Brian Curtin

Research Summary

Department of Mathematics, University of South Florida, Tampa, FL 33620
fax: (813) 974 - 2700 **e-mail:** bcurtin@math.usf.edu

My areas of research lie on the interface of algebra and combinatorics, and touch on several other areas of mathematics such as orthogonal polynomials and number theory. The main objects of my considerations are Bose-Mesner algebras, subconstituent algebras, Leonard pairs, spin models, and planar algebras. I am particularly interested in duality (Section 2), spin models (Section 3), algebraic characterizations of combinatorial properties (Section 4), and Leonard pairs (Sections 5 and 6) and tridiagonal pairs (Sections 7 and 8) in representation theory. In these areas I often focus on examples related to distance-regular graphs and connection to quantum groups. I have also studied some combinatorial planar algebras (Section 9). Recently I have used linear algebra to study Fibonacci numbers (Section 10).

1 Background: Bose-Mesner algebras

Bose-Mesner algebras arose independently in three areas during the 1950's and 1960's: statistical designs, centralizer algebras of permutation groups, and distance-transitive graphs. A period of growth in the subject occurred in the 1970's and 1980's after Delsarte and others unified these examples and demonstrated applications to error correcting codes, combinatorial designs, and found connections to orthogonal polynomials. The classification of finite simple groups also motivated and aided the study of distance-transitive and distance-regular graphs. Details can be found in [6, 7]. Since the 1990's connections have been developed to link invariants, quantum groups, maximal abelian subalgebras, and subfactors.

Throughout X shall denote a finite nonempty set of size n, and $\mathbb{M}_X$ shall denote the $\mathbb{C}$-algebra of matrices with entries in $\mathbb{C}$ whose rows and columns are indexed by X. For $A \in \mathbb{M}_X$ and for x, $y \in X$, let $A(x, y)$ denote the (x, y)-entry of A. For A, $B \in \mathbb{M}_X$, let $A \circ B$ denote the Hadamard (entry-wise) product of A and B: $(A \circ B)(x, y) = A(x, y)B(x, y)$ for all x, $y \in X$. The ordinary matrix product of A and B will be denoted by the juxtaposition AB. For $A \in \mathbb{M}_X$, let ${}^{t}A$ and $\overline{A}$ denote the transpose and complex conjugate of A, respectively. For $A \in \mathbb{M}_X$ with nonzero entries, let A^{-} denote the matrix in $\mathbb{M}_X$ whose (x, y) entry is $A(y, x)^{-1}$.

Definition 1.1 A *Bose-Mesner algebra* on X is a commutative subalgebra $\mathcal{M}$ of $\mathbb{M}_X$ which is closed under Hadamard product, which is closed under transposition, and which contains the identity matrix I and the all 1's matrix J.

Let $\mathcal{M}$ denote a $(d + 1)$-dimensional Bose-Mesner algebra on X. Then $\mathcal{M}$ has a unique *basis of Hadamard idempotents* $\{A_i\}_{i=0}^{d}$: $A_0 = I$, $A_i \circ A_j = \delta_{ij} A_i$ $(0 \le i, j \le d)$, $\sum_{i=0}^{d} A_i = J$, where δ_{ij} denotes the Kronecker symbol. In addition, $\mathcal{M}$ has a unique *basis of primitive idempotents* $\{E_i\}_{i=0}^{d}$: $E_0 = |X|^{-1}J$, $E_iE_j = \delta_{ij}E_i$ $(0 \le i, j \le d)$, $\sum_{i=0}^{d} E_i = I$.

Our work often treats the Bose-Mesner algebras which arise from distance-regular graphs. A finite simple connected graph Γ with vertex set X and diameter d is said to be *distance-regular* whenever there exist scalars p^h_{ij} $(0 \le h, i, j \le d)$ such that for all x, $y \in X$ with $\partial(x, y) = h$, the number of vertices z with $\partial(x, z) = i$ and $\partial(y, x) = j$ is p^h_{ij}. We abbreviate $c_0 = 0$, $c_i = p^i_{1i-1}$ $(1 \le i \le d)$, $a_i = p^i_{1i}$ $(0 \le i \le d)$, $b_i = p^i_{1i+1}$ $(0 \le i \le d-1)$, and $b_d = 0$. Suppose Γ is a distance-regular graph with vertex set X and diameter d. The i^{th} distance-matrix of Γ is the matrix $A_i \in \mathbb{M}_X$ with $A_i(x,y) = 1$ if $\partial(x,y) = i$ and 0 otherwise. Observe that $A := A_1$ is the adjacency matrix of Γ. Then $\{A_i\}_{i=0}^d$ is the basis of Hadamard idempotents of a Bose-Mesner algebra and A_i is a polynomial of degree i in $A = A_1$. For more on distance-regular graphs and Bose-Mesner algebras see [6, 7, 35].

The subconstituent algebra of a Bose-Mesner algebra was introduced by Terwilliger to refine the combinatorial data encoded by an algebra [52].

