tag{40}$$

where every element of $\mathscr{F}_{\mathcal{H}_{\mathcal{F}}}$ is a partite-isomorphic copy of $\mathcal{F}$ (cf. Definition 2.6). Repeat over all $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+$, of which there are at most $O(1)$ many (cf. (36)).

Output. Construct the family $\mathscr{F}_{\mathcal{H}} = \bigcup \left\{ \mathscr{F}_{\mathcal{H}_{\mathcal{F}}} : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+ \right\}$ in time $O(n^3)$. (That is, collect $O(m^3)$ copies of $\mathcal{F}_0$ over at most $O(1)$ indices $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+$ (cf. (36)).

It remains to check the correctness of the algorithm. To that end, the family $\mathscr{F}_{\mathcal{H}}$ was clearly constructed in time polynomial in n. Regarding the remaining details, we first prove that $\mathscr{F}_{\mathcal{H}}$ is an $\mathcal{F}_0$-packing of $\mathcal{H}$.

Proof that $\mathscr{F}_{\mathcal{H}}$ is an $\mathcal{F}_0$-packing of $\mathcal{H}$. Let $\mathcal{F}, \mathcal{F}' \in \mathscr{F}_{\mathcal{H}}$, and for contradiction, suppose $\mathcal{F} \cap \mathcal{F}' \neq \emptyset$. By construction, there exist $\hat{\mathcal{F}}, \hat{\mathcal{F}}' \in \binom{\mathcal{A}}{\mathcal{F}_0}^+$ so that $\mathcal{F} \in \mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}}}$ and $\mathcal{F}' \in \mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}'}}$. If $\hat{\mathcal{F}} = \hat{\mathcal{F}}'$, then

$\mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}}} = \mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}'}}$, and so $\mathcal{F} \cap \mathcal{F}' \neq \emptyset$ contradicts the Packing Lemma (which ensured that $\mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}}} = \mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}'}}$ was an $\mathcal{F}_0$-packing of $\mathcal{H}_{\hat{\mathcal{F}}} = \mathcal{H}_{\hat{\mathcal{F}}'}$). Henceforth, we assume $\hat{\mathcal{F}} \neq \hat{\mathcal{F}}'$.

Fix $\{x,y,z\} \in \mathcal{F} \cap \mathcal{F}'$. Clearly, this implies (cf. (38), (39)),

$$\{x,y,z\} \in \mathcal{H}_{\hat{\mathcal{F}}} \quad \text{and} \quad \{x,y,z\} \in \mathcal{H}_{\hat{\mathcal{F}}'}. \tag{41}$$

Since $\mathcal{F}, \mathcal{F}' \subset \mathcal{H}'$ (recall the notation in Theorem 2.12), there exist $1 \le h < i < j \le t$ and $1 \le a,b,c \le \ell$ so that

$$\{x,y,z\} \in \mathcal{K}_3(P_a^{hi} \cup P_b^{ij} \cup P_c^{hj}). \tag{42}$$

Write $A = \{p_a^{hi}, p_b^{ij}, p_c^{hj}\} \in \mathcal{A}$. By (39), we have

$$\mathcal{H}_{\hat{\mathcal{F}}} \cap \mathcal{K}_3(P^A) = \mathcal{H}^A_{\hat{\mathcal{F}}} \quad \text{and} \quad \mathcal{H}_{\hat{\mathcal{F}}'} \cap \mathcal{K}_3(P^A) = \mathcal{H}^A_{\hat{\mathcal{F}}'}. \tag{43}$$

But now, (41)–(43) imply that $\{x,y,z\} \in \mathcal{H}^A_{\hat{\mathcal{F}}} \cap \mathcal{H}^A_{\hat{\mathcal{F}}'}$, which contradicts the Slicing Lemma. (Indeed, $\mathcal{H}^A_{\hat{\mathcal{F}}}$ and $\mathcal{H}^A_{\hat{\mathcal{F}}'}$ are distinct classes (since $\hat{\mathcal{F}} \neq \hat{\mathcal{F}}'$) of a partition.) $\square$

Proof that $\mathscr{F}_{\mathcal{H}}$ has size promised in (29). By construction, we have

$$|\mathscr{F}_{\mathcal{H}}| = \sum \left\{ |\mathscr{F}_{\mathcal{H}_{\mathcal{F}}}| : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+ \right\} \overset{(40)}{\ge} (1-2\rho)\frac{m^3}{\ell^3} \sum \left\{ \psi_0(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+ \right\} \ge (1-2\rho)\frac{m^3}{\ell^3}\,|\psi_0|$$
$$\overset{(33)}{\ge} (1-2\rho)\frac{m^3}{\ell^3}\left(\nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) - \eta t^3 \ell^3\right) \overset{(32)}{\ge} (1-2\rho)\left((1-\xi)\nu^*_{\mathcal{F}_0}(\mathcal{H}) - (\alpha_0 + \delta + \tau + o(1))n^3 - \eta n^3\right)$$
$$\ge \nu^*_{\mathcal{F}_0}(\mathcal{H}) - (\xi + \alpha_0 + \delta + \tau + \eta + 2\rho + o(1))\,n^3 \overset{(22),(26)}{\ge} \nu^*_{\mathcal{F}_0}(\mathcal{H}) - 8\xi n^3 \overset{(19)}{=} \nu^*_{\mathcal{F}_0}(\mathcal{H}) - \zeta n^3,$$

as promised. $\square$

All that remains to prove Theorem 1.3 is the proof of (32).

3.1. **Proof of (32).** It suffices to produce a fractional $\mathcal{F}_0$-packing $\tilde{\psi}_0 : \binom{\mathcal{A}^\omega}{\mathcal{F}_0} \to [0,1]$ for which $m^3|\tilde{\psi}_0|/\ell^3$ has the lower bound promised in (32). To that end, we establish some notation and terminology. Write $\binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$ to denote the copies $\mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0}$ (cf. Lemma 1.7) for which $\mathcal{F} \subset \mathcal{H}'$ (cf. Theorem 2.12), and fix $\mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$. We define the following $\tilde{\mathcal{F}} = (\tilde{\phi}_1, \tilde{\phi}_2) \in \binom{\mathcal{A}}{\mathcal{F}_0}$ to be the *projection* of $\mathcal{F}$ onto $\mathcal{A}$. Define $\tilde{\phi}_1 : V(\mathcal{F}) \to [t]$ by $\tilde{\phi}_1(v) = i$, if, and only if, $v \in V_i$. Then $\tilde{\phi}_1$ is an injection since $\mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$ crosses Π_0 (cf. Lemma 1.7 and Remark 2.13). Write $\tilde{\Phi}_1 = \tilde{\phi}_1(V(\mathcal{F}))$, and for $i \in \tilde{\Phi}_1$, write $v_i = \tilde{\phi}_1^{-1}(i)$. Define $\tilde{\phi}_2 : \binom{\tilde{\Phi}_1}{2} \to [\ell]$ by $\tilde{\phi}_2(\{i,j\}) = a$ if, and only if, $\{v_i, v_j\} \in P_a^{ij}$. ($\tilde{\phi}_2$ is well-defined since $\{v_i, v_j\} \in K[V_i, V_j]$ (cf. Definition 2.10).) To check that $\tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}$ (cf. Definition 2.16), fix $\{u,v,w\} \in \mathcal{F}$ and write

$$\left(\tilde{\phi}_1(u), \tilde{\phi}_1(v), \tilde{\phi}_1(w)\right) = (h,i,j) \quad \text{and} \quad \left(\tilde{\phi}_2(\{h,i\}), \tilde{\phi}_2(\{i,j\}), \tilde{\phi}_2(\{h,j\})\right) = (a,b,c).$$

We show that $A = \{p_a^{hi}, p_b^{ij}, p_c^{hj}\} \in \mathcal{A}$. To that end, by construction we have that $\{u,v,w\} \in \mathcal{K}_3(P^A)$. Since $\{u,v,w\} \in \mathcal{F} \subset \mathcal{H}'$, the Regularity Lemma guarantees $\mathcal{H}^A$ (cf. (15)) is (α^A, δ)-minimal w.r.t. $\mathcal{K}_3(P^A)$ for some $\alpha^A \ge \alpha_0$. Then by (18), $A \in \mathcal{A}$.

Set (cf. (14))

$$\Delta = \max\left\{ |\mathcal{K}_3(P^A)| : A \in \mathcal{A} \right\}. \tag{44}$$

Define the function $\tilde{\psi}_0 : \binom{\mathcal{A}}{\mathcal{F}_0} \to [0,1]$ by the rule

$$\tilde{\psi}_0(\tilde{\mathcal{F}}) = \frac{1}{\Delta} \sum \left\{ \psi(F) : F \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \text{ has projection } \tilde{\mathcal{F}} \right\}. \tag{45}$$

To show that $\tilde{\psi}_0$ is a fractional $\mathcal{F}_0$-packing of $\mathcal{A}^\omega$, fix $A = \{p_a^{hi}, p_b^{ij}, p_c^{hj}\} \in \mathcal{A}$. Then (cf. Definition 2.16)

$$\begin{aligned}
&\sum\left\{\tilde{\psi}_0(\tilde{\mathcal{F}}) : \tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A\right\} \\
&\qquad = \frac{1}{\Delta}\sum\left\{\sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \text{ has projection } \tilde{\mathcal{F}}\right\} : \tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A\right\} \\
&\qquad \overset{(44)}{\le} \frac{1}{|\mathcal{K}_3(P^A)|}\sum\left\{\sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \text{ has projection } \tilde{\mathcal{F}}\right\} : \tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A\right\}.
\end{aligned}$$

Note that an element $\mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$ projects to some element $\tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A$ if, and only if, $\mathcal{F} \cap \mathcal{H}^A \neq \emptyset$ (cf. (15)). Therefore,

$$\begin{aligned}
\sum\left\{\tilde{\psi}_0(\tilde{\mathcal{F}}) : \tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A\right\} &\le \frac{1}{|\mathcal{K}_3(P^A)|}\sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \text{ satisfies } \mathcal{F} \cap \mathcal{H}^A \neq \emptyset\right\} \\
&= \frac{1}{|\mathcal{K}_3(P^A)|}\sum\left\{\sum\left\{\psi(\mathcal{F}) : \{u,v,w\} \in \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}\right\} : \{u,v,w\} \in \mathcal{H}^A\right\} \\
&\le \frac{1}{|\mathcal{K}_3(P^A)|}\sum\left\{\sum\left\{\psi(\mathcal{F}) : \{u,v,w\} \in \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}\right\} : \{u,v,w\} \in \mathcal{H}^A\right\} \\
&\overset{\text{cf.}(2)}{=} \frac{1}{|\mathcal{K}_3(P^A)|}\sum\left\{\sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\{u,v,w\}}\right\} : \{u,v,w\} \in \mathcal{H}^A\right\} \\
&\le \frac{|\mathcal{H}^A|}{|\mathcal{K}_3(P^A)|} = d_{\mathcal{H}}(P) = \alpha^A = \omega(A),
\end{aligned}$$

where in the last inequality, we used that ψ is a fractional $\mathcal{F}_0$-packing of $\mathcal{H}$, and in the last equality, we used that $A \in \mathcal{A}$ and (16) and (18).

To conclude the proof of (32), consider the quality $|\psi_{\Pi_0}| - \Delta|\tilde{\psi}_0|$. From (45), we see that $\Delta|\tilde{\psi}_0|$ equals

$$\sum\left\{\sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \text{ has projection } \tilde{\mathcal{F}}\right\} : \tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}\right\} = \sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}\right\},$$

since every $\mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$ projects to a unique $\tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}$. Thus,

$$\begin{aligned}
|\psi_{\Pi_0}| - \Delta\left|\tilde{\psi}_0\right| &= \sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0}\right\} - \sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}\right\} \\
&= \sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0} \setminus \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}\right\}.
\end{aligned}$$

Now, $\mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0} \setminus \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$ if, and only if, there exists $\{x,y,z\} \in \mathcal{F} \cap (\mathcal{H} \setminus \mathcal{H}')$ which crosses Π_0. Thus,

$$\begin{aligned}|\psi_{\Pi_0}| - \Delta\left|\tilde{\psi}_0\right| &\leq \sum\left\{\sum\left\{\psi(\mathcal{F}) : \{x,y,z\} \in \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0}\right\} : \{x,y,z\} \in \mathcal{H} \setminus \mathcal{H}' \text{ crosses } \Pi_0\right\}\\ &\leq \sum\left\{\sum\left\{\psi(\mathcal{F}) : \{x,y,z\} \in \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}\right\} : \{x,y,z\} \in \mathcal{H} \setminus \mathcal{H}' \text{ crosses } \Pi_0\right\}\\ &= \sum\left\{\sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\{x,y,z\}}\right\} : \{x,y,z\} \in \mathcal{H} \setminus \mathcal{H}' \text{ crosses } \Pi_0\right\}\\ &\leq |\{\{x,y,z\} : \{x,y,z\} \in \mathcal{H} \setminus \mathcal{H}' \text{ crosses } \Pi_0\}| \leq |\mathcal{H} \setminus \mathcal{H}'| \overset{\text{Thm.2.12}}{\leq} (\alpha_0 + \delta + o(1))\, n^3,\end{aligned}$$

where in the third to last inequality, we used that ψ is a fractional $\mathcal{F}_0$-packing of $\mathcal{H}$. Using Fact 2.2, the Triangle Counting Lemma (cf. (19), (25), (31)), the bounds above imply

$$(1+\tau)\frac{m^3}{\ell^3}\nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) \geq \Delta\nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) \geq \Delta\left|\tilde{\psi}_0\right| \geq |\psi_{\Pi_0}| - (\alpha_0 + \delta + o(1))\, n^3,$$

and so

$$\frac{m^3}{\ell^3}\nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) \geq \frac{1}{1+\tau}\left(|\psi_{\Pi_0}| - (\alpha_0 + \delta + o(1))\, n^3\right) \geq |\psi_{\Pi_0}| - (\alpha_0 + \delta + \tau + o(1))\, n^3,$$

as promised.

4. Proof of the Packing Lemma (Lemma 2.7)

The proof of the Packing Lemma (Lemma 2.7) will follow immediately from upcoming Theorem 4.1 (a well-known result of Grable [7]) and Lemma 4.2 (which we discuss in a moment). To present the former result, we review a few standard concepts. For a j-uniform hypergraph $\mathcal{J}$, a *matching* $\mathscr{J}$ in $\mathcal{J}$ is a family of pairwise disjoint edges from $\mathcal{J}$. For $x, x' \in V(\mathcal{J})$, define $N_{\mathcal{J}}(x) = \{I : I \cup \{x\} \in \mathcal{J}\}$ to be the *neighborhood* of x, and define $N_{\mathcal{J}}(x,x') = N_{\mathcal{J}}(x) \cap N_{\mathcal{J}}(x')$ to be the *co-neighborhood* of x and x'. Set $\deg_{\mathcal{J}}(x) = |N_{\mathcal{J}}(x)|$ to be the *degree* of x, and set $\deg_{\mathcal{J}}(x,x') = |N_{\mathcal{J}}(x,x')|$ to be the *co-degree* of x and x'. Grable's result may now be given as follows.

