Inheritance of hyper-duality in imprimitive Bose-Mesner algebras

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Abstract

We prove the following result concerning the inheritance of hyper-duality by block and quotient Bose-Mesner algebras associated with a hyper-dual pair of imprimitive Bose-Mesner algebras. Let $M$ and $\tilde{M}$ denote Bose-Mesner algebras. Suppose there is a hyper-duality $\psi$ from the subconstituent algebra of $M$ with respect to $p$ to the subconstituent algebra of $\tilde{M}$ with respect to $\tilde{p}$. Also suppose that $M$ is imprimitive with respect to a subset $I$ of Hadamard idempotents, so $\tilde{M}$ is dual imprimitive with respect to the subset $\Psi(I)$ of primitive idempotents, where $\Psi : M \rightarrow \tilde{M}$ is the formal duality associated with $\psi$. Let $B$ denote the block Bose-Mesner algebra of $M$ on the block containing $p$, and let $\tilde{Q}$ denote the quotient Bose-Mesner algebra of $\tilde{M}$ with respect to $\Psi(I)$. Then there is a hyper-duality from the subconstituent algebra of $B$ with respect to $p$ to the subconstituent algebra of $\tilde{Q}$ with respect to $\tilde{p}$.

Key words: block Bose-Mesner algebra, quotient Bose-Mesner algebra, subconstituent algebra, Terwilliger algebra

1 Introduction

We study the inheritance of hyper-duality by block and quotient Bose-Mesner algebras associated with imprimitive hyper-dual pairs of Bose-Mesner algebras. Let $M$ denote a Bose-Mesner algebra. Then $M$ is imprimitive with respect to some subset $I$ of its Hadamard idempotents whenever $I$ spans a proper nonzero subalgebra of $M$, and $M$ is dual imprimitive with respect to some subset $D$ of its primitive idempotents whenever $D$ spans a proper

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nonzero subalgebra with respect to entry-wise multiplication of \( \mathcal{M} \). If \( \mathcal{M} \) is imprimitive with respect to \( \mathcal{I} \), then restricting the Hadamard idempotents in \( \mathcal{I} \) to any one of certain subsets (blocks) of the underlying point set yields the Hadamard idempotents of a block Bose-Mesner algebra \( \mathcal{B} \). If \( \mathcal{M} \) is dual imprimitive with respect to \( \mathcal{D} \), then restricting the primitive idempotents in \( \mathcal{D} \) to certain subsets of the underlying point set (class representatives for a particular equivalence relation) and renormalizing yields the primitive idempotents of a quotient Bose-Mesner algebra \( \mathcal{Q} \).

In the presence of a formal duality, the dual nature of these constructions yields the following. Suppose \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) are Bose-Mesner algebras, and suppose \( \Psi : \mathcal{M} \to \tilde{\mathcal{M}} \) is a formal duality. If \( \mathcal{M} \) is imprimitive with respect to \( \mathcal{I} \), then \( \tilde{\mathcal{M}} \) is dual imprimitive with respect to \( \Psi(\mathcal{I}) \). Moreover, each block Bose-Mesner algebra \( \mathcal{B} \) of \( \mathcal{M} \) with respect to \( \mathcal{I} \) is formally dual to the quotient Bose-Mesner algebra \( \tilde{\mathcal{Q}} \) of \( \tilde{\mathcal{M}} \) with respect to \( \Psi(\mathcal{I}) \).

In this paper we extend the above fact to hyper-duality for Bose-Mesner algebras [6,10]. Hyper-duality is an extension of formal duality of Bose-Mesner algebras to their subconstituent (Terwilliger) algebras with respect to some base points. Hyper-duality implies formal duality; indeed, associated with each hyper-duality is a unique formal duality for the underlying Bose-Mesner algebras. Our main result is the following. Suppose there is a hyper-duality \( \psi \) from the subconstituent algebra of \( \mathcal{M} \) with respect to \( p \) to the subconstituent algebra of \( \tilde{\mathcal{M}} \) with respect to \( \tilde{p} \). Also suppose that \( \mathcal{M} \) is imprimitive with respect to \( \mathcal{I} \), so \( \tilde{\mathcal{M}} \) is dual imprimitive with respect to \( \Psi(\mathcal{I}) \), where \( \Psi : \mathcal{M} \to \tilde{\mathcal{M}} \) is the formal duality associated with \( \psi \). Let \( \mathcal{B} \) denote the block Bose-Mesner algebra of \( \mathcal{M} \) on the block containing \( p \), and let \( \tilde{\mathcal{Q}} \) denote the quotient Bose-Mesner algebra of \( \tilde{\mathcal{M}} \) with respect to \( \Psi(\mathcal{I}) \). Then there is a hyper-duality from subconstituent algebra of \( \mathcal{B} \) with respect to \( p \) to the subconstituent algebra of \( \tilde{\mathcal{Q}} \) with respect to \( \tilde{p} \).

This paper is organized as follows. In Section 2 we recall Bose-Mesner and subconstituent algebras. In Sections 3 – 5 we recall imprimitivity and the constructions of block and quotient Bose-Mesner algebras from an imprimitive Bose-Mesner algebra. In Section 6, we recall formal and hyper-duality for pairs of Bose-Mesner algebras. Most of this material up to this point is generally well-known. In Section 7 we prove our main results. In Section 8 we state the results for the special case of self-duality. Finally, in Section 9 we discuss the application of our results to the specific example of the folded Hamming cubes and the halved Hamming cubes. The reader may find it illustrative to work out the details of this example as going through the paper, although it assumes some knowledge of P- and Q-polynomial association schemes.
In this section we review Bose-Mesner and subconstituent algebras. The reader is referred to [2,3] for more details concerning Bose-Mesner algebras.

For any finite, nonempty set $X$, write $M_X$ to denote the complex algebra of matrices with complex entries whose rows and columns are indexed by $X$. For $A \in M_X$ and for $x, y \in X$, let $A(x,y)$ denote the $(x,y)$-entry of $A$. For $A, B \in M_X$, let $A \circ B$ denote the Hadamard (entry-wise, Schur) product of $A$ and $B$: $(A \circ B)(x,y) = A(x,y)B(x,y)$ for all $x, y \in X$. The ordinary matrix product of $A$ and $B$ will be denoted by juxtaposition: $AB$. For $A \in M_X$, let $t A$ and $\overline{A}$ denote the transpose and the complex conjugate, respectively.

A Bose-Mesner algebra on $X$ is a commutative subalgebra $\mathcal{M}$ of $M_X$ which is closed under Hadamard product, which is closed under transposition, and which contains the identity matrix $I$ and the all-ones matrix $J$. Let $\mathcal{M}$ denote a $(d+1)$-dimensional Bose-Mesner algebra on $X$. Then $\mathcal{M}$ has a unique basis $\{A_i\}_{i=0}^d$ such that

$$A_0 = I, \quad A_i \circ A_j = \delta_{ij} A_i \quad (0 \leq i, j \leq d), \quad \sum_{i=0}^d A_i = J, \quad (1)$$

where $\delta_{ij}$ denotes the Kronecker symbol. We call $\{A_i\}_{i=0}^d$ the basis of Hadamard idempotents of $\mathcal{M}$. Observe that each $A_i$ has entries in $\{0, 1\}$ since $A_i \circ A_i = A_i$.