Definition 1.2 Let $\mathcal{M}$ be a $(d+1)$-dimensional Bose-Mesner algebra on X. Fix a "base point" $p \in X$. For $A \in \mathcal{M}$, let $\rho(A) \in \mathbb{M}_X$ denote the diagonal matrix with (x,x)-entry $\rho(A)(x,x) = A(p,x)$. Let $\mathcal{M}^* = \{\rho(A) \,|\, A \in \mathcal{M}\}$. Then $\mathcal{M}^*$ is called the *dual Bose-Mesner algebra* of $\mathcal{M}$ with respect to the base point p. Set $E_i^* = \rho(A_i)$ and $A_i^* = \rho(nE_i)$ $(0 \le i \le d)$. Since ρ is a linear bijection, $\{E_i^*\}_{i=0}^d$ and $\{A_i^*\}_{i=0}^d$ are bases of $\mathcal{M}^*$. We refer to $\{E_i^*\}_{i=0}^d$ and $\{A_i^*\}_{i=0}^d$ as the *basis of dual idempotents* and the *basis of dual Hadamard idempotents* of $\mathcal{M}^*$, respectively. The dual Hadamard idempotents and dual idempotents inherit orderings from the primitive and Hadamard idempotents of $\mathcal{M}$, respectively. The subalgebra $\mathcal{T} = \mathcal{T}(p)$ of $\mathbb{M}_X$ generated by $\mathcal{M} \cup \mathcal{M}^*$ is called the *subconstituent* (or *Terwilliger*) *algebra* of $\mathcal{M}$ with respect to p. Note: The use of * in connection with subconstituent algebras does not refer to the operation of taking the conjugate-transpose of a matrix.

2 Research: Duality in Bose-Mesner algebras

We describe some of our wok concerning duality in Bose-Mesner algebras.

Notation 2.1 Let X and $\widetilde{X}$ denote finite nonempty sets of the same size $|X| = |\widetilde{X}| = n$. Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ denote $(d+1)$-dimensional Bose-Mesner algebras on X and $\widetilde{X}$, respectively. Fix $p \in X$ and $\widetilde{p} \in \widetilde{X}$, and let $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ denote the subconstituent algebras of $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ with respect to p and $\widetilde{p}$, respectively. Write $\widetilde{\ }$ with all objects associated with $\widetilde{\mathcal{T}}$.

Our starting point is a result of Neumaier.

Lemma 2.2 *[6, 7, 50] With Notation 2.1, let $\Psi : \mathcal{M} \to \widetilde{\mathcal{M}}$ denote a linear bijection. Then the following are equivalent.*

(i) *For all A, $B \in \mathcal{M}$, $\Psi(AB) = \Psi(A) \circ \Psi(B)$, $\Psi(A \circ B) = n^{-1}\Psi(A)\Psi(B)$.*

(ii) *There exist orderings $E_0, E_1, \ldots, E_d$ and $\widetilde{E}_0, \widetilde{E}_1, \ldots, \widetilde{E}_d$ of the primitive idempotents such that $\Psi(E_i) = \widetilde{A}_i$, $\Psi(A_i) = n^{\mathrm{t}}\widetilde{E}_i$ $(0 \le i \le d)$.*

Suppose (i) and (ii) hold. Then the map Ψ is called a formal duality *from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$. The map $\widetilde{\Psi} = n\Psi^{-1}$. is a formal duality from $\widetilde{\mathcal{M}}$ to $\mathcal{M}$. Moreover, $\widetilde{\Psi}\Psi = n\tau|_{\mathcal{M}}$ and $\Psi\widetilde{\Psi} = n\widetilde{\tau}|_{\widetilde{\mathcal{M}}}$. $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are said to be* formally dual *whenever there exists a formal duality from one to the other. Orderings of the idempotents as in (ii) are said to be* standard *for Ψ.*

We have recently studied an extension of formal duality in the subconstituent algebra. We begin by describing formal duality using the subconstituent algebra.

Theorem 2.3 (Curtin [11]) *With Notation 2.1, fix orderings $E_0, E_1, \ldots, E_d$ and $\widetilde{E}_0, \widetilde{E}_1, \ldots, \widetilde{E}_d$ of the primitive idempotents. Then the following are equivalent.*

(i) *$\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are formally dual and $E_0, E_1, \ldots, E_d$ and $\widetilde{E}_0, \widetilde{E}_1, \ldots, \widetilde{E}_d$ are standard orderings for some formal duality from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$.*

(ii) *There exists a linear bijection $\psi : \mathcal{M}^* \to \widetilde{\mathcal{M}}$ satisfying $\psi(A_i^*) = \widetilde{A}_i$, $\psi(E_i^*) = {}^{\mathrm{t}}\widetilde{E}_i$ $(0 \le i \le d)$.*

(iii) *There exists a linear bijection $\widetilde{\psi} : \widetilde{\mathcal{M}}^* \to \mathcal{M}$ satisfying $\widetilde{\psi}(\widetilde{E}_i^*) = E_i$, $\widetilde{\psi}(\widetilde{A}_i^*) = {}^{\mathrm{t}}A_i$ $(0 \le i \le d)$.*

Suppose (i)–(iii) hold. Let ψ and $\widetilde{\psi}$ be as in (ii) and (iii), respectively. Then ψ and $\widetilde{\psi}$ are algebra isomorphisms. Let $\Psi = n\psi\rho|_{\mathcal{M}}$ and $\widetilde{\Psi} = n\tau\widetilde{\psi}\widetilde{\rho}|_{\widetilde{\mathcal{M}}}$. Then Ψ is a formal duality from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$, where the fixed orderings of the primitive idempotents are standard for Ψ, and $\widetilde{\Psi}$ is a formal duality from $\widetilde{\mathcal{M}}$ to $\mathcal{M}$, where the fixed orderings of the primitive idempotents are standard for $\widetilde{\Psi}$. Moreover, $\Psi\widetilde{\Psi} = n\widetilde{\tau}|_{\widetilde{\mathcal{M}}}$, $\widetilde{\Psi}\Psi = n\tau|_{\mathcal{M}}$.

Our main result is the following.