Theorem 4.1 (Grable [7]). *For every integer $j \geq 2$ and for all $\lambda > 0$, there exists $\beta = \beta_{\text{Thm.4.1}}(j,\lambda) > 0$ so that the following holds. Let $\mathcal{J}$ be a j-graph with a sufficiently large vertex set $X = V(\mathcal{J})$ satisfying that, for some $\Delta > 0$,*

1. *for all $x \in X$, $\deg_{\mathcal{J}}(x) = (1 \pm \beta)\Delta$,*
2. *for all distinct $x, x' \in X$, $\deg_{\mathcal{J}}(x,x') < \Delta/\log^4|X|$.*

Then, there exists a matching $\mathscr{J}$ of $\mathcal{J}$ covering all but $\lambda|X|$ vertices of X. Moreover, $\mathscr{J}$ can be constructed in time polynomial in $|X|$.

In addition to Theorem 4.1, we will need the following Lemma 4.2, which we call the *Constructive Extension Lemma*. The Reader may already be familiar with 'extension lemmas' from hypergraph regularity literature, and indeed upcoming Lemma 4.2 is similar. However, as its title suggests, Lemma 4.2 adds a constructive element to such lemmas, which we need in the current paper. In the next section, we will prove Lemma 4.2, and we will discuss its relationship to these earlier lemmas.

Lemma 4.2 (Constructive Extension Lemma). *Let $\mathcal{F}_0$ be a fixed 3-graph with $V(\mathcal{F}_0) = [f]$. For all $\alpha_0, \gamma > 0$, there exists $\delta = \delta_{\text{Lem.4.2}}(\mathcal{F}_0, \alpha_0, \gamma) > 0$ so that, for all $d^{-1} \in \mathbb{N}$, there exists $\varepsilon = \varepsilon_{\text{Lem.4.2}}(\mathcal{F}_0, \alpha_0, \gamma, \delta, d) > 0$ so that the following holds.*

Let P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5 with $\mathcal{F}_0$ and some $\alpha \geq \alpha_0$, with $\delta, d, \varepsilon > 0$ above, and with a sufficiently large integer m. Then, one may construct, in time $O(m^f)$, a subhypergraph $\hat{\mathcal{G}} \subseteq \mathcal{G}$, where $|\hat{\mathcal{G}}| > (1-\gamma)|\mathcal{G}|$, so that every $\{u,v,w\} \in \hat{\mathcal{G}}$ belongs to within $(1 \pm \gamma)\alpha^{|\mathcal{F}_0|-1}d^{\binom{f}{2}-3}m^{f-3}$ many partite-isomorphic copies of $\mathcal{F}_0$ which reside entirely in the subhypergraph $\hat{\mathcal{G}}$.

4.1. **Proof of the Packing Lemma.** We begin the proof by defining the promised constants. Let $\mathcal{F}_0$ be a fixed 3-graph with $V(\mathcal{F}_0) = [f]$, and let $\alpha_0, \rho > 0$ be given. Set $\lambda = \rho/2$, and let $\beta = \beta_{\text{Thm.4.1}}(|\mathcal{F}_0|, \lambda = \rho/2)$ be the constant guaranteed by Theorem 4.1. Set $\gamma = \beta$ so that

$$\gamma = \beta \ll \lambda = \frac{\rho}{2}. \tag{46}$$

With $\gamma = \beta$, let $\delta_{\text{Lem.4.2}} = \delta_{\text{Lem.4.2}}(\mathcal{F}_0, \alpha_0, \gamma = \beta)$ be the constant guaranteed by the Extension Lemma (Lemma 4.2). Let $d^{-1} \in \mathbb{N}$ be given. Let $\varepsilon_{\text{Lem.4.2}} = \varepsilon_{\text{Lem.4.2}}(\mathcal{F}_0, \alpha_0, \gamma = \beta, \delta_{\text{Lem.4.2}}, d) > 0$ be the constant guaranteed by the Extension Lemma (Lemma 4.2). We define

$$\delta = \delta_{\text{Lem.2.7}} = \delta_{\text{Lem.4.2}} \quad \text{and} \quad \varepsilon = \varepsilon_{\text{Lem.2.7}} = \varepsilon_{\text{Lem.4.2}}. \tag{47}$$

Let P and $\mathcal{G}$ be given satisfying the hypothesis of Setup 2.5 with $\mathcal{F}_0$ and some $\alpha \geq \alpha_0$, with δ given in (47), with $d^{-1} \in \mathbb{N}$ given above, with ε given in (47), and with a sufficiently large integer m.

To construct the promised family $\mathscr{F}_{\mathcal{G}}$, we first apply the Extension Lemma (Lemma 4.2) to P and $\mathcal{G}$ (which is appropriate to do on account of our choice of constants in (47)). The Extension Lemma constructs, in time $O(m^f)$, a subhypergraph $\hat{\mathcal{G}} \subseteq \mathcal{G}$ satisfying $|\hat{\mathcal{G}}| > (1-\gamma)|\mathcal{G}|$ and satisfying that every edge $\{u, v, w\} \in \hat{\mathcal{G}}$ belongs to within $(1 \pm \gamma)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}$ many partite-isomorphic copies of $\mathcal{F}_0$ in $\hat{\mathcal{G}}$.

We now apply Theorem 4.1 to the following j-uniform hypergraph $\mathcal{J}$, where $j = |\mathcal{F}_0|$. Set $X = V(\mathcal{J}) = \hat{\mathcal{G}}$, i.e., the vertices of $\mathcal{J}$ are the edges of $\hat{\mathcal{G}}$. For $\{e_1, \dots, e_j\} \in \binom{X}{j}$, we put $\{e_1, \dots, e_j\} \in \mathcal{J}$ if, and only if, $\{e_1, \dots, e_j\}$ is the edge-set of a partite-isomorphic copy of $\mathcal{F}_0$ in $\hat{\mathcal{G}}$. Set $\Delta = \alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}$. In this language, the Packing Lemma (Lemma 4.2) implies that for all vertices $e \in X$,

$$\deg_{\mathcal{J}}(e) = (1 \pm \gamma)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} = (1 \pm \gamma)\Delta = (1 \pm \beta)\Delta.$$

Clearly, for distinct $e, e' \in X$, we have (with m sufficiently large) $\deg_{\mathcal{J}}(e, e') \leq m^{f-4} < \frac{\Delta}{\log^4 |X|}$, since $\Delta = \Theta(m^{f-3})$ and $|X| = \Theta(m^3)$. Theorem 4.1 constructs, in time polynomial in $|X| = \Theta(m^3)$, a matching $\mathscr{J}$ of $\mathcal{J}$ covering all but $\lambda|X|$ vertices of X. Then $\mathscr{J}$ corresponds to an $\mathcal{F}_0$-packing of $\hat{\mathcal{G}}$ covering all but $\lambda|\hat{\mathcal{G}}| \leq \lambda|\mathcal{G}|$ edges of $\hat{\mathcal{G}}$. Since $|\mathcal{G} \setminus \hat{\mathcal{G}}| < \gamma|\mathcal{G}|$, we have that $\mathscr{J}$ corresponds to an $\mathcal{F}_0$-packing of $\mathcal{G}$ covering all but $(\gamma + \lambda)|\mathcal{G}| \overset{(46)}{\leq} 2\lambda|\mathcal{G}| \overset{(46)}{=} \rho|\mathcal{G}|$ edges of $\mathcal{G}$. Since both $\hat{\mathcal{G}}$ and $\mathscr{J}$ were constructed in time polynomial in m, the proof is complete.

5. Proof of the Constructive Extension Lemma (Lemma 5.1)

We mentioned that Lemma 5.1 is a constructive version of earlier extension-type results. The first of these results is due to Haxell et al. [10] (see Theorem 10.12 there), and a generalization of this result (together with a simpler proof) was given by Cooley et al. [2] (see Lemma 5 there). The following Lemma 5.1 gives a representation of these results, and for purposes of distinction, we call it the *Traditional Extension Lemma.*

Lemma 5.1 (Traditional Extension Lemma [2, 10])**.** *Let $\mathcal{F}_0$ be a fixed 3-graph with $V(\mathcal{F}_0) = [f]$. For all $\alpha_0, \zeta > 0$, there exists $\delta = \delta_{\text{Lem.5.1}}(\mathcal{F}_0, \alpha_0, \zeta) > 0$ so that, for all $d^{-1} \in \mathbb{N}$, there exists $\varepsilon = \varepsilon_{\text{Lem.5.1}}(\mathcal{F}_0, \alpha_0, \zeta, \delta, d) > 0$ so that the following holds.*

Let P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5 with $\mathcal{F}_0$ and some $\alpha \geq \alpha_0$, with $\delta, d, \varepsilon > 0$ above, and with a sufficiently large integer m. Then, all but $\zeta|\mathcal{G}|$ elements $\{u, v, w\} \in \mathcal{G}$ belong to within $(1 \pm \zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}$ many partite-isomorphic copies of $\mathcal{F}_0$ in $\mathcal{G}$.

The Constructive Extension Lemma (Lemma 4.2) is not a trivial corollary of the Traditional Extension Lemma (Lemma 5.1). In the following remark, we pause to explain this reason, which may help motivate the proof of Lemma 4.2.

Remark 5.2. Roughly speaking, Lemma 5.1 says that most $\{u, v, w\} \in \mathcal{G}$ extend to essentially the expected number of copies of $\mathcal{F}_0$ in $\mathcal{G}$. The few edges which don't are exhaustively identified in time $O(m^f)$, and can be deleted from $\mathcal{G}$ in time $O(m^3)$, which yields a large subhypergraph $\mathcal{G}' \subseteq \mathcal{G}$. However, $\mathcal{G}'$ does not prove Lemma 4.2, since for a fixed $\{u, v, w\} \in \mathcal{G}'$, it is possible that every copy of $\mathcal{F}_0$ in $\mathcal{G}$ which contained $\{u, v, w\}$ also contained a triple we deleted. As such, these copies reside back in $\mathcal{G}$, but none would survive in $\mathcal{G}'$.

To overcome the difficulty above, we still use Lemma 5.1, but in a more careful way that also takes into account the underlying graph P. Strictly speaking, these required details will be standard, but they will also be somewhat technical. □

We mentioned at the start of the section that Lemma 5.1 is a *representative* of the results proven in [2, 10], but it is not identical to these results, as we now explain.

Remark 5.3. Lemma 5.1 differs from its predecessors by the condition of hypergraph regularity it assumes. In particular, Lemma 5.1 assumes the condition of (α, δ)-minimality in order to be compatible with the algorithmic Theorem 2.12. The predecessors in [2, 10] assume the condition of (δ, r)-regularity (not defined in this paper) in order to be compatible with the regularity lemma of Frankl and Rödl [6]. It is known (see [3]) that (δ, r)-regularity is a stronger condition than (α, δ)-minimality, and relatedly, the Frankl-Rödl regularity lemma currently admits no known algorithm. Since our efforts are focused on algorithms, we can not use the Frankl-Rödl regularity lemma here, and since their concepts of regularity are not supported by Theorem 2.12, we can not use the extension predecessors of [2, 10] here.

The incompatibility above is not, however, an essential problem. All 'traditional' extension lemmas are corollaries of a so-called *Counting Lemma*, which was proven for (α, δ)-minimality in [9], and for (δ, r)-regularity in [15]. With an appropriate counting lemma in place, the condition of hypergraph regularity assumed is no longer important for deriving a traditional extension lemma. For completeness, we sketch this derivation (the proof of Lemma 5.1) in Section 6. □

In the next subsection, we begin to prepare graph concepts motivated by Remark 5.2

5.1. **Graph concepts and facts.** In this section, we describe (within the context of Setup 2.5) 'terrible' vertices, pairs of vertices, triangles, and pairs of triangles. (Later, we also describe 'bad' versions of these, which aren't quite as bad as their terrible counterparts.) In what follows, for a vertex $v \in V(P)$, we write $N_P(v)$ for the *neighborhood* of v, and we write $N_{P,i}(v)$ for $N_P(v) \cap V_i$.

Definition 5.4 (terrible vertices, terrible pairs)**.** Let P be a graph satisfying the hypothesis of Setup 2.5.

(1) We say that $u \in V(P)$ is a *terrible vertex* if there exist at least $2f^2d^3m^2$ many $\{u, v, w\} \in \mathcal{K}_3(P)$.
(2) We say that $\{u, v\} \in P$ is a *terrible pair* if there exist at least $2fd^2m$ many $\{u, v, w\} \in \mathcal{K}_3(P)$.

We continue with concepts and facts about triangles. Fix $\{a, b, c\}, \{h, i, j\} \in \binom{[f]}{3}$. We call

$$\big(\{v_a, v_b, v_c\}, \{v_h, v_i, v_j\}\big) \in \mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac}) \times \mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj})$$

a *transversal pair of triangles* if $|\{v_a, v_b, v_c, v_h, v_i, v_j\} \cap V_k| \le 1$ for all $1 \le k \le f$. In this case, note that $|\{a, b, c\} \cap \{h, i, j\}| = |\{v_a, v_b, v_c\} \cap \{v_h, v_i, v_j\}| \stackrel{\text{def}}{=} s \in \{0, 1, 2, 3\}$, and so we say that $(\{v_a, v_b, v_c\}, \{v_h, v_i, v_j\})$ is *s-overlapping*. We then define

$$\mathrm{ext}_{K_f,P}(\{v_a, v_b, v_c\}, \{v_h, v_i, v_j\}) = \left| \left\{ U \in \binom{V(P)}{f} : \binom{U}{2} \subset P \text{ and } v_a, v_b, v_c, v_h, v_i, v_j \in U \right\} \right|$$

to be the number of cliques K_f in P extending (containing) both $\{v_a, v_b, v_c\}$ and $\{v_h, v_i, v_j\}$.

To motivate our next definition, we pause to make the following remark.

Remark 5.5. In Setup 2.5, suppose the graph $P = \bigcup_{1 \le i < j \le f} P^{ij}$ satisfies that each $P^{ij} = \mathbb{G}[V_i, V_j; d]$ is the binomial random bipartite graph with edge density d and bipartition $V_i \cup V_j$, where $1 \le i < j \le f$. (It is well-known that, with high probability, this graph satisfies the regularity hypothesis in Setup 2.5.) Then, for an s-overlapping triangle pair $(\{v_a, v_b, v_c\}, \{v_h, v_i, v_j\})$ from P, we have

$$\mathbb{E}\left[\mathrm{ext}_{K_f,P}(\{v_a, v_b, v_c\}, \{v_h, v_i, v_j\})\right] = d^{\binom{f}{2} - \left(\binom{3}{2} + \binom{3}{2} - \binom{s}{2}\right)} m^{f - (3+3-s)} = d^{\binom{f}{2} + \binom{s}{2} - 6} m^{f+s-6},$$

and it is quite unlikely that any of the $\Theta(m^{6-s})$ many s-overlapping triangle pairs of P would deviate much from the mean. (It is also quite unlikely for P to contain any terrible vertices, or terrible pairs of vertices, and so almost surely, all of its s-overlapping triangle pairs would be free of such objects.) $\square$

Definition 5.6 (terrible triangles)**.** Let P be a graph satisfying the hypothesis of Setup 2.5, and let $(\{u,v,w\},\{x,y,z\})$ be an s-overlapping triangle pair from P. We say $(\{u,v,w\},\{x,y,z\})$ is a *terrible pair of triangles* if $\{u,v,w,x,y,z\}$ contains any terrible vertices or any terrible pairs of vertices, or if

$$\mathrm{ext}_{K_f,P}(\{u,v,w\},\{x,y,z\}) > 2d^{\binom{f}{2}+\binom{s}{2}-6}m^{f+s-6}. \tag{48}$$

Relatedly, we also say that $\{x,y,z\}$ is

(1) (s,τ)*-terrible* if there exist at least $\sqrt{\tau}m^{3-s}$ many $\{u,v,w\} \in \mathcal{K}_3(P)$ where $(\{u,v,w\},\{x,y,z\})$ is a terrible pair of s-overlapping triangles;
(2) τ*-terrible* if it is (s,τ)-terrible for some s;
(3) *terrible* when the constant τ is clear from context.