In addition, $\mathcal{M}$ has a unique basis $\{E_i\}_{i=0}^d$ such that

$$E_0 = |X|^{-1} J, \quad E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d), \quad \sum_{i=0}^d E_i = I. \quad (2)$$

We call $\{E_i\}_{i=0}^d$ the basis of primitive idempotents of $\mathcal{M}$. It is known that $\overline{E_i} = E_i$ $(0 \leq i \leq d)$. Throughout this paper we shall assume that the Hadamard and primitive idempotents of any given Bose-Mesner algebra are ordered.

Let $\mathcal{M}$ denote a $(d+1)$-dimensional Bose-Mesner algebra on $X$. Let $\{A_i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$ denote orderings of the Hadamard and primitive idempotents of $\mathcal{M}$. Then for $0 \leq h, i, j \leq d$, the intersection numbers $p_{ij}^h$, the Krein parameters $q_{ij}^h$, the eigenvalues $P(j, i)$, and the dual eigenvalues $Q(j, i)$ of $\mathcal{M}$ are defined respectively by

$$A_i A_j = \sum_{h=0}^d p_{ij}^h A_h, \quad E_i \circ E_j = |X|^{-1} \sum_{h=0}^d q_{ij}^h E_h \quad (0 \leq i, j \leq d), \quad (3)$$

$$A_i = \sum_{j=0}^d P(j, i) E_j, \quad E_i = |X|^{-1} \sum_{j=0}^d Q(j, i) A_j \quad (0 \leq i \leq d). \quad (4)$$
Each of these four sets of parameters determines the other three [2]. The \((d+1) \times (d+1)\) matrices \(P\) and \(Q\) with \((j,i)\)-entries \(P(j,i)\) and \(Q(j,i)\) are called the eigenmatrix and dual eigenmatrix of \(\mathcal{M}\), respectively. The valency of \(A_i\) is the number \(k_i := P(0,i)\). It is well-known that \(k_i\) is equal to the number of ones in each row of \(A_i\). The multiplicity of \(E_i\) is the number \(m_i := Q(0,i)\). It is well-known that \(m_i\) is equal to the rank and trace of \(E_i\).

The above parameters carry all of the algebraic structure of a Bose-Mesner algebra. That is to say, any two Bose-Mesner algebras with the same parameters are isomorphic as Bose-Mesner algebras. It is well-known that the Hadamard idempotents of a Bose-Mesner algebra encode the combinatorial structure of an association scheme. There are many nonisomorphic association schemes that have isomorphic Bose-Mesner algebras.

We recall some formulas [2,3] concerning the intersection numbers and Krein parameters which we shall need later. First, note that transposition defines involutions on the Hadamard and primitive idempotents: Let \(i^t\) and \(i_t\) be such that \(A_{i^t} = t A_i\) and \(E_{i^t} = t E_i\). With this notation,

\[
k_h p_{ij}^h = k_j p_{ih}^j = k_i p_{hj}^i, \quad m_h q_{ij}^h = m_j q_{ih}^j = m_i q_{hj}^i \quad (0 \leq h, i, j \leq d),
\]

\[
\sum_{i=0}^d p_{ij}^h = k_j, \quad \sum_{i=0}^d q_{ij}^h = m_j \quad (0 \leq h, j \leq d).
\]

We now recall the subconstituent algebra of a Bose-Mesner algebra. See [24] for further details. Let \(\mathcal{M}\) be a \((d+1)\)-dimensional Bose-Mesner algebra on \(X\). Fix a “base point” \(p \in X\). For \(A \in \mathcal{M}\), let \(\rho(A) \in \mathbb{M}_X\) denote the diagonal matrix with \((x,x)\)-entry \(\rho(A)(x,x) = A(p,x)\). Let \(\mathcal{M}^* = \{\rho(A) | A \in \mathcal{M}\}\). Then \(\mathcal{M}^*\) is called the dual Bose-Mesner algebra of \(\mathcal{M}\) (with respect to the base point \(p\)). The map \(\rho: \mathcal{M} \rightarrow \mathcal{M}^*\) is a linear bijection. Set \(E_i^* = \rho(A_i)\) and \(A_i^* = \rho(|X|E_i)\) \((0 \leq i \leq d)\). Since \(\rho\) is a linear bijection, \(\{E_i^*\}_{i=0}^d\) and \(\{A_i^*\}_{i=0}^d\) are bases of \(\mathcal{M}^*\). We refer to \(\{E_i^*\}_{i=0}^d\) and \(\{A_i^*\}_{i=0}^d\) as the basis of dual idempotents and the basis of dual Hadamard idempotents of \(\mathcal{M}^*\), respectively. The dual and dual Hadamard idempotents inherit the orderings of the Hadamard and primitive idempotents of \(\mathcal{M}\), respectively. The subalgebra \(\mathcal{T} = \mathcal{T}(p)\) of \(\mathbb{M}_X\) generated by \(\mathcal{M} \cup \mathcal{M}^*\) is called the subconstituent (or Terwilliger) algebra of \(\mathcal{M}\) (with respect to \(p\)). The use of \(^*\) in connection with subconstituent algebras is not related to the operation of taking the conjugate-transpose of a matrix.

### 3 Imprimitive Bose-Mesner algebras

In this section we review some material concerning imprimitive Bose-Mesner algebras. The reader is referred to [2,3] for summaries of imprimitivity. Concerning our treatment of imprimitivity, we note the following. To describe the
associated subconstituent algebras, we will need to carefully track a base-point as we pass from an imprimitive Bose-Mesner algebra to the appropriate block Bose-Mesner algebra and the quotient Bose-Mesner algebra. Thus we have avoided the common practices of reordering the point-set and canonical bases \[2\] and of expressing certain elements of the imprimitive Bose-Mesner algebra as a Kronecker product of two matrices \[2,3\]. As a bridge from Bose-Mesner to subconstituent algebras, we formulate some of these results as statements concerning elements of the Bose-Mesner algebra whereas most references only state the equivalent results concerning intersection numbers and Krein parameters \[3,21\]. We have chosen to present brief proofs of these results. In doing so, we favored proofs which can be “dualized” in the sense which Bannai and Ito \[2\] have dubbed “Kawada-Delsarte duality” \[11\]. Finally, we note that our treatment of imprimitivity is by no means comprehensive. For example, the references \[2–4,17,21,25\] treat various aspects of imprimitivity which we do not discuss here.

Lemma 3.1 Let \(\mathcal{M}\) denote a \((d+1)\)-dimensional Bose-Mesner algebra on \(X\). Let \(\mathcal{I}\) denote a subset of \(\{0, 1, \ldots, d\}\). Then the following are equivalent.

(i) The subset \(\{A_i\}_{i \in \mathcal{I}}\) of the Hadamard idempotents of \(\mathcal{M}\) spans a matrix subalgebra with respect to the ordinary matrix product.

(ii) \(A_i A_j = \sum_{h \in \mathcal{I}} p_{ij}^h A_h\) for all \(i, j \in \mathcal{I}\).

(iii) \(p_{ij}^k = 0\) whenever \(i, j \in \mathcal{I}\) and \(k \notin \mathcal{I}\).

\(\mathcal{M}\) is said to be imprimitive with respect to \(\mathcal{I}\) whenever (i) – (iii) hold and \(\mathcal{I}\) is neither \(\{0\}\) nor \(\{0, 1, \ldots, d\}\). Moreover, if (i) – (iii) hold, then the following hold.