Theorem 2.4 (Curtin [11]) *With Notation 2.1, suppose $\psi : \mathcal{T} \to \widetilde{\mathcal{T}}$ is an algebra isomorphism. Then the following are equivalent.*

(i) *$\psi(\mathcal{M}) = \widetilde{\mathcal{M}}^*$, $\psi(\mathcal{M}^*) = \widetilde{\mathcal{M}}$, $\Psi = n\psi\rho|_{\mathcal{M}}$ is a formal duality from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$, $\widetilde{\Psi} = n\tau\psi^{-1}\widetilde{\rho}|_{\widetilde{\mathcal{M}}}$ is a formal duality from $\widetilde{\mathcal{M}}$ to $\mathcal{M}$, $\Psi\widetilde{\Psi} = n\widetilde{\tau}|_{\widetilde{\mathcal{M}}}$, and $\widetilde{\Psi}\Psi = n\tau|_{\mathcal{M}}$.*

(ii) *There exist orderings $E_0, E_1, \ldots, E_d$ and $\widetilde{E}_0, \widetilde{E}_1, \ldots, \widetilde{E}_d$ of the primitive idempotents such that $\psi({}^{\mathrm{t}}A_i) = \widetilde{A}_i^*$, $\psi(A_i^*) = \widetilde{A}_i$, $\psi(E_i) = \widetilde{E}_i^*$, $\psi(E_i^*) = {}^{\mathrm{t}}\widetilde{E}_i$ $(0 \le i \le d)$.*

Whenever (i) and (ii) hold, the map ψ is called a hyper-duality *from $\mathcal{T}$ to $\widetilde{\mathcal{T}}$. $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are said to be* hyper-dual *to one another whenever there exists a hyper-duality from one to the other and $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are said to be* hyper-dual *with respect to $(p, \widetilde{p})$.*

In the paper [11], we also characterized hyper-duality in terms of the irreducible modules for the Terwilliger algebras, and we have also specialized our results to hyper-self-duality.

We have shown that many examples of formally dual pairs of Bose-Mesner algebras are in fact hyper-dual. In [24] we showed that spin models give rise to hyper-self-dual Bose-Mesner algebras. We generalized this in [11] where we showed that the formally dual pair of Bose-Mesner algebras constructed by Jaeger, Matsumoto, and Nomura [42] are hyper-dual. In the paper [12], we showed that the Bose-Mesner algebras of formally self-dual strongly regular graphs are in fact hyper-self-dual. We have considered the inheritance of hyper-duality in several constructions.

In the paper [13] we studied the inheritance of hyper-duality by block and quotient Bose-Mesner algebras associated with imprimitive hyper-dual pairs of Bose-Mesner algebras. Suppose $\mathcal{M}$ is an imprimitive Bose-Mesner algebra. Then one may construct a block Bose-Mesner algebra $\mathcal{B}$ by restricting the point set and a quotient Bose-Mesner algebra $\mathcal{Q}$ by identifying points based upon this imprimitivity. Suppose $\widetilde{\mathcal{M}}$ is formally dual to $\mathcal{M}$, then it too is imprimitive and the corresponding block Bose-Mesner algebra $\widetilde{\mathcal{B}}$ is formally dual to $\mathcal{Q}$ and the corresponding quotient quotient Bose-Mesner algebra $\widetilde{\mathcal{Q}}$ is formally dual to $\mathcal{B}$. We have shown that taking great care in the choice of base-points this fact is also true for hyper-duality.

In a current project we show that the Kronecker product of hyper-dual Bose-Mesner algebras is hyper-dual and that fusions of hyper-dual Bose-Mesner algebras are hype-dual. We have a number of other results which remain to be written up. Among the open problems on hyper-duality are to show that various other examples of formally dual Bose-Mesner algebras are hyper-dual. We plan to re-examine some of Delsarte's work from the perspective of hyper-duality. We hyper-duality may be of useful in finding pairs maximal abelian subalgebras.

3 Research: Spin models

A spin model is the basic data for a statistical mechanical construction of link invariants due to V.F.R. Jones [43] which was later generalized in [5, 47]. Nomura and Jaeger independently gave an algebraic framework for the study of spin models by showing that each spin model W is contained in a Bose-Mesner algebra [41, 42, 51]. We have studied spin models from this perspective.

Definition 3.1 A *(matrix) spin model on* X is a matrix $W \in \mathbb{M}_X$ that has all entries non-zero, and which satisfies the following equations for all $a, b, c \in X$: $\sum_{x\in X} \frac{W(a,x)}{W(b,x)} = n\delta_{ab}$, $\sum_{x\in X} \frac{W(a,x)W(b,x)}{W(c,x)} = \sqrt{n}\,\frac{W(a,b)}{W(a,c)W(c,b)}$. Setting $b = c$ in in the last equation shows that every diagonal entry of W is the same; we refer this value as the *modulus* of W.

Theorem 3.2 ([42, 51]) *Let W be a spin model on X modulus α. For every pair $(b, c) \in X$, let Y_{bc} in $V = \mathbf{C}^X$ whose x-entry is given by $Y_{bc}(x) = \frac{W(x,b)}{W(x,c)}$ $(x \in X)$. Let $\mathcal{N}(W) = \{A \in \mathbb{M}_X \mid AY_{bc} \in \mathbf{C}Y_{bc}$ for all $b, c \in X\}$. For $A \in \mathcal{N}(W)$, let $\Psi(A) \in \mathbb{M}_X$ be defined by $AY_{bc} = \Psi(A)(b, c)Y_{bc}$ $(b, c \in X)$.*

(i) *$\mathcal{N}(W)$ is a Bose-Mesner algebra on X containing W.*

(ii) *$\Psi(A) \in \mathcal{N}(W)$ for all $A \in \mathcal{N}(W)$, and the map $A \mapsto \Psi(A)$ is a formal duality of $\mathcal{N}(W)$.*

(iii) *$\Psi(A) = \alpha^{-1}W \circ ({}^{t}W^{-}({}^{t}W \circ A)$ for all $A \in \mathcal{N}(W)$.*

Let $\mathcal{M}$ be a Bose-Mesner algebra such that $W \in \mathcal{M} \subseteq \mathcal{N}(W)$. Such a Bose-Mesner algebra $\mathcal{M}$ is said to *support* W. Any Bose-Mesner algebra which supports a spin model inherits a formal duality from $\mathcal{N}(W)$. In fact, we have shown that this is a hyper-duality.