The following fact is easily proven by standard graph regularity arguments involving the so-called *Graph Counting Lemma* (see upcoming Lemma 8.2). For simplicity, we omit these arguments.

Fact 5.7. *For all $d,\tau > 0$, there exists $\varepsilon = \varepsilon_{\text{Fact 5.7}}(d,\tau) > 0$ so that the following holds. Let P be a graph satisfying the hypothesis of Setup 2.5 with d and ε above and with m sufficiently large.*

(1) *For each $0 \le s \le 3$, at most $f^6 \times \tau m^{6-s}$ many s-overlapping pairs of triangles are terrible.*
(2) *There are at most $f^6 \times \sqrt{\tau}m^3$ many τ-terrible triangles.*

We now proceed to the proof of Lemma 4.2.

5.2. **Proof of Lemma 4.2.** We begin the proof of Lemma 4.2 by defining the promised constants. Let $\mathcal{F}_0$ be a fixed 3-graph with $V(\mathcal{F}_0) = [f]$ and let $\alpha_0, \gamma > 0$ be given. Define

$$\sqrt{\zeta} = \frac{\alpha^{|\mathcal{F}_0|-1}\gamma}{200f^6} \tag{49}$$

Let $\delta_{\text{Lem.5.1}} = \delta_{\text{Lem.5.1}}(\mathcal{F}_0, \alpha_0, \zeta) > 0$ be the constant guaranteed by Lemma 5.1. Set

$$\delta = \delta_{\text{Lem.4.2}} = \frac{\alpha}{2} \times \delta_{\text{Lem.5.1}}. \tag{50}$$

Let $d^{-1} \in \mathbb{N}$ be given. Let $\varepsilon_{\text{Lem.5.1}} = \varepsilon_{\text{Lem.5.1}}(\mathcal{F}_0, \alpha_0, \zeta, \delta_{\text{Lem.5.1}}, d) > 0$ be the constant guaranteed by Lemma 5.1. Define

$$\sqrt{\tau} = \frac{\alpha\zeta}{8f^6} d^{\binom{f}{2}}. \tag{51}$$

Let $\varepsilon_{\text{Fact 2.2}} = \varepsilon_{\text{Fact 2.2}}(d,\tau) > 0$ be the constant guaranteed by Fact 2.2. Let $\varepsilon_{\text{Fact 5.7}} = \varepsilon_{\text{Fact 5.7}}(f,d,\tau) > 0$ be the constant guaranteed by Fact 5.7. Set

$$\varepsilon = \varepsilon_{\text{Lem.4.2}} = \min\left\{\varepsilon_{\text{Lem.5.1}}, \varepsilon_{\text{Fact 2.2}}, \varepsilon_{\text{Fact 5.7}}\right\}. \tag{52}$$

Let P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5 with $\mathcal{F}_0$ and some $\alpha \ge \alpha_0$, with $\delta, d, \varepsilon > 0$ above, and with a sufficiently large integer m. To define the promised subhypergraph $\hat{\mathcal{G}} \subseteq \mathcal{G}$, we define several concepts. To begin, let $\mathcal{G}_{\text{terr}} \subseteq \mathcal{G}$ denote the set of edges $\{u,v,w\} \in \mathcal{G}$ so that $\{u,v,w\}$ is a terrible triangle (cf. Definition 5.6).

Claim 5.8. $|\mathcal{G}_{\text{terr}}| \le \frac{\gamma}{2}|\mathcal{G}|$.

Proof of Claim 5.8. Indeed, by Fact 5.7 (cf. (52)), $|\mathcal{G}_{\text{terr}}| \le f^6\sqrt{\tau}m^3$. However, from Setup 2.5 and Fact 2.2 (cf. (52)), we have

$$\begin{aligned}|\mathcal{G}| &= \sum\left\{\left|\mathcal{G}^{hij}\right| : \{h,i,j\} \in \mathcal{F}_0\right\} = \sum\left\{\alpha_{hij}|\mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj})| : \{h,i,j\} \in \mathcal{F}_0\right\} \\ &= (\alpha \pm \delta)\sum\left\{|\mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj})| : \{h,i,j\} \in \mathcal{F}_0\right\} = |\mathcal{F}_0| \times (\alpha \pm \delta)(1 \pm \tau)d^3m^3. \end{aligned} \tag{53}$$

Thus,

$$|\mathcal{G}_{\mathrm{terr}}| \le f^6\sqrt{\tau}m^3 \overset{(53)}{\le} f^6\sqrt{\tau}\frac{|\mathcal{G}|}{(\alpha-\delta)(1-\tau)d^3} \overset{(50)}{\le} 4f^6\frac{\sqrt{\tau}}{\alpha d^3}|\mathcal{G}| \overset{(51)}{\le} \frac{\gamma}{2}|\mathcal{G}|,$$

as promised. □

To define $\hat{\mathcal{G}}$, we will delete $\mathcal{G}_{\mathrm{terr}}$ from $\mathcal{G}$, but we delete other edges too. For $\{u,v,w\} \in \mathcal{G}$, define

$$\mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u,v,w\}) = \left|\left\{\mathcal{F} \in \binom{\mathcal{G}}{\mathcal{F}_0}_{\{u,v,w\}} : \mathcal{F} \text{ is a partite-isomorphic copy of } \mathcal{F}_0 \text{ in } \mathcal{G}\right\}\right|. \tag{54}$$

We consider the following concepts.

Definition 5.9 (bad triples, pairs and vertices)**.**

(1) We say $\{u,v,w\} \in \mathcal{G} \setminus \mathcal{G}_{\mathrm{terr}}$ is a *bad edge* if $\mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u,v,w\}) \ne (1\pm\zeta)\alpha^{|\mathcal{F}_0|-1}d^{\binom{f}{2}-3}m^{f-3}$.
(2) We say $\{x,y\} \in P$ is a *bad pair* if there exist at least $f\sqrt{\zeta}d^2m$ many bad edges $\{x,y,z\} \in \mathcal{G}$.
(3) We say $x \in V(P)$ is a *bad vertex* if there exist at least $f^2\sqrt{\zeta}d^3m^2$ many bad edges $\{x,y,z\} \in \mathcal{G}$.

Denote by $\mathcal{G}_{\mathrm{bad}}$ the set of all edges $\{u,v,w\} \in \mathcal{G} \setminus \mathcal{G}_{\mathrm{terr}}$ which are bad edges, or contain a bad pair, or contain a bad vertex.

Claim 5.10. $|\mathcal{G}_{\mathrm{bad}}| \le \frac{\gamma}{2}|\mathcal{G}|$.

Proof of Claim 5.10. Indeed, by Lemma 5.1 (cf. (50), (52)), the number of bad edges is at most $\zeta|\mathcal{G}|$. As such, we may estimate the number of edges containing a bad pair, or a bad vertex. To begin, there are at most $4f^2\sqrt{\zeta}dm^2$ bad pairs $\{x,y\} \in P$, since otherwise, there would be

$$4f^2\sqrt{\zeta}dm^2 \times f\sqrt{\zeta}d^2m \ge \zeta|\mathcal{F}_0|(\alpha+\delta)(1+\tau)d^3m^3 \overset{(53)}{\ge} \zeta|\mathcal{G}|$$

many bad edges. Similarly, there are at most $4f\sqrt{\zeta}m$ many bad vertices. Now, a bad pair $\{x,y\} \in P$ belongs to at most $2fd^2m$ many $\{x,y,z\} \in \mathcal{G}$. Indeed, by definition, 'many' of these are bad edges $\{x,y,z'\} \in \mathcal{G} \setminus \mathcal{G}_{\mathrm{terr}}$, which can't contain terrible pairs (cf. Definition 5.6). Similarly, a bad vertex belongs to at most $2f^2d^3m^2$ many $\{x,y,z\} \in \mathcal{G}$. We therefore conclude

$$\begin{aligned}|\mathcal{G}_{\mathrm{bad}}| &\le \zeta|\mathcal{G}| + \left(4f^2\sqrt{\zeta}dm^2 \times 2fd^2m\right) + \left(4f\sqrt{\zeta}m \times 2f^2d^3m^2\right) \le \zeta|\mathcal{G}| + 16f^3\sqrt{\zeta}d^3m^3 \\ &\overset{(53)}{\le} \zeta|\mathcal{G}| + 16f^3\frac{\sqrt{\zeta}}{(\alpha-\delta)(1-\tau)}|\mathcal{G}| \overset{(50)}{\le} \zeta|\mathcal{G}| + 64f^3\frac{\sqrt{\zeta}}{\alpha}|\mathcal{G}| \le 65f^3\frac{\sqrt{\zeta}}{\alpha}|\mathcal{G}| \overset{(49)}{\le} \frac{\gamma}{2}|\mathcal{G}|,\end{aligned} \tag{55}$$

as promised. □

To prove Lemma 4.2, define $\hat{\mathcal{G}} = \mathcal{G} \setminus (\mathcal{G}_{\mathrm{terr}} \cup \mathcal{G}_{\mathrm{bad}})$. It follows from Definitions 5.4, 5.6 and 5.9 that $\mathcal{G}_{\mathrm{terr}}$ and $\mathcal{G}_{\mathrm{bad}}$ can each be identified in time $O(m^f)$ (and deleted from $\mathcal{G}$ in time $O(m^3)$), and so $\hat{\mathcal{G}}$ is constructed in time $O(m^f)$, as promised in Lemma 4.2. It follows from Claims 5.8 and 5.10 (and $\mathcal{G}_{\mathrm{terr}} \cap \mathcal{G}_{\mathrm{bad}} = \emptyset$) that $|\hat{\mathcal{G}}| = |\mathcal{G}| - |\mathcal{G}_{\mathrm{terr}}| - |\mathcal{G}_{\mathrm{bad}}| \ge (1-\gamma)|\mathcal{G}|$, as promised in Lemma 4.2. We now verify the final property promised by Lemma 4.2. To that end, fix $\{v_h,v_i,v_j\} \in \hat{\mathcal{G}}$, where $\{v_h,v_i,v_j\} \in \mathcal{G}^{hij}$ for $\{h,i,j\} \in \mathcal{F}_0$. Define $\mathrm{ext}_{\mathcal{F}_0,\hat{\mathcal{G}}}(\{v_h,v_i,v_j\})$ analogously to (54), i.e.,

$$\mathrm{ext}_{\mathcal{F}_0,\hat{\mathcal{G}}}(\{v_h,v_i,v_j\}) = \left|\left\{\mathcal{F} \in \binom{\hat{\mathcal{G}}}{\mathcal{F}_0}_{\{v_h,v_i,v_j\}} : \mathcal{F} \text{ is a partite-isomorphic copy of } \mathcal{F}_0 \text{ in } \hat{\mathcal{G}}\right\}\right|.$$

Clearly, $\mathrm{ext}_{\mathcal{F}_0,\hat{\mathcal{G}}}(\{v_h,v_i,v_j\}) \le \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{v_h,v_i,v_j\})$. As such, since $\{v_h,v_i,v_j\} \notin \mathcal{G}_{\mathrm{bad}}$,

$$\mathrm{ext}_{\mathcal{F}_0,\hat{\mathcal{G}}}(\{v_h,v_i,v_j\}) \le \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{v_h,v_i,v_j\}) \le (1+\zeta)\alpha^{|\mathcal{F}_0|-1}d^{\binom{f}{2}-3}m^{f-3} \overset{(49)}{\le} (1+\gamma)\alpha^{|\mathcal{F}_0|-1}d^{\binom{f}{2}-3}m^{f-3}.$$

Thus, to complete the proof of Lemma 4.2, we will show

$$\mathrm{ext}_{\mathcal{F}_0,\hat{\mathcal{G}}}(\{v_h,v_i,v_j\}) \ge (1-\gamma)\alpha^{|\mathcal{F}_0|-1}d^{\binom{f}{2}-3}m^{f-3}. \tag{56}$$

To prove (56), observe that

$$\begin{aligned}\mathrm{ext}_{\mathcal{F}_0,\hat{\mathcal{G}}}(\{v_h,v_i,v_j\}) &\geq \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{v_h,v_i,v_j\}) - \sum_{\{v_a,v_b,v_c\}\in\mathcal{G}\setminus\hat{\mathcal{G}}} \mathrm{ext}_{K_f,P}(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\}) \\ &\geq (1-\zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} - \sum_{\{v_a,v_b,v_c\}\in\mathcal{G}\setminus\hat{\mathcal{G}}} \mathrm{ext}_{K_f,P}(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\}). \quad (57)\end{aligned}$$

For future reference, we make the following remark.

Remark 5.11.

(1) Summands in (57) are zero when $(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\})$ is not transversal.
(2) Since $\{v_h,v_i,v_j\}\in\hat{\mathcal{G}}$, every $\{v_a,v_b,v_c\}\in\mathcal{G}\setminus\hat{\mathcal{G}}$ satisfies $|\{v_a,v_b,v_c\}\cap\{v_h,v_i,v_j\}| = s\in\{0,1,2\}$.

□

In (57), we have $\mathcal{G}\setminus\hat{\mathcal{G}} = \mathcal{G}_{\mathrm{terr}}\cup\mathcal{G}_{\mathrm{bad}}$, where this union is disjoint. As such,

$$\begin{aligned}&\sum_{\{v_a,v_b,v_c\}\in\mathcal{G}\setminus\hat{\mathcal{G}}} \mathrm{ext}_{K_f,P}(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\}) \\ &= \sum_{\{v_a,v_b,v_c\}\in\mathcal{G}_{\mathrm{terr}}} \mathrm{ext}_{K_f,P}(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\}) + \sum_{\{v_a,v_b,v_c\}\in\mathcal{G}_{\mathrm{bad}}} \mathrm{ext}_{K_f,P}(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\}).\end{aligned}$$

We proceed with the following two claims.

Claim 5.12. $\sum_{\{v_a,v_b,v_c\}\in\mathcal{G}_{\mathrm{terr}}} \mathrm{ext}_{K_f,P}(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\}) \leq f^3\sqrt{\tau} m^{f-3}$.

Claim 5.13. $\sum_{\{v_a,v_b,v_c\}\in\mathcal{G}_{\mathrm{bad}}} \mathrm{ext}_{K_f,P}(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\}) \leq 41 f^6\sqrt{\zeta} d^{\binom{f}{2}-3} m^{f-3}$.

We defer the proofs of Claims 5.12 and 5.13 momentarily to finish the proof of (56). Returning to (57),

$$\begin{aligned}\mathrm{ext}_{\mathcal{F}_0,\hat{\mathcal{G}}}(\{v_h,v_i,v_j\}) &\geq (1-\zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} - f^3\sqrt{\tau}m^{f-3} - 41 f^6\sqrt{\zeta} d^{\binom{f}{2}-3} m^{f-3} \\ &\overset{(51)}{\geq} (1-\zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} - 42 f^6\sqrt{\zeta} d^{\binom{f}{2}-3} m^{f-3} = \left(1-\zeta-42f^6\frac{\sqrt{\zeta}}{\alpha^{|\mathcal{F}_0|-1}}\right)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \\ &\geq \left(1-43f^6\frac{\sqrt{\zeta}}{\alpha^{|\mathcal{F}_0|-1}}\right)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \overset{(49)}{\geq} (1-\gamma)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}.\end{aligned}$$

This proves (56). It remains to prove Claims 5.12 and 5.13.