(a) \(A_j \in \{A_h\}_{h \in \mathcal{I}}\) for all \(j \in \mathcal{I}\) (i.e. \(j \in \mathcal{I}\) implies \(j^t \in \mathcal{I}\)).

(b) \(I = A_0 \in \{A_h\}_{h \in \mathcal{I}}\) (i.e. \(0 \in \mathcal{I}\)).

(c) There exists a unique subset \(\mathcal{D}\) of \(\{0, 1, \ldots, d\}\) such that

\[
\sum_{i \in \mathcal{I}} A_i = k_{\mathcal{I}} \sum_{i \in \mathcal{D}} E_i,
\]

where \(k_{\mathcal{I}} = \sum_{i \in \mathcal{I}} k_i\).

Proof. The equivalence of (i) – (iii) is clear from (3).

(a): Let \(\lambda_1, \lambda_2, \ldots, \lambda_r\) denote the distinct entries in column \(i\) of the eigenmatrix \(P\) of \(\mathcal{M}\). Then \(A_i = \sum_{\ell=1}^r \lambda_\ell \sum_{j \in \{P_{j\ell} = \lambda_\ell\}} E_j\) by (4), and \(A_i = \sum_{\ell=1}^r \lambda_\ell \sum_{j \in \{P_{j\ell} = \lambda_\ell\}} E_j\). The \(E_j\) are idempotents, so \(A_i^k = \sum_{\ell=1}^r \lambda_\ell^k \sum_{j \in \{P_{j\ell} = \lambda_\ell\}} E_j\). The \(\lambda_\ell\) are distinct, so each sum \(\sum_{j \in \{P_{j\ell} = \lambda_\ell\}} E_j\) lies in the span of \(A_i, A_i^2, A_i^3, \ldots, A_i^r\) since the coefficient matrix is essentially Vandermonde. The result follows from (i).
(b): Observe that \( p_{i,i}^0 = k_i > 0 \), so \( 0 \in \mathcal{I} \) by (iii) and (a).

(c): Set \( S = k_{I}^{-1} \sum_{i \in \mathcal{I}} A_i \). By (3), (ii), and (6), \( S^2 = k_{I}^{-2} \sum_{i,j \in \mathcal{I}} \sum_{h=0}^{d} p_{ij}^h A_h = k_{I}^{-2} \sum_{i,j \in \mathcal{I}} p_{ij}^h A_h \). Note that \( p_{ij}^h = 0 \) if and only if \( p_{i,j}^h = 0 \) by (5). Thus by (a), (6), and (iii), \( S^2 = k_{I}^{-2} \sum_{h \in \mathcal{I}} \sum_{i,j \in \mathcal{I}} \sum_{h=0}^{d} p_{ij}^h A_h = S \). Since \( S \) is an idempotent, it must be a sum of primitive idempotents, \( S = \sum_{i \in D} E_i \) for some subset \( D \) of \( \{0, 1, \ldots, d\} \), as required. \( \square \)

**Lemma 3.2** Let \( \mathcal{M} \) denote a \((d+1)\)-dimensional Bose-Mesner algebra on \( X \). Let \( D \) denote a subset of \( \{0, 1, \ldots, d\} \). Then the following are equivalent.

(i) The subset \( \{E_i\}_{i \in \mathcal{D}} \) of the primitive idempotents of \( \mathcal{M} \) span a matrix subalgebra with respect to the Hadamard product.

(ii) \( E_i \circ E_j = |X|^{-1} \sum_{h \in \mathcal{D}} q_{ij}^h E_h \) for all \( i, j \in \mathcal{D} \).

(iii) \( q_{ij}^k = 0 \) whenever \( i, j \in \mathcal{D} \) and \( k \notin \mathcal{D} \).

\( \mathcal{M} \) is said to be dual imprimitive with respect to \( \mathcal{D} \) whenever (i) – (iii) hold and \( \mathcal{D} \) is neither \( \{0\} \) nor \( \{0, 1, \ldots, d\} \). Moreover, if (i) – (iii) hold, then the following hold.

(a) \( 1 E_j \in \{E_h\}_{h \in \mathcal{D}} \) for all \( j \in \mathcal{D} \) (i.e. \( j \in \mathcal{D} \) implies \( j_i \in \mathcal{D} \)).

(b) \( |X|^{-1} J = E_0 \in \{E_h\}_{h \in \mathcal{D}} \) (i.e. \( 0 \in \mathcal{D} \)).

(c) There exists a unique subset \( \mathcal{I} \) of \( \{0, 1, \ldots, d\} \) such that

\[
|X| \sum_{i \in \mathcal{D}} E_i = m_{\mathcal{D}} \sum_{i \in \mathcal{I}} A_i, \]

where \( m_{\mathcal{D}} = \sum_{i \in \mathcal{D}} m_i \).

**Proof.** Use the proof of Lemma 3.1 with the Kawada-Delsarte dual of each statement. \( \square \)

**Lemma 3.3** Let \( \mathcal{M} \) denote a \((d+1)\)-dimensional Bose-Mesner algebra on \( X \).

(i) If \( \mathcal{M} \) is imprimitive with respect to \( \mathcal{I} \), then \( \mathcal{M} \) is dual imprimitive with respect to the set \( \mathcal{D} \) of Lemma 3.1(c).

(ii) If \( \mathcal{M} \) is dual imprimitive with respect to \( \mathcal{D} \), then \( \mathcal{M} \) is imprimitive with respect to the set \( \mathcal{I} \) of Lemma 3.2(c).

As imprimitivity and dual imprimitivity coincide, we say in the above cases that \( \mathcal{M} \) is imprimitive with respect to \( (\mathcal{I}, \mathcal{D}) \).
Proof. (i): Note that $\sum_{h \in I} A_h = k_I \sum_{i \in D} E_i$ is an idempotent for Hadamard multiplication, so

$$k_I \sum_{i \in D} E_i = \left( k_I \sum_{i \in D} E_i \right)^{\circ 2} = k_I^2 \sum_{i \in D} E_i \circ E_j = k_I^2 \sum_{h=0}^{d} \left( |X|^{-1} \sum_{i,j \in D} q_{ij}^h \right) E_h.$$

Since the primitive idempotents are linearly independent, this implies that $\sum_{i,j \in D} q_{ij}^h = 0$ for $h \not\in D$. But the Krein parameters are nonnegative real numbers, so $q_{ij}^h = 0$ when $i, j \in D$ and $h \not\in D$. Thus $M$ is dual imprimitive with respect to $D$.

(ii): Use the proof of (i) with the Kawada-Delsarte dual of each statement. □

**Lemma 3.4** Suppose $M$ is a $(d+1)$-dimensional Bose-Mesner algebra on $X$ which is imprimitive with respect to $(I, D)$. Then $k_I m_D = |X|$.

**Proof.** Combine Lemmas 3.1(c) and 3.2(c). □

In the next two sections we recall the constructions of block and quotient Bose-Mesner algebras from imprimitive Bose-Mesner algebras. We shall use some nonstandard notation, namely $\lfloor \ \rfloor$ and $\lceil \ \rceil$ for the respective objects associated with a block and a quotient Bose-Mesner algebra. As a mnemonic, we suggest the similarity in shape between the letter “b” for block and the left side of $\lfloor \ \rfloor$ and the similarity in shape between the letter “q” for quotient and the right side of $\lceil \ \rceil$. We also find $\lfloor \ \rfloor$ and $\lceil \ \rceil$ suggestive of the dual nature of block and quotient Bose-Mesner algebras which we are stressing in this paper.