Theorem 3.3 (Curtin and Nomura [24]) *Let W denote a spin model on X and suppose that $\mathcal{M}$ is a Bose-Mesner algebra which supports W. Fix a base point $p \in X$, and let $\mathcal{T}$ denote the Terwilliger algebra of $\mathcal{M}$ with respect to p. Define $K \in \mathbb{M}_X$ by $K = \rho({}^{t}W){}^{t}W^{-}\rho(W)$. Then the map $\mathcal{T} \to \mathcal{T}$ defined by $A \mapsto K^{-1}AK$ is a hyper-self-duality of $\mathcal{T}$.*

We have focused most of our research on spin models on the special case that the spin model is supported by the Bose-Mesner algebra of a distance-regular graph. In this situation we have shown the following results. The distance-regular graph algebra is "thin" in the sense of Terwilliger (For every irreducible $\mathcal{T}$-module W, $\dim E_i^* W \leq 1$) [17]. With K. Nomura, we have shown that these distance-regular graphs are either almost bipartite or 1-homogeneous (for all vertices u, v, w with $\partial(u, v) = 1$, $\partial(u, w) = r$, $\partial(v, w) = s$, the number $|\Gamma_i(u) \cap \Gamma_j(v) \cap \Gamma_1(w)|$ is independent of the choice of u, v, w, depending only on r, s, i, j) [28]. Also with K. Nomura, we have given formulas for the intersection numbers of a distance-regular graph whose Bose-Mesner algebra supports a spin model in terms of two parameters. We conjecture that in fact, there are really just two one parameter families.

4 Research: Algebraic characterizations of graph regularity

We have studied some graph regularity properties from an algebraic perspective. We describe one such result to give taste of our work.

Definition 4.1 Let Γ denote a connected graph with diameter d. Fix a vertex x of Γ. Γ is said to be *t-homogeneous around x* whenever whenever for all i, j, k, ℓ $(0 \leq i, j, k, \ell \leq d)$ there exist nonnegative integers $\gamma^{i,j}_{k,\ell}(x)$ such that $|\Gamma_k(x) \cap \Gamma_1(y) \cap \Gamma_\ell(z)| = \gamma^{i,j}_{k,\ell}(x)$ for all vertices y, z of Γ with $\partial(x, y) = i$, $\partial(y, z) = j$, and $\partial(z, x) = t$.

In [23], we generalized the following well-known theorem of algebraic graph theory: A connected finite simple graph is regular (ie, each vertex is adjacent to the same number of vertices) if and only if the all-ones matrix spans an ideal of the adjacency algebra [38]. We have several results of the following sort.

Theorem 4.2 *Let Γ denote a connected graph with diameter d. Fix any vertex x of Γ, and let $E^*_i = E^*_i(x)$ $(0 \leq i \leq d)$. Let $\mathcal{S}$ be the algebra generated by A and E^*_i $(0 \leq i \leq d)$. Fix t $(0 \leq t \leq d(x))$. Then Γ is t-homogeneous around x if and only if the left ideal $\mathcal{S}E^*_t$ of $\mathcal{S}$ generated by E^*_t is linearly spanned by $\{E^*_i A_j E^*_t \,|\, 0 \leq i, j \leq d\}$.*

Recently with K. Nomura we extended this result for "almost" 1-homogeneous distance-regular graphs [27]–with the assumption of distance-regularity we have much stronger results. We have studied extensively the 2-homogeneous bipartite distance-regular graphs. All known examples of such graphs support a spin model. We have characterized these graphs as extremal in both combinatorial and spectral inequalities [18], and we have characterized them in terms of their subconstituent algebras [19, 21]. We have studied the local structure from an algebraic perspective [20]. One of our more interesting results is a connection between the 2-homogeneous bipartite distance-regular graphs and the quantum group $U_q(sl(2))$.

Theorem 4.3 *(Curtin [18]) Suppose Γ is not isomorphic to the d-cube. Then Γ is bipartite and 2-homogeneous if and only if there exists a complex scalar $q \notin \{0, 1, -1\}$ such that*

$$c_i = e_i\,[\,i\,], \qquad b_i = e_i\,[\,d - i\,] \qquad (0 \leq i \leq d), \tag{1}$$

where for all integers i

$$e_i = q^{i-1}(q^d + q^2)(q^d + q^{2i})^{-1}, \qquad [\,i\,] = (q^i - q^{-i})(q - q^{-1})^{-1}. \tag{2}$$

Theorem 4.4 (Curtin and Nomura [26]) *Suppose Γ is bipartite and 2-homogeneous but not isomorphic to the d-cube. Let $q \notin \{0, 1, -1\}$ be any complex scalar such that (1), (2) hold, and let e_i $(0 \leq i \leq d)$ be as in (2). Then the matrices*

$$X^- = \sum_{j=0}^{d-1} e_j^{-1} E^*_j A E^*_{j+1}, \qquad X^+ = \sum_{j=1}^{d} e_j^{-1} E^*_j A E^*_{j-1}, \qquad Y = \sum_{j=0}^{d} q^{d-2j} E^*_j$$

generate $\mathcal{T}$ and satisfy the relations of $U_q(sl(2))$:

$$YX^- = q^2 X^- Y, \quad YX^+ = q^{-2} X^+ Y, \quad X^- X^+ - X^+ X^- = \frac{Y - Y^{-1}}{q - q^{-1}}.$$

Ito and Terwilliger have been working on a generalization of this result connecting the affine $U_q(\widehat{sl(2)})$ to the bilinear forms and related graphs. We hope to contribute to this line of research.