Proof of Claim 5.12. For $\{a,b,c\}\in\mathcal{F}_0$, write $\mathcal{G}_{\mathrm{terr}}^{abc} = \mathcal{G}_{\mathrm{terr}}\cap\mathcal{G}^{abc}$. Note that

$$\sum_{\{v_a,v_b,v_c\}\in\mathcal{G}_{\mathrm{terr}}} \mathrm{ext}_{K_f,P}(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\}) = \sum_{\{a,b,c\}\in\mathcal{F}_0}\ \sum_{\{v_a,v_b,v_c\}\in\mathcal{G}_{\mathrm{terr}}^{abc}} \mathrm{ext}_{K_f,P}(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\}).$$

Fix $\{a,b,c\}\in\mathcal{F}_0$, and let $\{a,b,c\}\cap\{h,i,j\} = s\in\{0,1,2\}$ (cf. Remark 5.11). We claim

$$\sum_{\{v_a,v_b,v_c\}\in\mathcal{G}_{\mathrm{terr}}^{abc}} \mathrm{ext}_{K_f,P}(\{v_a,v_b,v_c\},\{v_h,v_i,v_j\}) \leq \sqrt{\tau} m^{f-3}.$$

Indeed, every term above is at most m^{f+s-6}. Moreover, since $\{v_h,v_i,v_j\}\in\hat{\mathcal{G}}$ is not an (s,τ)-terrible triangle (cf. Definition 5.6), at most $\sqrt{\tau}m^{3-s}$ indices $\{v_a,v_b,v_c\}\in\mathcal{G}_{\mathrm{terr}}^{abc}$ have non-vanishing terms (cf. Remark 5.11). Claim 5.12 now follows. □

Proof of Claim 5.13. We proceed along similar lines to the proof of Claim 5.12. Fix $\{a, b, c\} \in \mathcal{F}_0$, and let $\{a, b, c\} \cap \{h, i, j\} = s \in \{0, 1, 2\}$ (cf. Remark 5.11). Then,

$$\sum_{\{v_a,v_b,v_c\}\in\mathcal{G}^{abc}_{\mathrm{bad}}} \mathrm{ext}_{K_f,P}(\{v_a, v_b, v_c\}, \{v_h, v_i, v_j\})$$

$$= \sum \left\{\mathrm{ext}_{K_f,P}(\{v_a, v_b, v_c\}, \{v_h, v_i, v_j\}) : \{v_a, v_b, v_c\} \in \mathcal{G}^{abc}_{\mathrm{bad}} \text{ satisfies (48)}\right\}$$

$$+ \sum \left\{\mathrm{ext}_{K_f,P}(\{v_a, v_b, v_c\}, \{v_h, v_i, v_j\}) : \{v_a, v_b, v_c\} \in \mathcal{G}^{abc}_{\mathrm{bad}} \text{ fails (48)}\right\}.$$

The first sum has at most $\sqrt{\tau}m^{3-s}$ terms, since $\{v_h, v_i, v_j\} \in \hat{\mathcal{G}}$ is not an (s, τ)-terrible triangle (cf. Definition 5.6), and every of these terms is at most m^{f+s-6}. In the second sum above, every term is bounded by failing (48). Write $\#_s$ for the number of non-vanishing terms in the second sum. Thus,

$$\sum_{\{v_a,v_b,v_c\}\in\mathcal{G}^{abc}_{\mathrm{bad}}} \mathrm{ext}_{K_f,P}(\{v_a, v_b, v_c\}, \{v_h, v_i, v_j\}) \le \sqrt{\tau}m^{f-3} + 2d^{\binom{f}{2}+\binom{s}{2}-6}m^{f+s-6}\#_s. \tag{58}$$

We claim that

$$\#_s \le 20f^3\sqrt{\zeta}d^{3-\binom{s}{2}}m^{3-s}, \tag{59}$$

which we prove in cases (depending on $s \in \{0, 1, 2\}$).

Case 0 ($s = 0$). We use $\#_0 \le |\mathcal{G}^{abc}_{\mathrm{bad}}| \le |\mathcal{G}_{\mathrm{bad}}|$, and recall from (55) that

$$|\mathcal{G}_{\mathrm{bad}}| \le \zeta|\mathcal{G}| + 16f^3\sqrt{\zeta}d^3m^3 \overset{(53)}{\le} 4\zeta f^3d^3m^3 + 16f^3\sqrt{\zeta}d^3m^3 \le 20f^3\sqrt{\zeta}d^3m^3.$$

Thus, $\#_0 \le 20f^3d^3m^3$, which proves (59). □

Case 1 ($s = 1$). Without loss of generality, assume $a = h$ but $\{b, c\} \cap \{i, j\} = \emptyset$. Since $\{v_h, v_i, v_j\} \in \hat{\mathcal{G}}$, the vertex $v_h = v_a$ is not a bad vertex (cf. Definition 5.9), and so $\#_1 \le f^2\sqrt{\zeta}d^3m^2$, proving (59). □

Case 2 ($s = 2$). Without loss of generality, assume $\{a, b\} = \{h, i\}$ but $c \ne j$. Since $\{v_h, v_i, v_j\} \in \hat{\mathcal{G}}$, the pair $\{v_a, v_b\} = \{v_h, v_i\}$ is not a bad pair (cf. Definition 5.9), and so $\#_2 \le f\sqrt{\zeta}d^2m$. proving (59). □

Employing (59) in (58), we see

$$\sum_{\{v_a,v_b,v_c\}\in\mathcal{G}^{abc}_{\mathrm{bad}}} \mathrm{ext}_{K_f,P}(\{v_a, v_b, v_c\}, \{v_h, v_i, v_j\}) \le \sqrt{\tau}m^{f-3} + 40f^3\sqrt{\zeta}d^{\binom{f}{2}-3}m^{f-3} \overset{(51)}{\le} 41f^3\sqrt{\zeta}d^{\binom{f}{2}-3}m^{f-3},$$

which implies Claim 5.13. □

This completes the proof of Lemma 4.2.

6. Proof of the Traditional Extension Lemma (Lemma 5.1)

In this section, we prove Lemma 5.1 by deriving it from a 'Counting Lemma' of Haxell, Nagle and Rödl [9]. (This derivation resembles one of Cooley et al. [2].) For that, we consider the following setup which generalizes Setup 2.5, and which uses (for reasons seen in context) some alternative notation.

Setup 6.1 (Counting Setup)**.** Suppose $\alpha_0, \delta, d_0, \varepsilon > 0$ and $r, m \in \mathbb{N}$ are given together with a graph R and 3-graph $\mathcal{H}$ satisfying the following conditions:

(1) $R = \bigcup\{R^{ij} : 1 \le i < j \le r\}$ is an r-partite graph with vertex partition $V_1 \cup \cdots \cup V_r$, $|V_1| = \cdots = |V_r| = m$, where for each $1 \le i < j \le r$, $R^{ij} = R[V_i, V_j]$ is (d_{ij}, ε)-regular, for some $d_{ij} \ge d_0$;

(2) $\mathcal{H} = \bigcup\{\mathcal{H}^{hij} : 1 \le h < i < j \le r\} \subseteq \mathcal{K}_3(R)$ is a 3-graph satisfying that, for each $1 \le h < i < j \le r$, $\mathcal{H}^{hij} = \mathcal{H}[V_h, V_i, V_j]$ is (α_{hij}, δ)-minimal w.r.t. $R^{hi} \cup R^{ij} \cup R^{hj}$, for some $\alpha_{hij} \ge \alpha_0$, meaning (cf. Definition 2.3)

$$|\mathcal{K}^{(3)}_{2,2,2}(\mathcal{H}^{hij})| \le \alpha^8_{hij}(d_{hi}d_{ij}d_{hj})^4 \binom{m}{2}^3 (1+\delta). \tag{60}$$

In the context of Setup 6.1, let $\mathcal{K}_r(\mathcal{H}) = \binom{\mathcal{H}}{K^{(3)}_r}$ denote the family of r-cliques in $\mathcal{H}$.

Theorem 6.2 (Counting Lemma [9])**.** *For every $r \in \mathbb{N}$ and for all $\alpha_0, \xi > 0$, there exists $\delta = \delta_{\mathrm{Thm.6.2}}(r, \alpha_0, \xi) > 0$ so that, for all $d_0 > 0$, there exists $\varepsilon = \varepsilon_{\mathrm{Thm.6.2}}(r, \alpha_0, \xi, \delta, d_0) > 0$ so that the following holds.*

Let R and $\mathcal{H}$ satisfy the hypothesis of Setup 6.1 with $r, \alpha_0, \delta, d_0, \varepsilon > 0$ above, and with a sufficiently large integer m. Then, $|\mathcal{K}_r(\mathcal{H})| = (1 \pm \xi) \prod_{1 \le h < i < j \le r} \alpha_{hij} \times \prod_{1 \le h < i \le r} d_{ij} \times m^r$.

Remark 6.3. In [9], Theorem 6.2 was proven in the case when all $\alpha_{hij} = \alpha_0$, $1 \le h < i < j \le r$, and all $d_{ij} = d_0$, $1 \le h < i \le r$. It is well-known and standard to show that this case implies the Counting Lemma in full. For completeness, we sketch these details in Section 8. □

To prove Lemma 5.1, we also use the following version of the Cauchy-Schwarz inequality.

Fact 6.4 (Cauchy-Schwarz Inequality)**.** *For $a_1, \dots, a_t \ge 0$ and $\beta \ge 0$, suppose $\sum_{i=1}^t a_i \ge (1-\beta)at$ and $\sum_{i=1}^t a_i^2 \le (1+\beta)a^2 t$. Then, all but $2\beta^{1/3}t$ indices $1 \le i \le t$ satisfy $a_i = a(1 \pm 2\beta^{1/3})$.*

We also use the following easy fact.

Fact 6.5. *For all $\delta, d_0 > 0$, there exists $\varepsilon = \varepsilon_{\mathrm{Fact6.5}}(\delta, d_0) > 0$ so that the following holds. Let R satisfy the hypothesis of Setup 6.1 with $r = 3$, $d_0, \varepsilon > 0$ above, and m sufficiently large. Let $\mathcal{H} = \mathcal{K}_3(R)$. Then, $\mathcal{H}$ is necessarily $(1, \delta)$-minimal w.r.t. R.*

Proof. Indeed, it is well-known that for a suitable function $f(\varepsilon) \to 0$ as $\varepsilon \to 0$, the Graph Counting Lemma (see upcoming Lemma 8.2) implies that R contains within $(1 \pm f(\varepsilon))(d_{12}d_{23}d_{13})^4\binom{m}{2}^3$ many copies of the graph $K^{(2)}_{2,2,2}$. Thus, we take $\varepsilon > 0$ small enough so that $f(\varepsilon) < \delta$, and (60) is immediate. □

6.1. **Proof of Lemma 5.1.** Let $\mathcal{F}_0$ be a fixed 3-graph with $V(\mathcal{F}_0) = [f]$. Let $\alpha_0, \zeta > 0$ be given. Define

$$F = 2f - 3, \quad \text{and choose } \xi > 0 \text{ to satisfy} \quad 1 - \left(\frac{\zeta}{2}\right)^3 \le (1-\xi)^{3+f^3} \le (1+2\xi)^{2+2F^3} \le 1 + \left(\frac{\zeta}{2}\right)^3. \tag{61}$$

With $r = f$, let $\delta_{\mathrm{Thm.\ 6.2}}(f) = \delta_{\mathrm{Thm.\ 6.2}}(f, \alpha_0, \xi) > 0$ be the constant guaranteed by Theorem 6.2. With $r = F$, let $\delta_{\mathrm{Thm.6.2}}(F) = \delta_{\mathrm{Thm.6.2}}(F, \alpha_0, \xi) > 0$ be the constant guaranteed by Theorem 6.2. Set

$$\delta = \delta_{\mathrm{Lem.5.1}}(\mathcal{F}_0, \alpha_0, \zeta) = \min\left\{\alpha\zeta,\ \delta_{\mathrm{Thm.6.2}}(f),\ \delta_{\mathrm{Thm.6.2}}(F)\right\}. \tag{62}$$

Let $d^{-1} \in \mathbb{N}$ be given. With $r = 3$ and $d_0 = d$, let $\varepsilon_{\mathrm{Fact6.5}}(\delta, d_0) > 0$ be the constant guaranteed by Fact 6.5. With $r = f$ and $d_0 = d$, let $\varepsilon_{\mathrm{Thm.6.2}}(f) = \varepsilon_{\mathrm{Thm.6.2}}(f, \alpha_0, \xi, \delta, d) > 0$ be the constant guaranteed by Theorem 6.2. With $r = F$ and $d_0 = d$, let $\varepsilon_{\mathrm{Thm.6.2}}(F) = \varepsilon_{\mathrm{Thm.6.2}}(F, \alpha_0, \xi, \delta, d) > 0$ be the constant guaranteed by Theorem 6.2. Finally, with $\tau = \xi$, let $\varepsilon_{\mathrm{Fact2.2}} = \varepsilon_{\mathrm{Fact2.2}}(d, \xi) > 0$ be the constant guaranteed by the Triangle Counting Lemma. Set

$$\varepsilon = \varepsilon_{\mathrm{Lem.5.1}}(\mathcal{F}_0, \alpha_0, \zeta, \delta, d) = \min\left\{\varepsilon_{\mathrm{Fact6.5}},\ \varepsilon_{\mathrm{Fact2.2}},\ \varepsilon_{\mathrm{Thm.6.2}}(f),\ \varepsilon_{\mathrm{Thm.6.2}}(F)\right\}. \tag{63}$$

Let P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5 with $\mathcal{F}_0$ and some $\alpha \ge \alpha_0$, with $\delta, d, \varepsilon > 0$ above, and with a sufficiently large integer m. To prove Lemma 5.1, we prove that for each fixed $\{h, i, j\} \in \mathcal{F}_0$, all but $\zeta|\mathcal{G}^{hij}|$ elements $\{u, v, w\} \in \mathcal{G}^{hij}$ satisfy

$$\mathrm{ext}_{\mathcal{F}_0, \mathcal{G}}(\{u, v, w\}) = (1 \pm \zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \tag{64}$$

(recall the notation in (54)). Without loss of generality, we assume $\{h,i,j\} = \{1,2,3\} \in \mathcal{F}_0$. The proof of (64) will use the Counting Lemma (Theorem 6.2) to show that

$$\sum_{\{u,v,w\}\in\mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u,v,w\}) \geq \left(1-\left(\frac{\zeta}{2}\right)^3\right)\left(\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}\right) \times |\mathcal{G}^{123}|, \tag{65}$$

and

$$\sum_{\{u,v,w\}\in\mathcal{G}^{123}} \mathrm{ext}^2_{\mathcal{F}_0,\mathcal{G}}(\{u,v,w\}) \leq \left(1+\left(\frac{\zeta}{2}\right)^3\right)\left(\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}\right)^2 \times |\mathcal{G}^{123}|. \tag{66}$$

Using (65) and (66), the Cauchy-Schwarz Inequality (Fact 6.4) immediately concludes (64).