**4 Block Bose-Mesner algebras**

Given an imprimitive Bose-Mesner algebra, one may construct related block Bose-Mesner algebras. To describe them we define some equivalence relations on $X$ and on $\{0, 1, \ldots, d\}$.

**Lemma 4.1** Suppose $M$ is a $(d+1)$-dimensional Bose-Mesner algebra on $X$ which is imprimitive with respect to $(I, D)$. Then the following are equivalence relations.

(i) The relation $\sim$ on $X$ defined by $x \sim y$ if and only if $(\sum_{i \in I} A_i)(x, y) = 1$. Let $\lceil X \rceil$ denote the set of $\sim$ equivalence classes, and write $\lceil x \rceil$ to denote the equivalence class of $x \in X$. 

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(ii) The relation \( \approx \) on \( \{0, 1, \ldots, d\} \) defined by \( i \approx j \) if and only if \( q_{ih}^j \neq 0 \) for some \( h \in D \). Let \( [D] \) denote the set of \( \approx \) equivalence classes, and write \( [i] \) to denote the equivalence class of \( i \in \{0, 1, \ldots, d\} \).

**Proof.** (i): Observe that \( \sim \) is reflexive by Lemma 3.1(b) and symmetric by Lemma 3.1(a). Suppose \( x \sim y, y \sim z \). Then there exist \( i, j \in I \) such that \( A_i(x, y) = 1, A_j(y, z) = 1 \). Hence the \( (x, z) \)-entry of \( A_iA_j \) is nonzero. Since \( \{A_k\}_{k \in I} \) spans a matrix algebra, there exists \( k \in I \) such that \( A_k(x, z) = 1 \). Thus \( x \sim z \), so \( \sim \) is transitive.

(ii): Observe that \( q_{i0}^j \neq 0 \) by (5) since \( q_{i0}^j = m_i > 0 \). Now \( \approx \) is reflexive since \( 0 \in D \) by Lemma 3.2(b). Note that \( \approx \) is symmetric by Lemma 3.2(a) since if \( q_{ih}^j \neq 0 \) for some \( h \in D \), then \( q_{jh}^i \neq 0 \) by (5). Finally if \( i \approx j \) and \( j \approx k \), then there are \( h, h' \in D \) such that \( q_{ih}^j \neq 0 \) and \( q_{jh'}^k \neq 0 \). This implies that \( E_i(E_i \circ E_h) = |X|^{-1}q_{ih}^jE_j \neq 0 \) and \( E_k(E_j \circ E_{h'}) = |X|^{-1}q_{jh'}^kE_k = 0 \) since the summand \( |X|^{-1}q_{ih}^jE_j(E_i \circ E_{h'}) \neq 0 \) since \( 0 \neq E_k(E_i \circ E_{h'}) = |X|^{-1}\sum_{j=0}^d q_{ih}^jE_j(E_i \circ E_{h'}) \), there is some \( \ell \in D \) such that \( E_k(E_i \circ E_{\ell}) = |X|^{-1}q_{i\ell}^kE_k \neq 0 \). Thus \( i \approx k \). \( \square \)

**Lemma 4.2** With reference to Lemma 4.1(i), \( \|X\| = m_D \) and \( \|x\| = k_I \) for all \( \|x\| \in [X] \).

**Proof.** Every row of \( \sum_{i \in I} A_i \) contains \( \sum_{i \in I} k_i = k_I \) many ones, so \( \|x\| = k_I \). Now there are a total of \( |X| \) many points split evenly among blocks of size \( k_I \), so there are \( \|X\| = |X|/k_I = m_D \) many blocks by Lemma 3.4. \( \square \)

**Lemma 4.3** Suppose \( \mathcal{M} \) is a \((d + 1)\)-dimensional Bose-Mesner algebra on \( X \) which is imprimitive with respect to \((I, D)\). Fix any \( \sim \) equivalence class \( \|x\| \) of \( X \). Then there is a Bose-Mesner algebra \( \mathcal{M} \{\|x\|\} \) on \( \|x\| \) with Hadamard idempotents \( \{\|A_i\|\}_{i \in I} \) and primitive idempotents \( \{\|E_i\| := \|E\|_{\|i\|}\}_{i \in [D]} \) defined as follows. (In particular, \( |I| = \|D\| \)).

(i) For each \( i \in I \), the Hadamard idempotent \( \|A_i\| \in M_{\|x\|} \) has entries \( \|A_i\|(x, y) = A_i(x, y) \ (x, y \in \|x\|) \).

(ii) For each \( \|x\| \in [D] \), the primitive idempotent \( \|E_i\| \in M_{\|x\|} \) has entries \( \|E_i\|(x, y) = \sum_{i' \in \|i\|} E_{i'}(x, y) \ (x, y \in \|x\|) \).

\( \mathcal{M} \{\|x\|\} \) is called the \( \|x\| \)-block Bose-Mesner algebra of \( \mathcal{M} \) with respect to \((I, D)\). Moreover, the following hold.

(a) The intersection numbers of \( \mathcal{M} \{\|x\|\} \) are \( p_{ij}^h = p_{ij}^h \ (h, i, j \in I) \).
(b) The Krein parameters of $[\mathcal{M}]$ are

$$\{q^h_{ij}\} := \{q\}_{[i] [j]}^h = m_D^{-1} \sum_{i' \in [i], j' \in [j]} q^{h'}_{i'j'}, \quad ([h], [i], [j] \in \{\mathcal{D}\})$$

for any $h' \in [h]$.

**Proof.** Observe that $[[A_i]] \circ [[A_j]] = \delta_{ij} [[A_i]] ([i], [j] \in \mathcal{I})$ by (1) and construction. By Lemma 3.1, $[[A_i]] [A_j] = \sum_{h \in \mathcal{I}} p^h_{ij} [[A_h]] ([i], [j] \in \mathcal{I})$. Lemma 3.1 also implies that $\{[[A_i]]\}_{i \in \mathcal{I}}$ contains the identity and is closed under transposition. Finally, by Lemma 3.3 and the definition of $\sim$, the $[[A_i]] ([i] \in \mathcal{I})$ sum to the all-ones matrix of $M_{[x]}$. Thus the span of $\{[[A_i]]\}_{i \in \mathcal{I}}$ is a Bose-Mesner algebra $[\mathcal{M}]$ and $\{[[A_i]]\}_{i \in \mathcal{I}}$ is its set of Hadamard idempotents.

Now the Bose-Mesner algebra $[\mathcal{M}]$ has a set of $|\mathcal{I}|$-many primitive idempotents, say $\{[[F_i]]\}_{i=0}^{[\mathcal{I}]-1}$. To show that $\{[[F_i]]\}_{i=0}^{[\mathcal{I}]-1} = \{[[E_i]]\}_{i \in \mathcal{D}}$, we show that each $[[E_i]] ([i] \in \{\mathcal{D}\})$ is a sum of some distinct $[[F_i]]$ and vice versa.