5 Background: Leonard pairs

Leonard pairs were introduced by P. Terwilliger [52] as an algebraic abstraction of work of D. Leonard [49] (cf. [6]) concerning the orthogonal polynomials in the terminating branch of the Askey scheme [48]. Like the results of Askey and Wilson [3, 4] in the continuous case, Leonard shows describes these polynomials using (basic) hypergeometric series. They consist of the q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, quantum q-Krawtchouk, affine q-Krawtchouk, Racah, Hahn, dual Hahn, Krawtchouk, Bannai-Ito, and orphan polynomials.

These orthogonal polynomials arise frequently in the representation theory of groups, Lie algebras, quantum groups, and several algebras which arise in physics. Because of the equivalence between Leonard pairs and the appropriate orthogonal polynomials,we view the appearance of Leonard pairs in the representation theory of an algebra as one explanation of why these orthogonal polynomials appear.

We use the following terms. Throughout $\mathbb{K}$ shall denote a field. A square matrix over $\mathbb{K}$ is said to be *tridiagonal* whenever every nonzero entry appears on the diagonal, the superdiagonal, or the subdiagonal. A tridiagonal matrix is *irreducible* whenever the entries on the sub- and superdiagonals are all nonzero. Let V denote a vector space over $\mathbb{K}$ of finite positive dimension. By a *linear operator on* V we mean a $\mathbb{K}$-linear map from V to V. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all linear operators on V.

Definition 5.1 Let V denote a vector space over $\mathbb{K}$ of finite positive dimension. Let A, A^* denote an ordered pair of elements taken from $\mathrm{End}(V)$. We call this pair a *Leonard pair on* V whenever for each $B \in \{A, A^*\}$, there exists a basis of V with respect to which the matrix representing B is diagonal and the matrix representing the other operator in the pair is irreducible tridiagonal.

6 Research: Modular Leonard triples and spin Leonard pairs

We have studied an extension of a Leonard pair known as a Leonard triple. Leonard triples reveal a symmetry that was not obvious for Leonard pairs.

Definition 6.1 Let V denote a vector space over $\mathbb{K}$ of finite positive dimension. Let A, A^*, $A^\diamond$ denote an ordered triple of elements taken from $\mathrm{End}(V)$. We call this triple a *Leonard triple on* V whenever for each $B \in \{A, A^*, A^\diamond\}$, there exists a basis of V with respect to which the matrix representing B is diagonal and the matrices representing the other two operators in the triple are irreducible tridiagonal.

We are interested in the following type of Leonard triple. To describe it we need a definition. By an *antiautomorphism* of $\mathrm{End}(V)$, we mean a $\mathbb{K}$-linear bijection $\tau : \mathrm{End}(V) \to \mathrm{End}(V)$ such that $\tau(XY) = \tau(Y)\tau(X)$ for all $X, Y \in \mathrm{End}(V)$.

Definition 6.2 Let A, A^*, $A^\diamond$ denote a Leonard triple on V. Then this Leonard triple is said to be *modular* whenever for each $B \in \{A, A^*, A^\diamond\}$ there exists an antiautomorphism of $\mathrm{End}(V)$ which fixes B and swaps the other two operators in the triple.

We shown that a triple of linear operators on V is a modular Leonard triple if and only if relative to some basis the operators are represented by one of four triples of matrices. The entries of these matrices are given by up to four independent parameters. To give the flavor of this results we recall the most general of the four families of modular Leonard triple. The other three families can be derived from this one by certain limiting procedures.

Lemma 6.3 (Curtin [14]) *Let d denote a nonnegative integer. Set*

$$\begin{aligned}
A &= \operatorname{tridiag}\begin{pmatrix} b_0 & b_1 & \cdots & b_{d-1} & * \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ * & c_1 & \cdots & c_{d-1} & c_d \end{pmatrix}, \\
A^* &= \operatorname{diag}(\theta_0, \theta_1, \ldots, \theta_d), \\
A^\diamond &= \operatorname{tridiag}\begin{pmatrix} b_0\nu_1 & b_1\nu_2 & \cdots & b_{d-1}\nu_d & * \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ * & c_1/\nu_1 & \cdots & c_{d-1}/\nu_{d-1} & c_d/\nu_d \end{pmatrix},
\end{aligned}$$

where

$$\nu_i = \nu q^{i-1} \qquad (1 \le i \le d), \tag{3}$$

$$\theta_i = \theta_0 + h(1-q^i)(1-\nu^2 q^{i-1})q^{-i} \qquad (0 \le i \le d), \tag{4}$$

$$b_0 = -\frac{h(1-q^d)(1+\nu^3 q^{d-1})}{q^d(1-\nu)}, \tag{5}$$

$$b_i = -\frac{h(1-q^{d-i})(1-\nu^2 q^{i-1})(1+\nu^3 q^{d+i-1})}{q^{d-i}(1-\nu q^i)(1-\nu^2 q^{2i-1})} \qquad (1 \le i \le d-1), \tag{6}$$

$$c_i = \frac{h\nu(1-q^i)(1+\nu q^{d-i})(1-\nu^2 q^{d+i-1})}{q^{d-i+1}(1-\nu q^{i-1})(1-\nu^2 q^{2i-1})} \qquad (1 \le i \le d-1), \tag{7}$$

$$c_d = \frac{h\nu(1-q^d)(1+\nu)}{q(1-\nu q^{d-1})}, \tag{8}$$

$$a_i = \theta_0 - b_i - c_i \qquad (0 \le i \le d)\ (c_0 = 0,\ b_d = 0) \tag{9}$$

for some scalars θ_0, h, ν, q in $\mathbb{K}$ such that $h\nu q \neq 0$, $q^i \neq 1$ $(1 \le i \le d)$, $\nu^3 q^{2d-1-i} \neq -1$ $(1 \le i \le d)$, and $\nu^2 q^i \neq 1$ $(0 \le i \le 2d-2)$. Then A, A^*, $A^\diamond$ is a modular Leonard triple on $\mathbb{K}^{d+1}$.