Proof of (65). Write

$$\#\{\mathcal{F}_0 \subseteq_{\text{p.i.}} \mathcal{G}\} = \left|\left\{\mathcal{F} \in \binom{\mathcal{G}}{\mathcal{F}_0} : \mathcal{F} \text{ is a partite-isomorphic copy of } \mathcal{F}_0 \text{ in } \mathcal{G}\right\}\right|$$

for the number of partite-isomorphic copies of $\mathcal{F}_0$ in $\mathcal{G}$. Observe that $\sum_{\{u,v,w\}\in\mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u,v,w\}) = \#\{\mathcal{F}_0 \subseteq_{\text{p.i.}} \mathcal{G}\}$. To bound this quantity with the Counting Lemma, construct the following hypergraph $\mathcal{H}$ from $\mathcal{G}$: for $\{h,i,j\} \in \binom{[f]}{3}$, set

$$\mathcal{H}^{hij} = \begin{cases} \mathcal{G}^{hij} & \text{if } \{h,i,j\} \in \mathcal{F}_0, \\ \mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj}) & \text{if } \{h,i,j\} \notin \mathcal{F}_0 \end{cases}$$

and set $\mathcal{H} = \bigcup\{\mathcal{H}^{hij} : 1 \leq h < i < j \leq f\}$. Then, $\sum_{\{u,v,w\}\in\mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u,v,w\}) = \#\{\mathcal{F}_0 \subseteq_{\text{p.i.}} \mathcal{G}\} = |\mathcal{K}_f(\mathcal{H})|$. Since $\mathcal{G}$ and P satisfy the hypothesis of Setup 2.5, $\mathcal{H}$ and $R = P$ satisfy the hypothesis of Setup 6.1, specifically with $\alpha_{hij} = \alpha \pm \delta$ when $\{h,i,j\} \in \mathcal{F}_0$ and $\alpha_{hij} = 1$ otherwise, and all $d_{ij} = d$, $1 \leq i < j \leq f$ (cf. (62), (63) and Fact 6.5). An application of the Counting Lemma (Theorem 6.2) gives

$$\sum_{\{u,v,w\}\in\mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u,v,w\}) \geq (1-\xi) \prod_{1\leq h<i<j\leq f} \alpha_{hij} \times \prod_{1\leq i<j\leq f} d_{ij} \times m^f \geq (1-\xi)(\alpha-\delta)^{|\mathcal{F}_0|} d^{\binom{f}{2}} m^f.$$

To infer (65), we rewrite the inequality above. By the Triangle Counting Lemma (Fact 2.2),

$$|\mathcal{G}^{123}| = \alpha_{123}|\mathcal{K}_3(P^{12} \cup P^{23} \cup P^{13})| = (\alpha \pm \delta)(1 \pm \xi) d^3 m^3. \tag{67}$$

As such (using $(1+x)^{-1} \geq 1-x$),

$$\begin{aligned} \sum_{\{u,v,w\}\in\mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u,v,w\}) &\geq \frac{(1-\xi)(\alpha-\delta)^{|\mathcal{F}_0|}}{(1+\xi)(\alpha+\delta)} d^{\binom{f}{2}-3} m^{f-3} \times |\mathcal{G}^{123}| \\ &\geq (1-\xi)^2 \left(1-\frac{\delta}{\alpha}\right)^{|\mathcal{F}_0|+1} \alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \times |\mathcal{G}^{123}| \overset{(62)}{\geq} (1-\xi)^{3+f^3} \alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \times |\mathcal{G}^{123}| \end{aligned}$$

and so (65) follows from (61). □

Proof of (66). The proof is similar to its predecessor, but more involved. We begin by defining a graph R, and 3-graphs $\mathcal{F}_1$, $\mathcal{H}$ and $\hat{\mathcal{H}}$, and use the following notation: define $\phi : [F] \to [f]$ (cf. (61))

$$\phi(a) = \bar{a} = \begin{cases} a & \text{if } a \in [f], \\ a - f + 3 & \text{if } a \in [F] \setminus [f]. \end{cases}$$

For $a \in [F] \setminus [f]$, let V_a be a copy of $V_{\bar{a}}$ (where $V(P) = V(\mathcal{G}) = V_1 \cup \cdots \cup V_f$ from Setup 2.5).

Defining R. Set $V(R) = V_1 \cup \cdots \cup V_F$. For each $1 \le i < j \le f$, set $R^{ij} = P^{ij}$. For distint $a \in \{1, 2, 3, f+1, \ldots, F\}$ and $b \in \{f+1, \ldots, F\}$, let R^{ab} be a copy of $P^{\bar{a}\bar{b}}$ (which is defined on $V_{\bar{a}} \cup V_{\bar{b}}$) defined on $V_a \cup V_b$. Finally, for $a \in \{4, \ldots, f\}$ and $b \in \{f+1, \ldots, F\}$, set $R^{ab} = K[V_a, V_b]$. (Any complete bipartite graph is $(1,0)$-regular.) Set $R = \bigcup \left\{ R^{ab} : \{a,b\} \in \binom{[F]}{2} \right\}$. Note that $P \subseteq R$. By Setup 2.5 and the construction above, R has precisely $2\binom{f}{2} - 3$ many bipartite graphs R^{ab}, $1 \le a < b \le f$, which are (d, ε)-regular, and all remaining bipartite graphs R^{ab} of R are $(1, \varepsilon)$-regular.

Defining $\mathcal{F}_1$, $\mathcal{H}$ and $\hat{\mathcal{H}}$. Set $V(\mathcal{F}_1) = [F] = [2f-3]$ (cf. (61)). Set $\mathcal{F}_1 = \left\{ \{a,b,c\} \in \binom{[F]}{3} : \{\bar{a}, \bar{b}, \bar{c}\} \in \mathcal{F}_0 \right\}$. Note that $\mathcal{F}_0 \subseteq \mathcal{F}_1$, and moreover, that $(\mathcal{F}_1 \setminus \mathcal{F}_0) \cup \{1,2,3\}$ is isomorphic to $\mathcal{F}_0$. As such, $|\mathcal{F}_1| = 2|\mathcal{F}_0| - 1$. Set $V(\mathcal{H}) = V_1 \cup \cdots \cup V_F$. For each $\{h,i,j\} \in \mathcal{F}_0$, set $\mathcal{H}^{hij} = \mathcal{G}^{hij}$. For each $\{a,b,c\} \in \mathcal{F}_1 \setminus \mathcal{F}_0$, let $\mathcal{H}^{abc} \subseteq \mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})$ be a copy of $\mathcal{G}^{\bar{a}\bar{b}\bar{c}} \subseteq \mathcal{K}_3(P^{\bar{a}\bar{b}} \cup P^{\bar{b}\bar{c}} \cup P^{\bar{a}\bar{c}})$. Finally, for each $\{a,b,c\} \in \binom{[F]}{3} \setminus \mathcal{F}_1$, set $\mathcal{H}^{abc} = \mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})$. Set $\mathcal{H} = \bigcup \{\mathcal{H}^{abc} : \{a,b,c\} \in \mathcal{F}_1\}$ and $\hat{\mathcal{H}} = \bigcup \left\{ \mathcal{H}^{abc} : \{a,b,c\} \in \binom{[F]}{3} \right\}$. Note that $\mathcal{G} \subseteq \mathcal{H} \subseteq \hat{\mathcal{H}}$. By Setup 2.5 and the construction above, $\hat{\mathcal{H}}$ has precisely $|\mathcal{F}_1| = 2|\mathcal{F}_0| - 1$ many subhypergraphs $\mathcal{H}^{abc}$, $1 \le a < b < c \le F$, which are (α_{abc}, δ)-minimal, where $\alpha_{abc} = \alpha \pm \delta$, and all remaining subhypergraphs $\mathcal{H}^{abc}$ of $\hat{\mathcal{H}}$ are $(1, \delta)$-minimal (cf. Fact 6.5).

Observe that

$$\sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}^2_{\mathcal{F}_0, \mathcal{G}}(\{u,v,w\}) = \sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_1, \mathcal{H}}(\{u,v,w\}) = \# \{\mathcal{F}_1 \subseteq_{\text{p.i.}} \mathcal{H}\} = |\mathcal{K}_3(\hat{\mathcal{H}})|.$$

Since $\mathcal{F}_0$, P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5, R and $\hat{\mathcal{H}}$ satisfy the hypothesis of Setup 2.5, specifically with $\alpha_{abc} = \alpha \pm \delta$ when $\{a,b,c\} \in \mathcal{F}_1$ and $\alpha_{abc} = 1$ otherwise (cf. Fact 6.5), and all $d_{ab} \in \{d, 1\}$, $1 \le a < b \le F$ (cf. (62), (63)). As such, an application of the Counting Lemma (Theorem 6.2) gives

$$\begin{aligned} \sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}^2_{\mathcal{F}_0, \mathcal{G}}(\{u,v,w\}) &\le (1+\xi) \prod_{1 \le a < b < c \le F} \alpha_{abc} \times \prod_{1 \le a < b \le F} d_{ab} \times m^F \\ &\le (1+\xi)(\alpha+\delta)^{|\mathcal{F}_1|} d^{2\binom{f}{2} - 3} m^{2f-3} = (1+\xi)(\alpha+\delta)^{2|\mathcal{F}_0| - 1} d^{2\binom{f}{2} - 3} m^{2f-3} \\ &\overset{(67)}{\le} \frac{(1+\xi)}{(1-\xi)(\alpha-\delta)} (\alpha+\delta)^{2|\mathcal{F}_0| - 1} d^{2\binom{f}{2} - 6} m^{2f-6} |\mathcal{G}^{123}|. \end{aligned}$$

Using $(1-x)^{-1} \le 1 + 2x$ for $x \in [0, 1/2]$, we further infer

$$\begin{aligned} \sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}^2_{\mathcal{F}_0, \mathcal{G}}(\{u,v,w\}) &\le (1+2\xi)^2 \left(1 + 2\frac{\delta}{\alpha}\right)^{2|\mathcal{F}_0|} \left(\alpha^{|\mathcal{F}_0| - 1} d^{\binom{f}{2} - 3} m^{f-3}\right)^2 |\mathcal{G}^{123}| \\ &\overset{(62)}{\le} (1+2\xi)^{2+2F^3} \left(\alpha^{|\mathcal{F}_0| - 1} d^{\binom{f}{2} - 3} m^{f-3}\right)^2 |\mathcal{G}^{123}|, \end{aligned}$$

and so (66) follows from (61). □

7. Proof of the Slicing Lemma (Lemma 2.4)

In our proof of Lemma 2.4, we use the following two lemmas.

Lemma 7.1 (Auxiliary Slicing Lemma). *For all $\rho > 0$ and $s \in \mathbb{N}$, there exists $S_0 = S_{\text{Lem.7.1}}(\rho, s) \in \mathbb{N}$ so that the following statement holds.*

Let $K^{(3)}[A,B,C]$ be the complete 3-partite 3-graph with vertex partition $A \cup B \cup C$, $|A| = |B| = |C| = S \ge S_0$. Let $q_1, \ldots, q_s > 0$ be given where $\sum_{i=1}^s q_i \le 1$. Then, there exists a partition $K^{(3)}[A,B,C] = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_s$ so that, for each $1 \le i \le s$, $|\mathcal{J}_i| = (q_i \pm \rho) S^3$ and $|\mathcal{K}_{2,2,2}(\mathcal{J}_i)| \le q_i^8 \binom{S}{2}^3 (1+\rho)$.

Lemma 7.1 holds by the following standard probabilistic considerations. In the context above, independently for each $\{a, b, c\} \in K^{(3)}[A, B, C]$, place $\{a, b, c\} \in \mathcal{J}_i$ with probability q_i, $1 \le i \le s$, and place $\{a, b, c\} \in \mathcal{J}_0$ otherwise. To prove that the resulting partition has the desired properties, one appeals to the Chernoff and Janson inequalities (cf. [12]). For simplicity, we omit these details.

Lemma 7.2 (Inheritance Lemma)**.** *For all $\alpha_0, \delta_0 > 0$, there exists $\delta = \delta_{\mathrm{Lem.7.2}} > 0$ so that, for all $d > 0$, there exists $\varepsilon = \varepsilon_{\mathrm{Lem.7.2}} > 0$ so that the following statement holds.*

Let P be a triad (cf. Definition 2.1) with parameters $d, \varepsilon > 0$ above and sufficiently large integer m, and suppose $\mathcal{G} \subseteq \mathcal{K}_3(P)$ is (α, δ)-minimal w.r.t. P, for some $\alpha \ge \alpha_0$. Let $V_i' \subseteq V_i$, $1 \le i \le 3$, be given with $|V_1'| = |V_2'| = |V_3'| > \delta_0 m$. Then,

(1) *for each $1 \le i < j \le 3$, $P[V_i', V_j']$ is $(d, \varepsilon/\delta_0)$-regular;*
(2) *$\mathcal{G}[V_1', V_2', V_3']$ is (α', δ_0)-minimal w.r.t. $P[V_1', V_2', V_3']$ (cf. Definition 2.3), with $\alpha' = \alpha \pm \delta_0$.*

Statement (1) of Lemma 7.2 is well-known, but we prove Statement (2) in Section 7.2.

We shall also use the following easy fact. In what follows, $K^{(2)}_{2,2,2}$ denotes the complete 3-partite graph with 2 vertices in each vertex class. For a graph R, let $\mathcal{K}_{2,2,2}(R)$ denote the set of all copies of $K^{(2)}_{2,2,2}$ in R.

Fact 7.3. *For all $d > 0$, there exists $\varepsilon = \varepsilon_{\mathrm{Fact7.3}}(d) > 0$ so that the following statement holds. Suppose $R = R^{12} \cup R^{13} \cup R^{23}$ is a 3-partite graph with 3-partition $U_1 \cup U_2 \cup U_3$, where each R^{ij}, $1 \le i < j \le 3$, is (d, ε)-regular and where each $|U_i|$, $1 \le i \le 3$, is sufficiently large. Then, $|\mathcal{K}_{2,2,2}(R)| \le 2d^{12}\binom{|U_1|}{2}\binom{|U_2|}{2}\binom{|U_3|}{2}$.*

Fact 7.3 is a standard and well-known consequence of the Graph Counting Lemma (Lemma 8.2 in the Appendix). We omit the proof.