Observe that by Lemma 3.1(c) and (3),

$$\sum_{i' \in [i]} E_{i'} \circ \sum_{j \in \mathcal{I}} A_j = k_{\mathcal{I}} \sum_{i' \in [i]} E_{i'} \circ \sum_{j \in \mathcal{D}} E_j = \frac{k_{\mathcal{I}}}{|\mathcal{X}|} \sum_{i' \in [i]} \sum_{j \in \mathcal{D}} \sum_{h=0}^{d} h_j E_h.$$

But $k_{\mathcal{I}}/|\mathcal{X}| = m_D^{-1}$ by Lemma 4.2 and the only nonzero $q^h_{i'j}$ are those with $h \in [i]$ by Lemma 4.1(ii) since each $i' \in [i]$ and $j \in \mathcal{D}$. Thus $\left(\sum_{i' \in [i]} E_{i'}\right) \circ \left(\sum_{j \in \mathcal{I}} A_j\right) = m_D^{-1} \sum_{h \in [i]} \left(\sum_{j \in \mathcal{D}} \sum_{i' \in [i]} q^h_{i'j}\right) E_h$. Now since $h \in [i]$, Lemma 4.1(ii) and (6) give $\sum_{i' \in [i]} q^h_{i'j} = \sum_{i' \in [i]} q^h_{i'j} = m_j$. Thus $\sum_{j \in \mathcal{D}} \sum_{i' \in [i]} q^h_{i'j} = m_D$. Hence $\sum_{i' \in [i]} E_{i'} \circ \sum_{j \in \mathcal{I}} A_j = \sum_{i' \in [i]} E_{i'}$. Thus $\sum_{i' \in [i]} E_{i'} \in \text{span} \{A_j\}_{j \in \mathcal{I}}$, so $[[E_i]] \in [\mathcal{M}]$ by restricting to $[x]$. Observe that $[[E_i]] [E_j] = [[E_i]] [E_j]$ as $\left(\sum_{i' \in [i]} E_{i'}\right) \left(\sum_{j' \in [j]} E_{j'}\right) = \delta_{[i] [j]} \sum_{i' \in [i]} E_{i'}$. Hence each $[[E_i]] ([i] \in \{\mathcal{D}\})$ is the sum of some distinct primitive idempotents of $[\mathcal{M}]$.

We now show that each primitive idempotent is the sum of some of the $[[E_i]]$. Since the nonzero entries of the $A_i$ ($i \in \mathcal{I}$) are all indexed by elements of $[x]$, each $[[F_i]]$ has $(u, v)$-entry $[[F_i]](u, v) = \sum_{i' \in [i]} E_{i'}(u, v)$ ($u, v \in [x]$) for some subset $[i]$ of $\{0, 1, \ldots, d\}$. Compute as above

$$\sum_{i' \in [i]} E_{i'} \circ \sum_{j \in \mathcal{I}} A_j = k_{\mathcal{I}} \sum_{i' \in [i]} E_{i'} \circ \sum_{j \in \mathcal{D}} E_j = \frac{k_{\mathcal{I}}}{|\mathcal{X}|} \sum_{i' \in [i]} \sum_{j \in \mathcal{D}} \sum_{h=0}^{d} q^h_{i'j} E_h$$

$$= m_D^{-1} \sum_{i' \in [i]} \sum_{j \in \mathcal{D}} \sum_{h \in [i']} q^h_{i'j} E_h$$

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Restricting the left side of this equation to vertices indexed by \(\lfloor \lfloor x \rfloor \rfloor\) gives \(\lfloor \lfloor F_i \rfloor \rfloor \circ \lfloor \lfloor J \rfloor \rfloor = \lfloor \lfloor F_i \rfloor \rfloor\), so the right side must also restrict to give \(\lfloor \lfloor F_i \rfloor \rfloor\). Thus, in light of (5) and the definition of \(\approx\), the set \([i]\) must be a union of some of the classes in \(\lfloor \lfloor D \rfloor \rfloor\). That is to say, each primitive idempotent is the sum of some of the \(\lfloor \lfloor E_i \rfloor \rfloor\), as required.

Observe that by Lemma 3.3, \(\lfloor \lfloor E_0 \rfloor \rfloor = k\lfloor \lfloor J \rfloor \rfloor\), where \(\lfloor \lfloor J \rfloor \rfloor\) denotes the all-ones matrix of \(M_{\lfloor \lfloor x \rfloor \rfloor}\). Furthermore, \(I = \sum_{h=0}^d E_h = \sum_{[h] \in [I]} \sum_{h' \in [h]} E_{h'}\), so restricting to points in \(\lfloor \lfloor x \rfloor \rfloor\) gives \(\sum_{[h] \in [I]} \lfloor \lfloor E_h \rfloor \rfloor = \lfloor \lfloor I \rfloor \rfloor\), the identity of \(M_{\lfloor \lfloor x \rfloor \rfloor}\). Now the Krein parameters of \(\lfloor \lfloor M \rfloor \rfloor\) are computed from the equation \(\lfloor \lfloor E_i \rfloor \rfloor \circ \lfloor \lfloor E_j \rfloor \rfloor = k^{-1} \sum d_{ij}^h \lfloor \lfloor E_h \rfloor \rfloor\) by first computing the product \(\sum_{i' \in [i]} E_{i'} \circ \sum_{j' \in [j]} E_{j'}\), using (3) and then restricting to \(\lfloor \lfloor x \rfloor \rfloor\).

The terminology "\(\lfloor \lfloor x \rfloor \rfloor\)-block Bose-Mesner algebra of \(M\)" is nonstandard. The literature generally uses "association subscheme" to refer to the association scheme related to a block Bose-Mesner algebra, but has no succinct means of referring to the Bose-Mesner algebra of a particular association subscheme. This distinction is generally not necessary as all block Bose-Mesner algebras with respect to a fixed set \(I\) of an imprimitive Bose-Mesner algebra are isomorphic as Bose-Mesner algebras, even though the related association schemes need not be isomorphic. We need this distinction as there is no reason to expect that the associated subconstituent algebras are isomorphic.

**Lemma 4.4** Suppose \(\mathcal{M}\) is a \((d + 1)\)-dimensional Bose-Mesner algebra on \(X\) which is imprimitive with respect to \((I, D)\). Fix \(p \in X\), and let \(\lfloor \lfloor \mathcal{M} \rfloor \rfloor\) denote the \(\lfloor \lfloor p \rfloor \rfloor\)-block Bose-Mesner algebra \(\mathcal{M}\) with respect to \((I, D)\). Let \(\lfloor \lfloor \mathcal{M}^* \rfloor \rfloor\) denote the dual Bose-Mesner algebra of \(\lfloor \lfloor \mathcal{M} \rfloor \rfloor\) with respect to \(p\).

(i) For each \(i \in I\), the dual idempotent \(\lfloor \lfloor E_i^* \rfloor \rfloor = \rho(\lfloor \lfloor A_i \rfloor \rfloor) \in M_{\lfloor \lfloor p \rfloor \rfloor}\) of \(\lfloor \lfloor \mathcal{M}^* \rfloor \rfloor\) has entries \(\lfloor \lfloor E_i^* \rfloor \rfloor(x, x) = E_i^*(x, x)\) \((x \in \lfloor \lfloor p \rfloor \rfloor)\).

(ii) For each \([i] \in [D]\), the dual Hadamard idempotent \(\lfloor \lfloor A_i^* \rfloor \rfloor = \rho(k\lfloor \lfloor E_i \rfloor \rfloor) \in M_{\lfloor \lfloor p \rfloor \rfloor}\) of \(\lfloor \lfloor \mathcal{M}^* \rfloor \rfloor\) has entries \(\lfloor \lfloor A_i^* \rfloor \rfloor(x, x) = m^{-1}_D \sum_{i' \in [i]} A_{i'}^*(x, x)\) \((x \in \lfloor \lfloor p \rfloor \rfloor)\).