We note that there is modular group (in fact braid group) action on each isomorphism class of modular Leonard triple–hence the name. Modular Leonard triples are closely related to another object.

Definition 6.4 We say that a Leonard pair S, S^* on V is a *spin Leonard pair* whenever there exist invertible linear operators U, U^* in $\operatorname{End}(V)$ such that

$$US = SU, \tag{10}$$

$$U^*S^* = S^*U^*, \tag{11}$$

$$US^*U^{-1} = U^{*-1}SU^*. \tag{12}$$

In this case, we refer to the ordered pair U, U^* as a *Boltzmann pair* for S, S^*.

If Γ is a distance-regular graph whose Bose-Mesner algebra supports a spin model, then every irreducible module of it subconstituent algebra has a natural structure as a spin Leonard pair.

Theorem 6.5 (Curtin [15]) *If A, A^*, $A^\diamond$ is a modular Leonard triple, then any two of them form a spin Leonard pair. Conversely, if S, S^* is a spin Leonard pair with Boltzmann pair U, U^*, then S, S^*, $U^{-1}S^*U$ is a modular Leonard triple.*

We are currently considering modular Leonard triples in representation theory. We have shown the following:

Theorem 6.6 (Curtin) *Every modular Leonard triple as in Lemma 6.3 satisfies*

$$\begin{aligned}
[A, A^*]_q &= xA^\diamond + yI + z(A + A^*), \\
[A^*, A^\diamond]_q &= xA + yI + z(A^* + A^\diamond), \\
[A^\diamond, A]_q &= xA^* + yI + z(A^\diamond + A),
\end{aligned}$$

for some scalars x, y, z, where $x \neq 0$.

The remaining families of modular Leonard triples satisfy similar relations. We are studying the algebra $\mathcal{P}$ with generators A, A^*, $A^\diamond$ which satisfy these three relations. We have shown that any finite-dimensional representation of $\mathcal{P}$ on which A, A^*, $A^\diamond$ act in a diagonalizable and multiplicity-free manner is a modular Leonard triple.

There are a number of open problems concerning Leonard triples and modular Leonard triples. We plan to study arbitrary Leonard triples. Preliminary results suggest that almost every Leonard pair can be extended to a Leonard triple.

7 Background: Tridiagonal pairs

Tridiagonal pairs we introduced by T. Ito, K. Tanabe, and P. Terwilliger [39] to describe an algebraic situation associated with distance-regular graphs and to generalize Leonard pairs. Instances of tridiagonal pairs arise in connection with distance-regular graphs, representations of $U_q(\widehat{sl(2)})$, and representations of the Onsager algebra. TD pairs are also related to the classical and quantum quadratic Askey-Wilson algebras introduced by Granovskii et al. [36].

Definition 7.1 By a *Tridiagonal pair* on V, we mean an ordered pair A, A^* in $\mathrm{End}(V)$ which satisfy the following four conditions.

(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering V_0, V_1, $\ldots$, V_d of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \tag{13}$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

(iii) There exists an ordering V_0^*, V_1^*, $\ldots$, V_δ^* of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta), \tag{14}$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(iv) There is no a subspace W of V such that both $AW \subseteq W$, $A^*W \subseteq W$, other than $W = 0$ and $W = V$.

A Leonard pair on V is simply a tridiagonal pair on V for which $\dim V_i = 1$, $\dim V_1^* = 1$ for $0 \leq i \leq d$.

Lemma 7.2 [39] *Let A, A^* denote a tridiagonal pair on V of diameter d. Fix orderings of the eigenspaces V_0, V_1, $\ldots$, V_d of A and of the eigenspaces V_0^*, V_1^*, $\ldots$, V_d^* of A^* which satisfy (13) and (14). Define U_i $(0 \leq i \leq d)$ by*

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d).$$

Then $V = U_0 + U_1 + \cdots + U_d$ (direct sum). The sequence U_0, U_1, $\ldots$, U_d is called the split decomposition *of V relative to the fixed standard orderings. For notational convenience we define $U_{-1} = 0$ and $U_{d+1} = 0$.*

8 Research: Mild tridiagonal pairs and $U_q(\widehat{sl(2)})$

We have studied a special family of tridiagonal pairs with our recently graduated Ph.D. student Hasan Alnajjar.

Let A, A^* denote a TD pair of diameter d on V. We say that A, A^* is mild whenever it is not a Leonard pair, but that $\dim V_0^* = \dim V_d = 1$ and the vector space V is of the form $V = Mv^* + M^*v$ for some $v^* \in V_0^*$, $v \in V_d$. In [1] Alnajjar and Curtin constructed a distinguished basis $\mathcal{B} = \{v_0^*, v_1, v_1^*, v_2, v_2^*, \ldots, v_{d-1}, v_{d-1}^*, v_d\}$ with v_i, $v_i^* \in U_i$ for the underlying vector space V of a mild tridiagonal pair. We explicitly described the action of mild tridiagonal pairs of q-Serre type in terms of 5 independent parameters.