7.1. **Proof of Lemma 2.4.** This proof involves quite a few constants, which we summarize below in (75). Let $\alpha_0, \delta' > 0$ be given. Let

$$0 < \rho < \frac{1}{16}\min\{\alpha_0, \delta'\} \quad \text{satisfy} \quad \left(1 + 16\frac{\rho}{\alpha_0}\right)^9 \le 1 + \frac{\delta'}{2}. \tag{68}$$

With $\rho > 0$ given above, and for an integer $1 \le s \le 1/\alpha_0$, let $S_0(s) = S_{\mathrm{Lem.7.1}}(\rho, s)$ be the constant guaranteed by the Auxiliary Slicing Lemma (Lemma 7.1). Set $S_0 = \max_{1 \le s \le 1/\alpha_0} S_0(s)$, and set

$$S = \max\left\{S_0, \left\lceil \frac{2^9 \cdot 105}{\delta' \alpha_0^8} \right\rceil\right\}. \tag{69}$$

Set

$$3\tau = \xi = \frac{\rho}{3} \quad \text{and let} \quad \delta_{\mathrm{Thm.6.2}} = \delta_{\mathrm{Thm.6.2}}(r = 6, \alpha_0/2, \xi) \tag{70}$$

be the constant guaranteed by the Counting Lemma (Theorem 6.2). Set

$$0 < \delta_0 \le \min\left\{\frac{1}{2S}, \delta_{\mathrm{Thm.6.2}}\right\} \quad \text{to satisfy} \quad \left(1 + \frac{\delta_0}{\alpha_0}\right)^8 \le 1 + \xi. \tag{71}$$

The Inheritance Lemma guarantees constant

$$\delta_{\mathrm{Lem.7.2}} = \delta_{\mathrm{Lem.7.2}}(\alpha_0, \delta_0) \quad \text{and set} \quad \delta = \delta_{\mathrm{Lem.2.4}}(\alpha_0, \delta') = \delta_{\mathrm{Lem.7.2}}. \tag{72}$$

Let $d > 0$ be given. Let

$$\begin{aligned} \varepsilon_{\mathrm{Fact2.2}} = \varepsilon_{\mathrm{Fact2.2}}(d, \tau), \quad \varepsilon_{\mathrm{Fact6.5}} = \varepsilon_{\mathrm{Fact6.5}}(\delta, d), \quad \varepsilon_{\mathrm{Thm.6.2}} = \varepsilon_{\mathrm{Thm.6.2}}(r = 6, \alpha_0, \xi, \delta_0, d), \\ \varepsilon_{\mathrm{Lem.7.2}} = \varepsilon_{\mathrm{Lem.7.2}}(\alpha_0, \delta_0, \delta, d), \quad \varepsilon_{\mathrm{Fact7.3}} = \varepsilon_{\mathrm{Fact7.3}}(d), \end{aligned} \tag{73}$$

be the constants guaranteed by the Triangle Counting Lemma (Fact 2.2), Fact 6.5, the Counting Lemma (Theorem 6.2), the Inheritance Lemma (Lemma 7.2) and Fact 7.3, respectively. Set

$$\varepsilon = \varepsilon_{\mathrm{Lem.2.4}}(\alpha_0, \delta', \delta, d) = \delta_0 \times \min\left\{\varepsilon_{\mathrm{Fact2.2}}, \varepsilon_{\mathrm{Fact6.5}}, \varepsilon_{\mathrm{Thm.6.2}}, \varepsilon_{\mathrm{Lem.7.2}}, \varepsilon_{\mathrm{Fact7.3}}\right\}. \tag{74}$$

In all that follows, let m be a sufficiently large integer. The constants above can be summarized by the following hierarchies:

$$\begin{aligned} &\alpha_0, \delta' \gg \rho > \xi > \tau; \\ &\qquad \rho \gg \frac{1}{S},\ \delta_{\mathrm{Thm.6.2}} \gg \delta_0 \gg \delta_{\mathrm{Lem.7.2}} = \delta; \\ &\qquad\qquad d \gg \varepsilon_{\mathrm{Fact2.2}}, \varepsilon_{\mathrm{Fact6.5}}, \varepsilon_{\mathrm{Thm.6.2}}, \varepsilon_{\mathrm{Lem.7.2}}, \varepsilon_{\mathrm{Fact7.3}} \gg \varepsilon \ggg \frac{1}{m}. \end{aligned} \tag{75}$$

Let $P = P^{12} \cup P^{23} \cup P^{13}$ be a triad (cf. Definition 2.1) with parameters d, ε, m above, and let $\mathcal{G} \subseteq \mathcal{K}_3(P)$ be (α, δ)-minimal w.r.t. P, for some $\alpha \geq \alpha_0$. Let $\sigma_1, \dots, \sigma_s \geq \alpha_0$ be given with $\sum_{i=1}^{s} \sigma_i \leq \alpha$. To define the partition promised by Lemma 2.4, we make two preparations.

First, we prepare an application of the Auxiliary Slicing Lemma (Lemma 7.1). Let $A = \{a_1, \dots, a_S\}$, $B = \{b_1, \dots, b_S\}$, $C = \{c_1, \dots, c_S\}$ be auxiliary sets (cf. (69)). Set $q_i = \sigma_i/\alpha > 0$, $1 \leq i \leq s$, so that

$$\sum_{i=1}^{s} \sigma_i \leq \alpha \quad \implies \quad \sum_{i=1}^{s} q_i \leq 1. \tag{76}$$

For $q_1, \dots, q_s > 0$ defined above, let $K^{(3)}[A, B, C] = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \dots \cup \mathcal{J}_s$ be the partition guaranteed by Lemma 7.1. (This partition can be found by an exhaustive search in time depending on $S = O(1)$.) Second, we construct a vertex partition of $V(\mathcal{G}) = V(P)$ which 'mirrors' $A \cup B \cup C$. Fix $1 \leq i \leq 3$, and let

$$V_i = V_{i,0} \cup \bigcup_{x=1}^{S} V_{i,x} \tag{77}$$

be any partition satisfying $|V_{i,1}| = \dots = |V_{i,S}| = \lfloor m/S \rfloor$. (Clearly, such a partition is constructed in linear time $O(m)$.)

We now define the promised partition $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_s$. Fix $0 \leq a, b, c \leq S$. If $\{a, b, c\} \cap \{0\} \neq \emptyset$, put $\mathcal{G}[V_{1,a}, V_{2,b}, V_{3,c}] \subseteq \mathcal{G}_0$. Otherwise, for $0 \leq k \leq s$, put

$$\mathcal{G}[V_{1,a}, V_{2,b}, V_{3,c}] \subset \mathcal{G}_k \quad \iff \quad \{a, b, c\} \in \mathcal{J}_k. \tag{78}$$

Since $s \leq 1/\alpha_0$ and S (cf. (69)) are constants, the partition $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_s$ is constructed in time $O(m^3)$. It therefore remains to show that it has the desired property. For that, and for the remainder of the proof, fix $1 \leq k \leq s$. We show that $\mathcal{G}_k$ is (α_k, δ')-minimal w.r.t. P, where $d_{\mathcal{G}_k}(P) \stackrel{\text{def}}{=} \alpha_k = \sigma_k \pm \delta'$.

It is fairly easy to see that $d_{\mathcal{G}_k}(P) = \alpha_k = \sigma_k \pm \delta'$. Indeed,

$$|\mathcal{G}_k| = \sum_{\{a,b,c\} \in \mathcal{J}_k} |\mathcal{G}[V_{1,a}, V_{2,b}, V_{3,c}]| = \sum_{\{a,b,c\} \in \mathcal{J}_k} \alpha_{abc} |\mathcal{K}_3(P[V_{1,a}, V_{2,b}, V_{3,c}])|,$$

where we wrote, for each $\{a, b, c\} \in \mathcal{J}_k$, $\alpha_{abc} = |\mathcal{G}[V_{1,a}, V_{2,b}, V_{3,c}]|/|\mathcal{K}_3(P[V_{1,a}, V_{2,b}, V_{3,c}])|$. Now, for each fixed $\{a, b, c\} \in \mathcal{J}_k$, we will apply the Inheritance Lemma (Lemma 7.2). Since $|V_{1,a}| = |V_{2,b}| = |V_{3,c}| = \lfloor m/S \rfloor > \delta_0 m$ (cf. (71)), Lemma 7.2 (cf. (72)–(74)) guarantees that $\alpha_{abc} = \alpha \pm \delta_0$ and that each of $P[V_{1,a}, V_{2,b}]$, $P[V_{1,a}, V_{3,c}]$, $P[V_{2,b}, V_{3,c}]$ is $(d, \varepsilon/\delta_0)$-regular. Applying the Triangle Counting Lemma (Fact 2.2) (cf. (73), (74)), we have

$$|\mathcal{G}_k| = (\alpha \pm \delta_0) \sum_{\{a,b,c\} \in \mathcal{J}_k} |\mathcal{K}_3(P[V_{1,a}, V_{2,b}, V_{3,c}])| = (\alpha \pm \delta_0)(1 \pm \tau) d^3 \left\lfloor \frac{m}{S} \right\rfloor^3 |\mathcal{J}_k|.$$

Applying Lemma 7.1 (cf. (69)) and recalling $\alpha q_k = \sigma_k$, we see

$$|\mathcal{G}_k| = (1 \pm o(1))(q_k \pm \rho)(\alpha \pm \delta_0)(1 \pm \tau) d^3 m^3 \stackrel{(75)}{=} (\sigma_k \pm 7\rho)\, d^3 m^3.$$

Finally, by the Triangle Counting Lemma (Fact 2.2) (cf. (73), (74)), we see[2]

$$\alpha_k \stackrel{\text{def}}{=} d_{\mathcal{G}_k}(P) = \frac{|\mathcal{G}_k|}{|\mathcal{K}_3(P)|} = \frac{(\sigma_k \pm 7\rho)\, d^3 m^3}{(1\pm\tau)d^3m^3} \stackrel{(70)}{=} \sigma_k \pm 8\rho \stackrel{(68)}{=} \sigma_k \pm \delta', \tag{79}$$

as promised.

It remains to show that $\mathcal{G}_k$ is (α_k, δ')-minimal w.r.t. P, i.e.,

$$|\mathcal{K}_{2,2,2}(\mathcal{G}_k)| \le \alpha_k^8 d^{12} \binom{m}{2}^3 (1+\delta'), \tag{80}$$

for which we establish some notation and terminology. Let $\mathcal{O} = \{a, a', b, b', c, c'\} \in \mathcal{K}_{2,2,2}(\mathcal{J}_k)$ be an *octohedron* of $\mathcal{J}_k$, where we understand $a, a' \in A$, $b, b' \in B$, $c, c' \in C$. Define $\mathcal{G}_{\mathcal{O}} = \mathcal{G}[V_{1,a} \cup V_{1,a'}, V_{2,b} \cup V_{2,b'}, V_{3,c} \cup V_{3,c'}]$ to be the subhypergraph of $\mathcal{G}$ induced on $V_{1,a} \cup V_{1,a'} \cup V_{2,b} \cup V_{2,b'} \cup V_{3,c} \cup V_{3,c'}$. Note that, since $\mathcal{O} \subseteq \mathcal{J}_k$, we have $\mathcal{G}_{\mathcal{O}} \subseteq \mathcal{G}_k$. We call an element $\hat{\mathcal{O}} = \{v_{1,a}, v_{1,a'}, v_{2,b}, v_{2,b'}, v_{3,c}, v_{3,c'}\} \in \mathcal{K}_{2,2,2}(\mathcal{G}_{\mathcal{O}})$ *standard* if

$$v_{1,a} \in V_{1,a}, \quad v_{1,a'} \in V_{1,a'}, \quad v_{2,b} \in V_{2,b}, \quad v_{2,b'} \in V_{2,b'}, \quad v_{3,c} \in V_{3,c}, \quad v_{3,c'} \in V_{3,c'}, \tag{81}$$

where, necessarily, $a \ne a'$, $b \ne b'$, $c \ne c'$. We write

$$\mathcal{K}^+_{2,2,2}(\mathcal{G}_{\mathcal{O}}) = \left\{\hat{\mathcal{O}} \in \mathcal{K}_{2,2,2}(\mathcal{G}_{\mathcal{O}}) : \hat{\mathcal{O}} \text{ standard}\right\}, \quad \mathcal{K}^+_{2,2,2}(\mathcal{G}_k) = \bigcup \left\{\mathcal{K}^+_{2,2,2}(\mathcal{G}_{\mathcal{O}}) : \mathcal{O} \in \mathcal{K}_{2,2,2}(\mathcal{J}_k)\right\},$$
$$\text{and} \quad \mathcal{K}^-_{2,2,2}(\mathcal{G}_k) = \mathcal{K}_{2,2,2}(\mathcal{G}_k) \setminus \mathcal{K}^+_{2,2,2}(\mathcal{G}_k).$$

We proceed with the following claims.

Claim 7.4. *For each $\mathcal{O} \in \mathcal{K}_{2,2,2}(\mathcal{J}_k)$, $|\mathcal{K}^+_{2,2,2}(\mathcal{G}_{\mathcal{O}})| \le \alpha^8 d^{12} \lfloor m/S \rfloor^6 (1+\rho)$.*

Claim 7.5. $\left|\mathcal{K}^-_{2,2,2}(\mathcal{G}_k)\right| \le \frac{13}{S} d^{12} m^6$.

We postpone the proofs of Claims 7.4 and 7.5 in favor of finishing the proof of (80).

By Claims 7.4 and 7.5,

$$\begin{aligned}|\mathcal{K}_{2,2,2}(\mathcal{G}_k)| = \left|\mathcal{K}^-_{2,2,2}(\mathcal{G}_k)\right| + \left|\mathcal{K}^+_{2,2,2}(\mathcal{G}_k)\right| &= \left|\mathcal{K}^-_{2,2,2}(\mathcal{G}_k)\right| + \sum \left\{\left|\mathcal{K}^+_{2,2,2}(\mathcal{G}_{\mathcal{O}})\right| : \mathcal{O} \in \mathcal{K}_{2,2,2}(\mathcal{J}_k)\right\} \\ &\le \frac{13}{S} d^{12} m^6 + |\mathcal{K}_{2,2,2}(\mathcal{J}_k)| \times \alpha^8 d^{12} \lfloor m/S \rfloor^6 (1+\rho)\,.\end{aligned}$$

Applying Lemma 7.1 (cf. (69)) and recalling $\alpha q_k = \sigma_k \le \alpha_k + 8\rho$ (cf. (79)), we see

$$\begin{aligned}|\mathcal{K}_{2,2,2}(\mathcal{G}_k)| &\le (q_k\alpha)^8 d^{12} \frac{m^6}{8} (1+\rho)^2 + \frac{13}{S} d^{12} m^6 \le \left((1+o(1))\sigma_k^8 (1+3\rho) + \frac{105}{S}\right) d^{12} \binom{m}{2}^3 \\ &\stackrel{(79)}{\le} \left((\alpha_k + 8\rho)^8 (1+4\rho) + \frac{105}{S}\right) d^{12} \binom{m}{2}^3 \le \left(\left(1 + 8\frac{\rho}{\alpha_k}\right)^9 + \frac{105}{S\alpha_k^8}\right) \alpha_k^8 d^{12} \binom{m}{2}^3.\end{aligned}$$

From (79), we have that $\alpha_k \ge \sigma_k - 8\rho \ge \alpha_0 - 8\rho \stackrel{(68)}{\ge} \alpha_0/2$ (recall, by hypothesis, $\sigma_k \ge \alpha_0$). Therefore,

$$|\mathcal{K}_{2,2,2}(\mathcal{G}_k)| \le \left(\left(1 + 16\frac{\rho}{\alpha_0}\right)^9 + 2^8 \frac{105}{S\alpha_0^8}\right) \alpha_k^8 d^{12} \binom{m}{2}^3 \stackrel{(68),(69)}{\le} (1+\delta')\, \alpha_k^8 d^{12} \binom{m}{2}^3,$$

which proves (80).