**Proof.** Straight forward from construction and Lemma 4.3. \(\square\)

5 Quotient Bose-Mesner algebras

Quotient Bose-Mesner algebra are a second type of Bose-Mesner algebra which can be constructed from an imprimitive Bose-Mesner algebra. They are developed in a manner dual to that of the block Bose-Mesner algebras.

**Lemma 5.1** Suppose \(\mathcal{M}\) is a \((d + 1)\)-dimensional Bose-Mesner algebra on \(X\)
which is imprimitive with respect to $(\mathcal{I}, \mathcal{D})$. Then the following are equivalence relations.

(i) The relation $\sim$ on $X$ defined by $x \sim y$ if and only if $E_i(w, x) = E_i(w, y)$ for all $i \in \mathcal{D}$ and all $w \in X$. Let $[,]$ denote the set of equivalence classes, and write $[x]$ to denote the equivalence class of $x \in X$.

(ii) The relation $\sim$ on $\{0, 1, \ldots, d\}$ defined by $i \sim j$ if and only if $p_{ih}^j$ for some $h \in \mathcal{I}$. Let $[,]$ denote the set of equivalence classes, and write $[i]$ to denote the equivalence class of $i \in \{0, 1, \ldots, d\}$.

Proof. (i): The relation $\sim$ is clearly reflexive, symmetric, and transitive. (ii): Use the proof of Lemma 4.1(ii) with the Kawada-Delsarte dual of each statement. $\square$

Lemma 5.2 With reference to Lemma 5.1, the equivalence relations $\sim$ of Lemma 4.1(i) and $\approx$ of Lemma 5.1(i) on $X$ coincide.

Proof. Observe that $(\sum_{i \in \mathcal{I}} A_i)(z, w) = 1$ if $z \sim w$ and 0 otherwise. Thus by Lemma 3.3, $(\sum_{i \in \mathcal{D}} E_i)(z, w) = k_{\mathcal{I}}^{-1}$ if $z \sim w$ and 0 otherwise. Suppose $x \sim y$. Then columns $x$ and $y$ of $\sum_{i \in \mathcal{D}} E_i$ are the same. Let $V$ denote the set of complex column vectors whose entries are indexed by $X$, and for all $z \in X$, let $\hat{z} \in V$ denote the vector with all entries 0 except for that indexed by $z$ which is 1. Then $\sum_{i \in \mathcal{D}} (E_i \hat{x}) = (\sum_{i \in \mathcal{D}} E_i \hat{x}) = (\sum_{i \in \mathcal{D}} E_i \hat{y}) = (\sum_{i \in \mathcal{D}} (E_i \hat{y}) \hat{x}$. Since the $E_i$ are projections onto orthogonal subspaces, it follows that $E_i \hat{x} = E_i \hat{y}$ for all $i \in \mathcal{D}$. Hence $x \approx y$. Now suppose $x \approx y$. Then $E_i \hat{x} = E_i \hat{y}$ for all $i \in \mathcal{D}$, so $\sum_{i \in \mathcal{D}} E_i \hat{x} = (\sum_{i \in \mathcal{D}} E_i \hat{y}) \hat{y}$. It follows that columns $x$ and $y$ of $\sum_{i \in \mathcal{I}} A_i$ are the same, so $x \sim y$. $\square$

Lemma 5.3 With reference to Lemma 5.1(i), $[X] = m_{\mathcal{D}}$ and $[x] = k_{\mathcal{I}}$ for all $[x] \in [X]$.

Proof. Immediate from Lemmas 4.2 and 5.2. $\square$

Lemma 5.4 Suppose $\mathcal{M}$ is a $(d + 1)$-dimensional Bose-Mesner algebra on $X$ which is imprimitive with respect to $(\mathcal{I}, \mathcal{D})$. Then there is a Bose-Mesner algebra $[,]$ on $[X]$ with Hadamard idempotents $\{[A_i] := [A_i]_{[i]} for [i] \in [\mathcal{I}]\}$ and primitive idempotents $\{[E_i]\}_{i \in \mathcal{D}}$ defined as follows. (In particular $|\mathcal{D}| = |[\mathcal{I}]|$.

(i) For each $i \in \mathcal{D}$, the primitive idempotent $[E_i] \in M_{|[X]}$ has entries $[E_i]([x], [y]) = k_{\mathcal{I}}E_i(x', y') ([x], [y] \in [X])$ for any $x' \in [x]$, $y' \in [y]$.

(ii) For each $[i] \in [\mathcal{I}]$, the Hadamard idempotent $[A_i] \in M_{|[X]}$ has entries
\[ [A_i](x, y) = \sum_{x' \in [x]} A_i(x', y') (x, y) \in [X]) \text{ for any } x' \in [x], y' \in [y]. \]

\[ [M] \text{ is called the quotient Bose-Mesner algebra of } M \text{ with respect to } (I, D). \]

Moreover, the following hold.

(a) The Krein parameters of \([M]\) are \([q^h_{ij}] = q^h_{ij} \quad (h, i, j \in D). \]

(b) The intersection numbers of \([M]\) are

\[ [p^h_{ij}] := [p]_{[i][j]} = k^{-1}_{[I]} \sum_{i' \in [i], j' \in [j]} p^h_{i'j'} \quad ([h], [i], [j] \in [I]) \]

for any \(h' \in [h].\)

**Proof.** Use the proof of Lemma 4.3 with the Kawada-Delsarte dual of each statement. \(\square\)

**Lemma 5.5** Suppose \(M\) is a \((d + 1)\)-dimensional Bose-Mesner algebra on \(X\) which is imprimitive with respect to \((I, D)\). Let \([M]\) denote the quotient Bose-Mesner algebra of \(M\) with respect to \((I, D)\). Let \([M^*]\) denote the dual Bose-Mesner algebra of \([M]\) with respect to \([p]\).

(i) For each \(i \in D\), the dual Hadamard idempotent \([A_i^*] = \rho(m_D[A_i]) \in M_{[X]}\) of \([M^*]\) has entries \([A_i^*](x, x) = A_i^*(x, x) \quad ([x] \in [X]).\]

(ii) For each \([i] \in [I]\), the dual Hadamard idempotent \([E_i^*] = \rho([A_i]) \in M_{[X]}\) of \([M^*]\) has entries \([E_i^*](x, x) = \sum_{x' \in [x]} E_i^*(x', x) \quad ([x] \in [X]).\]

**Proof.** Clear from construction and Lemma 5.4. \(\square\)

Although we have described generators of the subconstituent algebra of block and quotient Bose-Mesner algebras associated with an imprimitive Bose-Mesner algebra (Lemmas 4.3, 4.4, 5.4, and 5.5), we have not described the structure of these subconstituent algebras. Since subconstituent algebras are semisimple, this is generally done by describing the isomorphism classes of the irreducible modules and the multiplicity with which they appear in the standard module. In a few cases the subconstituent algebras of block and quotient Bose-Mesner algebras have been related to that of the imprimitive Bose-Mesner algebra [5].
6 Duality of Bose-Mesner algebras

We recall some descriptions of formal and hyper-duality of Bose-Mesner algebras. See [2,3,20] for summaries of formal duality and [6,10] for more on hyper-duality. We work in the following setting.