Definition 8.1 Let A, A^* denote a tridiagonal pair on V. Then A, A^* is said to be of *q-Serre type* whenever the following hold:

$$A^3A^* - [3]A^2A^*A + [3]AA^*A^2 - A^*A^3 = 0,$$
$$A^{*3}A - [3]A^{*2}AA^* + [3]A^*AA^{*2} - AA^{*3} = 0.$$

These relations are called the *q-Serre relations* and are among the defining relations of $U_q(\widehat{sl_2})$ [8, 9], which we now recall.

Definition 8.2 [8, 9] The *quantum affine algebra* $U_q(\widehat{sl_2})$ is the associative $\mathbb{K}$-algebra with generators $e_i^{\pm}$, $K_i^{\pm}$, $(i = 0, 1)$ and relations: $K_iK_i^{-1} = K_i^{-1}K_i = 1$, $K_0K_1 = K_1K_0$, $K_ie_i^{\pm}K_i^{-1} = q^{\pm 2}e_i^{\pm}$, $K_ie_j^{\pm}K_i^{-1} = q^{\mp 2}e_j^{\pm}$, $i \neq j$, $[e_i^+, e_i^-] = \frac{K_i - K_i^{-1}}{q - q^{-1}}$, $[e_0^{\pm}, e_1^{\mp}] = 0$, $(e_i^{\pm})^3e_j^{\pm} - [3](e_i^{\pm})^2e_j^{\pm}e_i^{\pm} + [3]e_i^{\pm}e_j^{\pm}(e_i^{\pm})^2 - e_j^{\pm}(e_i^{\pm})^3 = 0 \qquad (i \neq j)$.

Let $F_i : V \to V$ $(0 \leq i \leq d)$ denote the the projection map from V onto U_i. Set $K = \sum_{i=0}^{d} q^{2i-d}F_i$. Set $R = A - \sum_{i=0}^{d} \theta_iF_i$, $L = A^* - \sum_{i=0}^{d} \theta_i^*F_i$. Then $RU_i \subseteq U_{i+1}$ and $LU_i \subseteq U_{i-1}$ $(0 \leq i \leq d)$. The maps R and L are referred to as the *raising* and *lowering* maps with respect to the split decomposition. Our main result is the following.

Theorem 8.3 (Alnajjar and Curtin [2]) *Let A, A^* be a mild tridiagonal pair on V of q-Serre type with diameter $d \geq 3$. Then there exist linear operators ℓ and r on V such that V supports a $U_q(\widehat{sl_2})$-module structure on which the following linear operators vanish on V: $e_1^+ - l$, $e_0^+ - r$, $e_1^- - R$, $e_0^- - L$, $K_0 - K$, and $K_1 - K^{-1}$. Moreover, this module structure is irreducible.*

T. Ito and P. Terwilliger [40] have recently announced that for any tridiagonal pair of q-Serre type there exist linear operators K, ℓ, and r which together with R and L behave as in Theorem 8.3. The result of Ito and Terwilliger is more general than ours, but in our result we explicitly describe the action of the generators on an "attractive" basis.

In future work, we shall use our construction to describe the associated $U_q(\widehat{sl_2})$-module structure in terms of a tensor product of evaluation modules for $U_q(\widehat{sl_2})$ [8, 9].

9 Research: Planar algebras constructed from graphs

Planar algebras were introduced by V.F.R. Jones as a characterization of an invariant associated with subfactors of type II_1 [45]. They arose from the deep connections between knot theory and subfactors [44] . A planar algebra is a graded vector space $\mathcal{V} = \cup_{n \in \mathbb{Z}^+ \cup \{+,-\}} \mathcal{V}_n$ over $\mathbb{C}$ which is closed under certain operators. True to its operator algebra origins, an emphasis is placed upon the interactions of the operators. These operators are defined diagrammatically by objects known as planar tangles.

To describe our work we don't need to give the (very technical) definition of a planar tangle; instead we give some sense what one is by analogy to skein theory [10] (local knot diagrams which may have loose ends at some boundary). Traditionally, strings interact in pairs and in one of two ways–over and under crossing. In planar algebras possibly many strings interact in possibly many different ways. A planar tangle is a skein diagram of noncrossing strings which enter (and then exit) various interactions, which we draw as a labeled box. The net result of certain collections of interactions may be the same as another collection interaction, like the Reidemeister moves in knot theory. We also allow formal linear combinations of these diagrams. Our interest is in combinatorially constructed planar algebras which are generated by a single type of interaction between pairs of strings [46]. The construction is similar to statistical mechanical one used to compute the link invariant from a spin model. We shall side-step this description and give an equivalent combinatorial one.

Fix a nonnegative integer n. By an *open graph of order* n, we mean a triple $\Delta = (V, E, \vec{b})$, where (V, E) is a multigraph and $\vec{b}$ is an n-tuple of elements of V, called the *boundary vector* of the open graph. Let $\Delta = (V, E, \vec{b})$ denote an open graph of order n. Δ is said to be *planar* if the multigraph (V, E) has a plane embedding (no crossing edges) into the interior of an n-gon with clockwise ordered vertices b'_1, b'_2, $\ldots$, b'_n such that each b_i can be joined to b'_i by a smooth curve which does not cross any edge of Δ, the n-gon, or other added curve. We think of a planar open graph as a patch of a planar open graph which has been cut out of a plane embedded planar graph: The boundary vertices may have neighbors in the larger graph, while all neighbors of non-boundary vertices must appear in the open graph. Let $\Gamma = (X, E)$ denote a finite simple graph. To each open graph $\Delta = (V, E, \vec{b})$ of order k, we associate a function $Z_\Delta^\Gamma : X^k \to \mathbb{C}$ as follows. The evaluation of Z_Δ^Γ at $(x_1, x_2, \ldots x_k) \in X^k$ is the number graph homomorphisms of Γ into Δ which map x_i to b_i.