[2] We write

$$x = \frac{a \pm b}{c \pm d} \quad \Longleftrightarrow \quad \frac{a-b}{c+d} \le x \le \frac{a+b}{c-d}.$$

Proof of Claim 7.4. Fix $\mathcal{O} = \{a, a', b, b', c, c'\} \in \mathcal{K}_{2,2,2}(\mathcal{J}_k)$. We apply the Counting Lemma (Theorem 6.2) to $\mathcal{G}_{\mathcal{O}}$, in the following way. Define a 6-partition $V_{1,a} \cup V_{1,a'} \cup V_{2,b} \cup V_{2,b'} \cup V_{3,c} \cup V_{3,c'}$, where $|V_{1,a}| = |V_{1,a'}| = |V_{2,b}| = |V_{2,b'}| = |V_{3,c}| = |V_{3,c'}| = \lfloor \frac{m}{S} \rfloor$, which will be the vertex set of the following 6-partite graph R and 3-graph $\mathcal{H} \subseteq \mathcal{K}_3(R)$. For each $(a_0, b_0, c_0) \in \{a, a'\} \times \{b, b'\} \times \{c, c'\}$, define $R^{a_0 b_0} = P[V_{1,a_0}, V_{2,b_0}]$, $R^{b_0 c_0} = P[V_{2,b_0}, V_{3,c_0}]$, $R^{a_0 c_0} = P[V_{1,a_0}, V_{3,c_0}]$. By the Inheritance Lemma (Lemma 7.2) (cf. (73), (74)), each of these 12 bipartite graphs is $(d, 2S\varepsilon)$-regular. Define $R^{a,a'} = K[V_{1,a}, V_{1,a'}]$, $R^{bb'} = K[V_{2,b}, V_{2,b'}]$, $R^{cc'} = K[V_{3,c}, V_{3,c'}]$. Each of these bipartite graphs is $(1, o(1))$-regular. Set

$$R = \bigcup \left\{ R^{xy} : \{x, y\} \in \binom{\{a, a', b, b', c, c'\}}{2} \right\}. \tag{82}$$

For each $(a_0, b_0, c_0) \in \{a, a'\} \times \{b, b'\} \times \{c, c'\}$, define $\mathcal{H}^{a_0 b_0 c_0} = \mathcal{G}[V_{1,a_0}, V_{2,b_0}, V_{3,c_0}]$. By the Inheritance Lemma (Lemma 7.2) (cf. (72)–(74)), we have that $\mathcal{H}^{a_0 b_0 c_0}$ is $(\alpha_{a_0 b_0 c_0}, \delta_0)$-minimal with respect to $R[V_{1,a_0}, V_{2,b_0}, V_{3,c_0}]$, for some $\alpha_{a_0 b_0 c_0} = \alpha \pm \delta_0$. We now define 12 further 3-partite 3-graphs. To that end, fix $\{x, y, z\} \in \binom{\{a,a',b,b',c,c'\}}{3}$ where, for some $d \in \{a, b, c\}$, we have $\{d, d'\} \subset \{x, y, z\}$. Define $\mathcal{H}^{xyz} = \mathcal{K}_3(R^{xy} \cup R^{yz} \cup R^{xy})$ so that, by Fact 6.5, (cf. (73), (74)) $\mathcal{H}^{xyz}$ is $(1, \delta)$-minimal w.r.t. $R^{xy} \cup R^{yz} \cup R^{xz}$. Set

$$\mathcal{H} = \bigcup \left\{ \mathcal{H}^{xyz} : \{x, y, z\} \in \binom{\{a, a', b, b', c, c'\}}{3} \right\}. \tag{83}$$

By the construction (82) and (83), every clique K_6 in $\mathcal{H}$ corresponds to a copy $\hat{\mathcal{O}} \in \mathcal{K}^{+}_{2,2,2}(\mathcal{G}_{\mathcal{O}})$, and vice-versa. Moreover, by construction, R and $\mathcal{H}$ satisfy the hypothesis of Theorem 6.2 with $r = 6$, α_0, δ_0, d, $2S\varepsilon$ (cf. (71)–(74)). Thus, by the Counting Lemma (recall $\alpha \geq \alpha_0$),

$$\begin{aligned} |\mathcal{K}^{+}_{2,2,2}(\mathcal{G}_{\mathcal{O}})| = |\mathcal{K}_6(\mathcal{H})| &\leq (1 + \xi) \prod \left\{ \alpha_{a_0 b_0 c_0} : (a_0, b_0, c_0) \in \{a, a'\} \times \{b, b'\} \times \{c, c'\} \right\} \times d^{12} \left\lfloor \frac{m}{S} \right\rfloor^6 \\ &\leq (1 + \xi)(\alpha + \delta_0)^8 d^{12} \left\lfloor \frac{m}{S} \right\rfloor^6 \leq (1 + \xi) \left(1 + \frac{\delta_0}{\alpha_0}\right)^8 \alpha^8 d^{12} \left\lfloor \frac{m}{S} \right\rfloor^6 \overset{(71)}{\leq} (1 + \xi)^2 \alpha^8 d^{12} \left\lfloor \frac{m}{S} \right\rfloor^6, \end{aligned}$$

and so Claim 7.4 follows from (70). □

Proof of Claim 7.5. Indeed, note that

$$\begin{aligned} \hat{\mathcal{O}} = \{v_{1,a}, v_{1,a'}, v_{2,b}, v_{2,b'}, v_{3,c}, v_{3,c'}\} \in \mathcal{K}^{-}_{2,2,2}(\mathcal{G}_k) \quad &\Longleftrightarrow \\ \text{either } \hat{\mathcal{O}} \cap V_{i,0} \neq \emptyset \text{ for some } i \in [3], \text{ or (81) holds with } &a = a' \text{ or } b = b' \text{ or } c = c'. \end{aligned}$$

Clearly, at most $3Sm^5 = O(m^5)$ elements $\hat{\mathcal{O}} \in \mathcal{K}_{2,2,2}(\mathcal{G}_k)$ can satisfy $\hat{\mathcal{O}} \cap (V_{1,0} \cup V_{2,0} \cup V_{3,0}) \neq \emptyset$. To enumerate $\hat{\mathcal{O}} \in \mathcal{K}^{-}_{2,2,2}(\mathcal{G}_k)$ of the latter variety, fix $1 \leq a, a', b, b', c, c' \leq S$ where, w.l.o.g., $a = a'$. We use Fact 7.3 to estimate

$$|\mathcal{K}_{2,2,2}(\mathcal{G}_k[V_{1,a}, V_{2,b} \cup V_{2,b'}, V_{3,c} \cup V_{3,c'}])| \leq |\mathcal{K}_{2,2,2}(P[V_{1,a}, V_{2,b} \cup V_{2,b'}, V_{3,c} \cup V_{3,c'}])|.$$

To apply Fact 7.3, set $U_1 = V_{1,a}$, $U_2 = V_{2,b} \cup V_{2,b'}$, $U_3 = V_{3,c} \cup V_{3,c'}$ and $R^{ij} = P[U_i, U_j]$ for all $1 \leq i < j \leq 3$. Since each $|U_i| \geq \lfloor m/S \rfloor$, $1 \leq i \leq s$, the Inheritance Lemma (Lemma 7.2) (cf. (73), (74)) guarantees that each R^{ij}, $1 \leq i < j \leq 3$, is $(d, 2S\varepsilon)$-regular. Fact 7.3 (cf. (73), (74)) then guarantees

$$|\mathcal{K}_{2,2,2}(P[V_{1,a}, V_{2,b} \cup V_{2,b'}, V_{3,c} \cup V_{3,c'}])| \leq 2d^{12} \binom{|V_{1,a}|}{2} \binom{|V_{2,b} \cup V_{2,b'}|}{2} \binom{|V_{3,c} \cup V_{3,c'}|}{2} \leq 4d^{12} \frac{m^6}{S^6}.$$

Summing over all $3S^5$ many 5-element indices $1 \leq a, a', b, b', c, c' \leq S$, we conclude

$$|\mathcal{K}^{-}_{2,2,2}(\mathcal{G}_k)| \leq O(m^5) + 12S^5 d^{12} \frac{m^6}{S^6} \leq \frac{13}{S} d^{12} m^6,$$

as promised. □

7.2. **Proof of Lemma 7.2.** We prove Statement (2) of the Inheritance Lemma, and use the following important concept of Frankl and Rödl [6].

Definition 7.6 ((α,δ)-regularity)**.** Let P be a triad (cf. Definition 2.1) and let $\mathcal{G} \subseteq \mathcal{K}_3(P)$ satisfy $d_{\mathcal{G}}(P) = \alpha$. For $\delta > 0$, we say that $\mathcal{G}$ is (α,δ)-*regular* if, for all $Q \subseteq P$ with $|\mathcal{K}_3(Q)| > \delta|\mathcal{K}_3(P)|$, we have $|d_{\mathcal{G}}(Q) - \alpha| < \delta$.

It was shown by Nagle, Poerschke, Rödl and Schacht (see Theorem 2.1 and Corollary 2.1 in [14]) that, with suitably defined constants, (α,δ)-minimality and (α,δ)-regularity are equivalent concepts.

Theorem 7.7 (Nagle, Poerschke, Rödl, Schacht [14])**.** *For all $\alpha_0, \hat{\delta} > 0$, there exists $\delta = \delta_{\mathrm{Thm.7.7}}(\alpha_0, \hat{\delta}) > 0$ so that, for all $d > 0$, there exists $\varepsilon = \varepsilon_{\mathrm{Thm.7.7}}(\alpha_0, \hat{\delta}, \delta, d) > 0$ so that the following statement holds.*

Let P be a triad with parameters d, ε and a sufficiently large integer m, and let $\mathcal{G} \subseteq \mathcal{K}_3(P)$ satisfy $d_{\mathcal{G}}(P) = \alpha \geq \alpha_0$.

1. *If $\mathcal{G}$ is (α,δ)-minimal w.r.t. P, then $\mathcal{G}$ is $(\alpha,\hat{\delta})$-regular w.r.t. P.*
2. *If $\mathcal{G}$ is (α,δ)-regular w.r.t. P, then $\mathcal{G}$ is $(\alpha,\hat{\delta})$-minimal w.r.t. P.*

By using Theorem 7.7, the proof of Lemma 7.2 is a formality.

Proof of Lemma 7.2. Let $\alpha_0, \delta_0 > 0$ be given. With $\hat{\delta} = \delta_0$, Theorem 7.7 ensures constant

$$\delta_1 = \delta_{\mathrm{Thm.7.7}}(\alpha_0 - \delta_0, \delta_0) > 0, \quad \text{and set} \quad \delta_2 = \frac{1}{4}\delta_0^3\delta_1. \tag{84}$$

With $\hat{\delta} = \delta_2$, Theorem 7.7 ensures constant

$$\delta_3 = \delta_{\mathrm{Thm.7.7}}(\alpha_0, \delta_2) > 0, \quad \text{and set} \quad \delta = \delta_{\mathrm{Lem.7.2}}(\alpha_0, \delta_0) = \delta_3. \tag{85}$$

Let $d > 0$ be given. With $\tau = 1/2$, let

$$\varepsilon_0 = \varepsilon_{\mathrm{Fact2.2}}(d, \tau = 1/2) > 0 \tag{86}$$

be the constant guaranteed by the Triangle Counting Lemma (Fact 2.2). With $\hat{\delta} = \delta_0$, let

$$\varepsilon_1 = \varepsilon_{\mathrm{Thm7.7}}(\alpha_0 - \delta_0, \delta_0, \delta_1, d) > 0 \tag{87}$$

be the constant guaranteed by Theorem 7.7. With $\hat{\delta} = \delta_2$, let

$$\varepsilon_2 = \varepsilon_{\mathrm{Thm7.7}}(\alpha_0, \delta_2, \delta, d) > 0 \tag{88}$$

be the constant guaranteed by Theorem 7.7. Set

$$\varepsilon = \varepsilon_{\mathrm{Lem.7.2}}(\alpha_0, \delta_0, \delta, d) = \min\{\delta_0\varepsilon_0, \delta_0\varepsilon_1, \varepsilon_2\}. \tag{89}$$

In all that follows, let m be a sufficiently large integer.

Let P be a triad (cf. Definition 2.1) with parameters d, ε, m above, and suppose $\mathcal{G} \subseteq \mathcal{K}_3(P)$ is (α,δ)-minimal w.r.t. P. Let $V_i' \subseteq V_i$, $1 \leq i \leq 3$, be given with $|V_1'| = |V_2'| = |V_3'| > \delta_0 m$. To simplify notation in the argument below, we write $P' = P[V_1', V_2', V_3']$, $\mathcal{G}' = \mathcal{G} \cap \mathcal{K}_3(P')$, $\alpha' = d_{\mathcal{G}'}(P')$. By Statement (1) of Lemma 7.2, each bipartite graph $P[V_i', V_j']$, $1 \leq i < j \leq 3$, is $(d, \varepsilon/\delta_0)$-regular, and so by (89), each such $P[V_i', V_j']$ is (d, ε_0)-regular and (d, ε_1)-regular.

We first apply Theorem 7.7 to $\mathcal{G}$ and P. To that end, recall from our hypothesis that $\mathcal{G}$ is (α,δ)-minimal w.r.t. P, where $\alpha \geq \alpha_0$ and where each P^{ij}, $1 \leq i < j \leq 3$, is (d,ε)-regular (cf. (85), (88) and (89)). As such, Theorem 7.7 ensures that $\mathcal{G}$ is (α, δ_2)-regular w.r.t. P. We proceed with the following claim.

Claim 7.8. *$\mathcal{G}'$ is (α', δ_1)-regular w.r.t. P', where $\alpha' = \alpha \pm \delta_2$.*

Proof of Claim 7.8. We first check that $\alpha' = \alpha \pm \delta_2$. Using Statement (1) of Lemma 7.2, it follows from (two applications of) the Triangle Counting Lemma (Fact 2.2) that

$$|\mathcal{K}_3(P')| > \frac{1}{2}d^3|V_1'||V_2'||V_3'| > \frac{1}{2}\delta_0^3 d^3 |V_1||V_2||V_3| > \frac{1}{4}\delta_0^3|\mathcal{K}_3(P)| \overset{(84)}{>} \delta_2|\mathcal{K}_3(P)|. \tag{90}$$

Setting $Q = P'$ in Definition 7.6, we conclude from the (α, δ_2)-regularity of $\mathcal{G}$ w.r.t. P that $|\alpha' - \alpha| < \delta_2$, as promised.

Now, let $Q \subseteq \mathcal{K}_3(P')$ be given satisfying $|\mathcal{K}_3(Q)| > \delta_1 |\mathcal{K}_3(P')| \overset{(84)}{=} (4\delta_2/\delta_0^3)|\mathcal{K}_3(P')|$. By the penultimate bound of (90), we see $|\mathcal{K}_3(Q)| > \delta_2 |\mathcal{K}_3(P)|$, and so by the (α, δ_2)-regularity of $\mathcal{G}$ w.r.t. P, we have $|d_{\mathcal{G}}(Q) - \alpha| < \delta_2$, where clearly, $d_{\mathcal{G}}(Q) = d_{\mathcal{G}'}(Q)$. By the Triangle Inequality, $|d_{\mathcal{G}'}(Q) - \alpha'| \leq 2\delta_2 \overset{(84)}{<} \delta_1$, concluding the proof. □

We now apply Theorem 7.7 to $\mathcal{G}'$ and P'. To that end, we have from Claim 7.8 that $\mathcal{G}'$ is (α', δ_1)-regular w.r.t. P', where $\alpha' \geq \alpha - \delta_2 \geq \alpha_0 - \delta_0$, and where each constituent bipartite graph $P[V_i', V_j']$, $1 \leq i < j \leq 3$, is (d, ε_1)-regular (cf. (84), (87), and (89)). As such, Theorem 7.7 ensures that $\mathcal{G}'$ is (α', δ_0)-minimal w.r.t. P', as promised. □

8. Appendix: Proof of Theorem 6.2

As we mentioned in Remark 6.3, the Counting Lemma was proven in [9] in the following special case.