**Notation 6.1** Let \( X \) and \( \tilde{X} \) denote finite nonempty sets of the same size \(|X| = |	ilde{X}| = n\). Let \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) denote \((d + 1)\)-dimensional Bose-Mesner algebras on \( X \) and \( \tilde{X} \), respectively. Fix \( p \in X \) and \( \tilde{p} \in \tilde{X} \), and let \( T \) and \( \tilde{T} \) denote the subconstituent algebras of \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) with respect to \( p \) and \( \tilde{p} \), respectively. Write \( \tilde{\cdot} \) with all objects associated with \( \tilde{T} \), but otherwise use the same notation introduced for Bose-Mesner and their subconstituent algebras. Write \( \tau \) and \( \tilde{\tau} \) to denote the transposition maps on \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \), respectively. Write \( \mathbb{M}_{X,\tilde{X}} \) to denote the set of complex matrices with rows indexed by \( X \) and columns indexed by \( \tilde{X} \).

**Lemma 6.2** [2,3,20] With Notation 6.1, let \( \Psi : \mathcal{M} \rightarrow \tilde{\mathcal{M}} \) denote a linear bijection. Then the following are equivalent.

(i) For all \( A, B \in \mathcal{M} \),
\[
\Psi(AB) = \Psi(A) \circ \Psi(B), \quad \Psi(A \circ B) = n^{-1}\Psi(A)\Psi(B).
\]
(ii) There exist orderings of the Hadamard and primitive idempotents of \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) such that
\[
\Psi(E_i) = \tilde{\tilde{A}}_i, \quad \Psi(A_i) = n^i\tilde{\tilde{E}}_i \quad (0 \leq i \leq d).\]

Suppose (i) and (ii) hold. Then the map \( \Psi \) is called a formal duality from \( \mathcal{M} \) to \( \tilde{\mathcal{M}} \). \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) are said to be formally dual whenever there exists a formal duality from one to the other. Orderings of the Hadamard and primitive idempotents which satisfy (8) are said to be standard for \( \Psi \).

**Lemma 6.3** [2,3,20] With Notation 6.1, the following are equivalent.

(i) \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) are formally dual and their Hadamard and primitive idempotents are in a standard order for some formal duality from \( \mathcal{M} \) to \( \tilde{\mathcal{M}} \).
(ii) \( p_i^j = \tilde{q}_{i}^{j} \) and \( q_i^h = \tilde{p}_{i}^{h} \) \((0 \leq h, i, j \leq d)\).

**Theorem 6.4** [6] With Notation 6.1, suppose \( \psi : T \rightarrow \tilde{T} \) is an algebra isomorphism. Then the following are equivalent.

(i) \( \psi(\mathcal{M}) = \tilde{\mathcal{M}}^* \), \( \psi(\mathcal{M}^*) = \tilde{\mathcal{M}} \), \( \Psi = n\psi \rho |_{\mathcal{M}} \) is a formal duality from \( \mathcal{M} \) to \( \tilde{\mathcal{M}} \), \( \tilde{\Psi} = n\tilde{\tau} \rho^{-1} |_{\mathcal{M}} \) is a formal duality from \( \tilde{\mathcal{M}} \) to \( \mathcal{M} \), \( \Psi \tilde{\Psi} = n\tilde{\tau} |_{\mathcal{M}} \), and \( \tilde{\Psi} \Psi = n\tau |_{\mathcal{M}} \).
There exist orderings of the Hadamard and primitive idempotents of \( M \) and \( \tilde{M} \) such that for all \( i \) (0 \( \leq i \leq d \))

\[
\psi(tA_i) = \tilde{A}_i^*, \quad \psi(A_i^*) = \tilde{A}_i, \quad \psi(E_i) = \tilde{E}_i^*, \quad \psi(E_i^*) = t\tilde{E}_i. \tag{9}
\]

Suppose (i) and (ii) hold. Then the map \( \psi \) is called a hyper-duality from \( T \) to \( \tilde{T} \). \( T \) and \( \tilde{T} \) are said to be hyper-dual to one another whenever there exists a hyper-duality from one to the other. Orderings of the Hadamard and primitive idempotents which satisfy (9) are said to be standard for \( \psi \). Standard orderings for \( \psi \) are standard for the formal dualities \( \Psi \) and \( \tilde{\Psi} \) of (i).

In the presence of formal duality, it is not necessary to check all four conditions in Equation (9).

**Lemma 6.5** [6] With Notation 6.1, assume that \( M \) and \( \tilde{M} \) are formally dual to one another and that the Hadamard and primitive idempotents are in a standard order for some formal duality of \( M \) and \( \tilde{M} \).

(i) Suppose \( \psi : M^* \to \tilde{M} \) is a linear map. Then \( \psi(E_i^*) = t\tilde{E}_i \) (0 \( \leq i \leq d \)) if and only if \( \psi(A_i^*) = \tilde{A}_i \) (0 \( \leq i \leq d \)).

(ii) Suppose \( \psi : M \to M^* \) is a linear map. Then \( \psi(tA_i) = \tilde{A}_i^* \) (0 \( \leq i \leq d \)) if and only if \( \psi(E_i) = \tilde{E}_i^* \) (0 \( \leq i \leq d \)).

**Definition 6.6** With Notation 6.1, suppose that \( \psi : T \to \tilde{T} \) is a hyper-duality. We say that an invertible matrix \( H \in M_{X,\bar{X}} \) represents the hyper-duality \( \psi \) whenever \( \psi(A) = H^{-1}AH \) for all \( A \in T \). In this case we say that \( \psi \) is representable, and that \( T \) and \( \tilde{T} \) are representably hyper-dual to one another.

**7 Duality and imprimitive Bose-Mesner algebras**

We describe the inheritance of formal and hyper-duality by block and quotient Bose-Mesner algebras from an imprimitive Bose-Mesner algebra. We will be considering a Bose-Mesner algebra \( M \) which is imprimitive with respect to \( (I, D) \) and block and quotient Bose-Mesner algebras \( [M] \) and \( \tilde{[M]} \) of \( M \) with respect to \( (I, D) \). We also consider the formal dual \( \tilde{M} \) of \( M \) which turns out to be imprimitive with respect to \( (D, I) \), so there are block and quotient Bose-Mesner algebras \( [\tilde{M}] \) and \( \tilde{[M]} \) of \( \tilde{M} \) with respect to \( (D, I) \). The notation aside, the inheritance of formal duality is straight forward and well-known (cf. [3]).

**Theorem 7.1** With Notation 6.1, assume that \( \Psi : M \to \tilde{M} \) is a formal duality. Suppose \( M \) is imprimitive with respect to \( (I, D) \). Then the following hold.
(i) $\tilde{M}$ is imprimitive with respect to $(D,I)$.

(ii) Let $[M]$ denote any block Bose-Mesner algebra of $\mathcal{M}$ with respect to $(I,D)$, and let $[\tilde{M}]$ denote the quotient Bose-Mesner algebra of $\mathcal{M}$ with respect to $(D,I)$. Then the linear map $[\Psi] : [M] \rightarrow [\tilde{M}]$ satisfying $[\Psi]([A_i]) = n_i[\tilde{E}_i]$ ($i \in I$) is a formal duality, where $\{A_i\}_{i \in I}$ and $\{\tilde{E}_i\}_{i \in I}$ are in a standard order for $\Psi$.