We consider three planar algebras associated with Γ. The first is $\mathcal{F}^\Gamma = \cup \mathcal{F}_n^\Gamma$, where $\mathcal{F}_n^\Gamma$ consist of those functions from X^n to $\mathbb{C}$ which are constant on the orbits of X^n under the action of $\mathfrak{Aut}(\Gamma)$. The second is the planar algebra generated by Γ, $\mathcal{P}^\Gamma = \cup \mathcal{P}_k^\Gamma$, where $\mathcal{P}_k^\Gamma$ is the linear span of all such functions Z^Γ_Δ as Δ runs over all planar open graphs of order k. The third is the open graph planar algebra of Γ, $\mathcal{O}^\Gamma = \cup \mathcal{O}^\Gamma_k$, where $\mathcal{O}^\Gamma_k$ is defined as $\mathcal{P}^\Gamma_k$ except that Δ runs over all open graphs of order k, not just those which are planar.

It turns out that $\mathcal{P}^W_n \subseteq \mathcal{O}^W_n \subseteq \mathcal{F}^W_n$. From theoretical perspective, $\mathcal{P}^\Gamma$ is the most interesting of these planar algebras, but also the most difficult to study. It is easy to compute $\dim \mathcal{F}^W_n$ using the Cauchy-Frobenius-Burnside formula for group characters. To take advantage of this we considered in [22] when $\mathcal{P}^\Gamma_n = \mathcal{O}^\Gamma_n = \mathcal{F}^\Gamma_n$.

Theorem 9.1 (Curtin [22]) *With the above notation,*

(i) $\mathcal{O}^\Gamma_n = \mathcal{F}^\Gamma_n$.

(ii) $\mathcal{P}^\Gamma_n = \mathcal{O}^\Gamma_n$ *if and only if* $Z^\Gamma_\Theta \in \mathcal{P}^\Gamma_4$, *where* Θ *is the open graph consisting of two disconnected points* x, y *and boundary* (x, y, x, y).

Our work has a number of applications to graph rewriting, graph homomorphisms, and graph reconstruction from homomorphic images. Relations among open graphs induced by the evaluation function can be viewed as the local rules of a graph rewriting system, with the graph Γ giving an invariant of the system in the same way that spin models give invariants for the Reidemeister moves. We have considered these problems from time to time, and have some preliminary results. We plan to return to this area of research soon.

10 Research: Fibonacci vectors

We are currently collaborating with our former Masters student Ena Salter and her undergraduate advisor David Stone to publish the results of her thesis. While the techniques are elementary, the lead to a number of new results. In [29], we derive the following identity involving the Fibonacci numbers F_n. For all positive integers d and for all integers n and m,

$$\sum_{j=0}^{d-1} F_{n+j}F_{m+j} = \begin{cases} F_d F_{n+m+d-1} & \text{if } d \text{ is even,} \\ \frac{1}{5}\left(L_d L_{n+m+d-1} - (-1)^n L_{m-n}\right) & \text{if } d \text{ is odd.} \end{cases}$$

We also derive similar shifted sum formulae $\sum_{j=0}^{d-1} L_{n+j}L_{m+j}$, and $\sum_{j=0}^{d-1} F_{n+j}L_{m+j}$ and the shfited convolutions $\sum_{j=0}^{d-1} F_{n+j}F_{d-m-j}$, $\sum_{j=0}^{d-1} L_{n+j}L_{d-m-j}$, and $\sum_{j=0}^{d-1} F_{n+j}L_{d-m-j}$ for positive integers d and arbitrary integers n and m.

Our derivation proceeds as follows. Fix a positive integer d to be the length of all vectors in the following. For all integers n, define a vector $\vec{f_n} = [F_n, F_{n+1}, \cdots, F_{n+d-1}]$. We refer to $\vec{f_n}$ as the *n-th Fibonacci vector (of length d)*. Also define $\vec{a} = \left[1, \alpha, \alpha^2, \cdots, \alpha^{d-1}\right]$ $\vec{b} = \left[1, \beta, \beta^2, \cdots, \beta^{d-1}\right]$. Then Binet's formula for the Fibonacci numbers generalizes to the following formula. For all integers n,

$$\vec{f_n} \;=\; \frac{1}{\alpha - \beta}\left(\alpha^n \vec{a} - \beta^n \vec{b}\right),$$

Now

$$\sum_{j=0}^{d-1} F_{n+j}F_{m+j} = \vec{f_n} \cdot \vec{f_m} = \frac{1}{\alpha-\beta}\left(\alpha^n\vec{a} - \beta^n\vec{b}\right) \cdot \frac{1}{\alpha-\beta}\left(\alpha^m\vec{a} - \beta^m\vec{b}\right).$$

The computation reduces to that of $\vec{a}\cdot\vec{a}$, $\vec{a}\cdot\vec{b}$, and $\vec{b}\cdot\vec{b}$ by linearity of the dot product. These dot products are computed by a simple application of the sum of a finite geometric series. The resulting formulae are quite interesting. This motivates us to further study the Fibonacci vectors.

By the above, the Fibonacci and Lucas vectors lie in a single plane. In [30] we shall describe a number of bases for this plane, and in [31] we study the geometry of the Fibonacci and Lucas vectors in this plane. In [32] we generalize some of the standard linear algebraic considerations of the Fibonacci numbers to Fibonacci vectors, and in [33] we generalize the split identity of for Fibonacci numbers. In [34] we generalize some of the above formulae to $\sum_{j=0}^{d-1} F_{n+js}^{p} F_{m+jt}^{q}$ for all integers n, m, s, t and all positive integers p, q. Many of these results generalize to several other related situations.

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