Theorem 8.1 (Haxell, Nagle, Rödl [9])**.** *For every $r \in \mathbb{N}$ and for all $\alpha, \mu > 0$, there exists $\delta = \delta_{\text{Thm. 8.1}}(r, \alpha, \mu) > 0$ so that, for all $d > 0$, there exists $\varepsilon = \varepsilon_{\text{Thm. 8.1}}(r, \alpha, \mu, \delta, d) > 0$ so that the following holds.*

Let $R = P$ and $\mathcal{H} = \mathcal{G}$ satisfy the hypothesis of Setup 6.1 with $r, \alpha_0 = \alpha, \delta, d_0 = d, \varepsilon > 0$ above, where all $\alpha_{hij} = \alpha$, $1 \leq h < i < j \leq r$, and all $d_{ij} = d$, $1 \leq i < j \leq r$, and where $m \in \mathbb{N}$ is sufficiently large. Then, $|\mathcal{K}_r(\mathcal{G})| = (1 \pm \mu)\alpha^{\binom{r}{3}} d^{\binom{r}{2}} m^r$.

It is not difficult to show that Theorem 8.1 implies Theorem 6.2, and we sketch this proof now. Our proof closely follows lines from [15, 16], where these ideas were used for different notions of regularity.

Before we start, we state, without proof, two very well-known results referenced earlier in this paper: the *Graph Counting Lemma* and the *Graph Extension Lemma*. To state these results, we use the following notation (for a graph R satisfying the hypothesis of Setup 6.1): $\mathcal{K}_r(R) = \binom{R}{K_r^{(2)}}$ and for each $\{u, v, w\} \in \mathcal{K}_3(R)$, $\text{ext}_{K_r^{(2)}, R}(\{u, v, w\}) = |\{U \in \mathcal{K}_r(R) : \{u, v, w\} \subseteq U\}|$.

Lemma 8.2 (Graph Counting and Extension Lemma)**.** *For every $r \in \mathbb{N}$ and for all $d_0, \tau > 0$, there exists $\varepsilon = \varepsilon_{\text{Lem.8.2}}(r, d_0, \tau) > 0$ so that the following holds.*

Let R be a graph satisfying the hypothesis of Setup 6.1 with $r, d_0, \varepsilon > 0$ above, and where $m \in \mathbb{N}$ is sufficiently large. Then,

1. $|\mathcal{K}_r(R)| = (1 \pm \tau) \prod_{1 \leq i < j \leq r} d_{ij} \times m^r$*;*
2. *for each $1 \leq h < i < j \leq r$, all but τm^3 triangles $\{v_h, v_i, v_j\} \in \mathcal{K}_3(R^{hi} \cup R^{ij} \cup R^{hj})$ satisfy*
$$\text{ext}_{K_r^{(2)}, R}(\{v_h, v_i, v_j\}) = (1 \pm \tau) \frac{1}{d_{hi} d_{ij} d_{hj}} \prod_{1 \leq a < b \leq r} d_{ab} \times m^r.$$

8.1. **Proof-sketch of Theorem 6.2.** We omit a formal description of constants and refer to the hierarchy $1/r, \alpha_0, \xi \gg \delta \gg d_0 \gg \varepsilon \ggg 1/m$, which is consistent with the quantification of Theorem 6.2. In the hierarchy above, we introduce further constants μ, α, d, τ (cf. Remark 8.4):
$$\frac{1}{r}, \alpha_0, \xi \gg \mu, \alpha \gg \delta \gg d_0 \gg d, \tau \gg \varepsilon \ggg \frac{1}{m}, \tag{91}$$
which is consistent with the quantification of Theorem 8.1. Now, let R and $\mathcal{H}$ satisfy the hypothesis of Setup 6.1 with the constants $\alpha_0, \delta, d_0, \varepsilon, m$. We show
$$|\mathcal{K}_r(\mathcal{H})| = (1 \pm \xi) \prod_{1 \leq h < i < j \leq r} \alpha_{hij} \times \prod_{1 \leq i < j \leq r} d_{ij} \times m^r. \tag{92}$$

The main idea for proving (92) is to 'randomly' partition the graph R and 3-graph $\mathcal{H}$ in a standard but suitable way. Fix $1 \leq h < i < j \leq r$, and set $p_{hij} = \alpha/\alpha_{hij}$ (cf. (91)) and $A_{hij} = \lfloor 1/p_{hij} \rfloor$. Define $\mathcal{H}^{hij} = \mathcal{H}_0^{hij} \cup \mathcal{H}_1^{hij} \cup \cdots \cup \mathcal{H}_{A_{hij}}^{hij}$ by the following rule: for every $1 \leq a \leq A_{hij}$ and $\{v_h, v_i, v_j\} \in \mathcal{H}^{hij}$, put $\{v_h, v_i, v_j\} \in \mathcal{H}_a^{hij}$ independently with probability p_{hij}. We define a random partition of R similarly. Fix $1 \le i < j \le r$, and set $q_{ij} = d/d_R(V_i, V_j)$ and $B_{ij} = \lfloor 1/q_{ij} \rfloor$. Define $R^{ij} = R_0^{ij} \cup R_1^{ij} \cup \cdots \cup R_{B_{ij}}^{ij}$ by the following rule: for every $1 \le b \le B_{ij}$ and $\{v_i, v_j\} \in R^{ij}$, put $\{v_i, v_j\} \in R_b^{ij}$ independently with probability q_{ij}. We set

$$\mathcal{H}_0 = \bigcup \left\{ \mathcal{H}_0^{hij} : 1 \le h < i < j \le r \right\} \quad \text{and} \quad R_0 = \bigcup \left\{ R_0^{ij} : 1 \le i < j \le r \right\}. \tag{93}$$

We have the following standard probabilistic fact.

Fact 8.3. *With probability $1 - o(1)$, where $o(1) \to 0$ as $m \to \infty$, the following hold:*

(1) *for all $1 \le i < j \le r$ and $1 \le b \le B_{ij}$, the graph R_b^{ij} is $(d, 2\varepsilon)$-regular;*
(2) *for all $\{h, i, j\} \in \binom{[r]}{3}$ and $(a, b_{hi}, b_{ij}, b_{hj}) \in [A_{hij}] \times [B_{hi}] \times [B_{ij}] \times [B_{hj}]$, the 3-graph $\mathcal{H}_a^{hij} \cap \mathcal{K}_3(P_{b_{hi}}^{hi} \cup P_{b_{ij}}^{ij} \cup P_{b_{hj}}^{hj})$ is $(\alpha \pm o(1), 2\delta)$-minimal w.r.t. $P_{b_{hi}}^{hi} \cup P_{b_{ij}}^{ij} \cup P_{b_{hj}}^{hj}$.*

With probability at least 1/6, the following hold:

(3) *the graph R_0 satisfies $|R_0| \le 3 \sum_{1 \le i < j \le r} q_{ij} |R^{ij}| = 3\binom{r}{2} d m^2$.*
(4) *for each $\{h, i, j\} \in \binom{[r]}{3}$, $|\mathcal{H}_0^{hij}| \le 2\binom{r}{3} p_{hij} |\mathcal{H}^{hij}| = 2\binom{r}{3}\alpha |\mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj})| \le 4 r^3 \alpha d_{hi} d_{ij} d_{hj} m^3$.*

We omit the proof of Fact 8.3, but indicate its key ingredients. Statement (1) is a well-known and routine application of the Chernoff inequality. Statement (2) is a routine application of the Janson inequality. Statements (3) and (4) are immediate applications of the Markov inequality. The last inequality in Statement (4) follows from the Triangle Counting Lemma (Lemma 8.2) with $r = 3$.

We may now argue (92). We call a function $\boldsymbol{a} : \binom{[r]}{3} \to \mathbb{N}$ an *$\mathcal{H}$-pattern* if, for every $1 \le h < i < j \le r$, we have $\boldsymbol{a}(\{h, i, j\}) \stackrel{\text{def}}{=} \boldsymbol{a}_{hij} \le A_{hij}$. We call a function $\boldsymbol{b} : \binom{[r]}{2} \to \mathbb{N}$ an *R-pattern* if, for every $1 \le i < j \le r$, we have $\boldsymbol{b}(\{i, j\}) \stackrel{\text{def}}{=} \boldsymbol{b}_{ij} \le B_{ij}$. We call a pair of functions $(\boldsymbol{a}, \boldsymbol{b})$ an *$(\mathcal{H}, R)$-pattern* if $\boldsymbol{a}$ is an $\mathcal{H}$-pattern and $\boldsymbol{b}$ is an R-pattern. For an $(\mathcal{H}, R)$-pattern $(\boldsymbol{a}, \boldsymbol{b})$, we write

$$\mathcal{H}_{\boldsymbol{a}} = \bigcup \left\{ \mathcal{H}_{\boldsymbol{a}_{hij}}^{hij} : \{h, i, j\} \in \binom{[r]}{3} \right\}, \quad R_{\boldsymbol{b}} = \bigcup \left\{ R_{\boldsymbol{b}_{ij}}^{ij} : \{i, j\} \in \binom{[r]}{2} \right\}, \quad \mathcal{H}_{\boldsymbol{ab}} = \mathcal{H}_{\boldsymbol{a}} \cap \mathcal{K}_3(R_{\boldsymbol{b}}).$$

Since there are (exactly)

$$\prod_{1 \le h < i < j \le r} A_{hij} \times \prod_{1 \le i < j \le r} B_{ij} = \prod_{1 \le h < i < j \le r} \left\lfloor \frac{\alpha_{hij}}{\alpha} \right\rfloor \times \prod_{1 \le i < j \le r} \left\lfloor \frac{d_R(V_i, V_j)}{d} \right\rfloor \le \alpha^{-\binom{r}{3}} d^{-\binom{r}{2}} = O(1) \tag{94}$$

many $(\mathcal{H}, R)$-patterns $(\boldsymbol{a}, \boldsymbol{b})$, Fact 8.3 implies that every one of them yields a pair $\mathcal{G} = \mathcal{H}_{\boldsymbol{ab}}$ and $P = R_{\boldsymbol{b}}$ which satisfy the hypothesis of Theorem 8.1 with the constants $r, \alpha \pm o(1), 2\delta, d, 2\varepsilon, m$ (cf. (91)). As such, we may immediately conclude the lower bound in (92):

$$\begin{aligned}
|\mathcal{K}_r(\mathcal{H})| &\ge \sum \{ |\mathcal{K}_r(\mathcal{H}_{\boldsymbol{ab}})| : (\boldsymbol{a}, \boldsymbol{b}) \text{ is an } (\mathcal{H}, R)\text{-pattern} \} \\
&\stackrel{\text{Thm.8.1}}{\ge} (1 - \mu) \alpha^{\binom{r}{3}} d^{\binom{r}{2}} m^r \times \prod_{1 \le h < i < j \le r} A_{hij} \times \prod_{1 \le i < j \le r} B_{ij} \\
&\stackrel{(94)}{\ge} (1 - \mu) \prod_{1 \le h < i < j \le r} (\alpha_{hij} - \alpha) \times \prod_{1 \le i < j \le r} (d_R(V_i, V_j) - d) \times m^r \\
&\stackrel{(91)}{\ge} (1 - \xi) \prod_{1 \le h < i < j \le r} \alpha_{hij} \times \prod_{1 \le i < j \le r} d_{ij} \times m^r,
\end{aligned}$$

where we used $d_R(V_i, V_j) \ge d_{ij} - \varepsilon$, $1 \le i < j \le r$. It remains to prove the upper bound in (92).

Observe that

$$|\mathcal{K}_r(\mathcal{H})| \leq \sum \{|\mathcal{K}_r(\mathcal{H}_{\boldsymbol{ab}})| : (\boldsymbol{a},\boldsymbol{b}) \text{ is an } (\mathcal{H},R)\text{-pattern}\} + \left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{2} \cap R_0 \neq \emptyset\right\}\right| + \left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{3} \cap \mathcal{H}_0 \neq \emptyset\right\}\right|. \quad (95)$$

An application of Theorem 8.1 (cf. (91)) yields

$$\sum \{|\mathcal{K}_r(\mathcal{H}_{\boldsymbol{ab}})| : (\boldsymbol{a},\boldsymbol{b}) \text{ is an } (\mathcal{H},R)\text{-pattern}\} \leq (1+\mu)\alpha^{\binom{r}{3}}d^{\binom{r}{2}}m^r \times \prod_{1\leq h<i<j\leq r} A_{hij} \times \prod_{1\leq i<j\leq r} B_{ij} \overset{(94)}{\leq} (1+\mu) \prod_{1\leq h<i<j\leq r} \alpha_{hij} \times \prod_{1\leq i<j\leq r} d_{ij} \times m^r. \quad (96)$$

Clearly,

$$\left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{2} \cap R_0 \neq \emptyset\right\}\right| \leq |R_0|m^{r-2} \overset{\text{Fact8.3}}{\leq} 3r^2dm^r. \quad (97)$$

Momentarily, we will prove the following bound (see Remark 8.4):

$$\left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{3} \cap \mathcal{H}_0 \neq \emptyset\right\}\right| \leq r^3\tau m^r + 8r^6\alpha \prod_{1\leq i<j\leq r} d_{ij} \times m^r. \quad (98)$$

Combining (95)–(98) and applying $\alpha_{hij} \geq \alpha_0$ and $d_{ij} \geq d_0$, $\{h,i,j\} \in \binom{[r]}{3}$, we have

$$|\mathcal{K}_r(\mathcal{H})| \leq \prod_{1\leq h<i<j\leq r} \alpha_{hij} \times \prod_{1\leq i<j\leq r} d_{ij} \times m^r \left(1 + \mu + 3r^2d\alpha_0^{-\binom{r}{3}}d_0^{-\binom{r}{2}} + r^3\tau\alpha_0^{-\binom{r}{3}}d_0^{-\binom{r}{2}} + 8r^6\alpha\alpha_0^{-\binom{r}{3}}\right) \overset{(91)}{\leq} (1+\xi) \prod_{1\leq h<i<j\leq r} \alpha_{hij} \times \prod_{1\leq i<j\leq r} d_{ij} \times m^r, \quad (99)$$

as promised.

Remark 8.4. In (96), (98) and (99), the constants in (91) play a subtle role. We introduced the constant α in (91) to satisfy $\delta \ll \alpha \ll \alpha_0$. Indeed, we used take $\alpha \ll \alpha_0$ in (99), but we used $\delta \ll \alpha$ in (96) (for applying Theorem 8.1). The quantification of Theorem 6.2 allows $d_0 \ll \delta$, so we must accept $\alpha \gg d_0$, which makes (98) necessarily more delicate than (97). □

Proof of (98). Note that

$$\left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{3} \cap \mathcal{H}_0 \neq \emptyset\right\}\right| \leq \sum_{1\leq h<i<j\leq r} \left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{3} \cap \mathcal{H}_0^{hij} \neq \emptyset\right\}\right|. \quad (100)$$

Now, fix $1 \leq h < i < j \leq r$. Clearly

$$\left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{3} \cap \mathcal{H}_0^{hij} \neq \emptyset\right\}\right| = \sum_{\{v_h,v_i,v_j\}\in\mathcal{H}_0^{hij}} \mathrm{ext}_{K_r^{(3)},\mathcal{H}}(\{v_h,v_i,v_j\})$$

$$\leq \sum_{\{v_h,v_i,v_j\}\in\mathcal{H}_0^{hij}} \mathrm{ext}_{K_r^{(2)},R}(\{v_h,v_i,v_j\}) \overset{\text{Lem.8.2}}{\leq} \tau m^r + 2\frac{1}{d_{hi}d_{ij}d_{hj}} \prod_{1\leq a<b\leq r} d_{ab} \times m^{r-3} \times |\mathcal{H}_0^{hij}| \overset{\text{Fact8.3}}{\leq} \tau m^r + 8r^3\alpha \prod_{1\leq a<b\leq r} d_{ab} \times m^r. \quad (101)$$

Applying (101) to (100) yields (98). □

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