(iii) Let $[\tilde{M}]$ denote the quotient Bose-Mesner algebra of $\mathcal{M}$ with respect to $(I,D)$, and let $[\hat{M}]$ denote any block Bose-Mesner algebra of $\mathcal{M}$ with respect to $(D,I)$. Then the linear map $[\Psi] : [\hat{M}] \rightarrow [\tilde{M}]$ satisfying $[\Psi]([E_i]) = [\tilde{A}_i]$ ($i \in D$) is a formal duality, where $\{E_i\}_{i \in D}$ and $\{\tilde{A}_i\}_{i \in D}$ are in a standard order for $\Psi$.

Proof. Assume that the Hadamard and primitive idempotents are in a standard order for $\Psi$. Then $p^h_{ij} = \tilde{q}^h_{ij}$ and $q^h_{ij} = \tilde{p}^h_{ij}$ by Lemma 6.3. In particular, $p^h_{ij} = 0$ if and only if $\tilde{q}^h_{ij} = 0$. Thus $\mathcal{M}$ is imprimitive with respect to $(D,I)$.

Moreover, $[p^h_{ij}] = [\tilde{q}^h_{ij}]$ and $[q^h_{ij}] = [\tilde{p}^h_{ij}]$ by Lemmas 4.3 and 5.4. Thus (ii) and (iii) hold by Lemma 6.3. □

The formally dual Bose-Mesner algebras $[M]$ and $[\tilde{M}]$ have respective point sets $[x]$ and $[\tilde{x}]$, both of size $k_I = \widetilde{m}_I$. Similarly, the formally dual Bose-Mesner algebras $[\hat{M}]$ and $[\tilde{M}]$ have respective point sets $[X]$ and $[\tilde{x}]$, both of size $m_D = \tilde{k}_D$.

Theorem 7.2 With Notation 6.1, suppose that $\psi : T \rightarrow \tilde{T}$ is a hyper-duality. Further suppose that $\mathcal{M}$ is imprimitive with respect to $(I,D)$ (so $\mathcal{M}$ is imprimitive with respect to $(D,T)$).

(i) Let $[T]$ denote the subconstituent algebra with respect to $p$ of the $[p]$-block Bose-Mesner algebra $[M]$ of $\mathcal{M}$ with respect to $(I,D)$. Let $[\tilde{T}]$ denote the subconstituent algebra with respect to $[\hat{p}]$ of the quotient Bose-Mesner algebra $[\hat{M}]$ of $\tilde{M}$ with respect to $(D,I)$. Then there is a hyper-duality $[\psi] : [T] \rightarrow [\tilde{T}]$ satisfying $[\psi](\lceil A_i \rceil) = [\tilde{A}_i]$, $[\psi](\lceil E_i \rceil) = [\tilde{E}_i]$ ($i \in I$), where $\{A_i\}_{i \in I}$ and $\{\tilde{E}_i\}_{i \in I}$ are in a standard order for $\psi$.

(ii) Let $[T]$ denote the subconstituent algebra with respect to $[\hat{p}]$ of the $[p]$-block Bose-Mesner algebra $[\hat{M}]$ of $\tilde{M}$ with respect to $(D,I)$. Let $[\tilde{T}]$ denote the subconstituent algebra with respect to $\tilde{p}$ of the $[\hat{p}]$-block Bose-Mesner algebra $[\hat{M}]$ of $\tilde{M}$ with respect to $(D,I)$. Then there is a hyper-duality $[\psi] : [T] \rightarrow [\tilde{T}]$ satisfying $[\psi](\lceil E_i \rceil) = [\tilde{E}_i]$, $[\psi](\lceil A_i \rceil) = [\tilde{A}_i]$ ($i \in D$), where $\{E_i\}_{i \in D}$ and $\{\tilde{A}_i\}_{i \in D}$ are in a standard order for $\psi$.

Proof. (i): Assume that the Hadamard and primitive idempotents of $\mathcal{M}$ and $\tilde{M}$ are in standard orders for the hyper-duality $\psi$. Let $T_I$ denote the
subalgebra of \( T \) generated by \( \{A_i\}_{i \in I} \cup \{E_i^\ast\}_{i \in I} \). The nonzero entries of all of these generators are indexed by elements of \( [p] \) by Lemmas 3.1, 4.3, and 4.4. Hence the map \( \lambda : [T] \to T \) which takes \( \lambda([A_i]) = A_i \) and \( \lambda([E_i^\ast]) = E_i^\ast \) \( (i \in I) \) is an isomorphism. Now \( \psi(\lambda([A_i])) = A_i^\ast \) and \( \psi(\lambda([E_i^\ast])) = E_i^\ast \) \( (i \in I) \). Observe that by Lemmas 5.4 and 5.5, the map \( \mu : \widehat{T} \to \widehat{T} \) which restricts the matrices to a principal minor whose rows and columns are indexed by class representatives for \( [X] \) and multiplies by \( k_T \) is an algebra homomorphism from \( \psi(\lambda([T])) = \operatorname{span}\{\widehat{E}_i\}_{i \in I} \) onto \( \widehat{T} \) satisfying \( \mu(\widehat{A}_i) = \widehat{A}_i^\ast \) and \( \mu(\widehat{E}_i) = \widehat{E}_i^\ast \). Now set \( \widehat{\psi} = \mu \psi \lambda \), composing right to left. Then \( \widehat{\psi} \) is an algebra homomorphism from \( [T] \) to \( \widehat{T} \) by construction, and it is a bijection since it carries a set of generators for \( [T] \) to one for \( \widehat{T} \).

By Theorems 6.4 and 7.1, \( [M] \) and \( \widehat{M} \) are formally dual and the orderings Hadamard and primitive idempotents induced by \( \widehat{\psi}(\lambda([A_i])) = \widehat{A}_i^\ast \) and \( \widehat{\psi}(\lambda([E_i^\ast])) = \widehat{E}_i^\ast \) \( (i \in I) \) are part of a standard ordering for some formal duality. Thus Lemma 6.5 implies that \( \widehat{\psi}([E_i]) = \widehat{E}_i^\ast \) and \( \widehat{\psi}([A_i^\ast]) = \widehat{A}_i^\ast \) for some orderings of \( \{E_i\}_{i \in \bar{D}} \) and \( \{A_i^\ast\}_{i \in \bar{D}} \).

(ii): Similar to (i). First invert the restriction and multiplication by \( k_T \) which gave \( [T] \) from \( T \). Then apply \( \psi \), and finally restrict \( \widehat{T} \) to \( \widehat{T} \). \( \square \)

**Theorem 7.3** With reference to Theorem 7.2, suppose that the matrix \( H \in M_{x,\widehat{x}} \) represents the hyper-duality \( \psi : T \to \widehat{T} \).

(i) Let \( [H] \in M_{[p],\widehat{X}} \) have \((x,\widehat{x})\)-entry

\[
[H](x,\widehat{x}) = \sum_{x' \in \widehat{x}} H(x, x') \quad (x \in [p], \widehat{x} \in \widehat{X}).
\]

Then \( [H] \) represents the hyper-duality \( \widehat{\psi} \) of Theorem 7.2(i).

(ii) Let \( [H] \in M_{[X],\widehat{p}} \) have \((\bar{p},x)\)-entry

\[
[H](\bar{p}, x) = \sum_{x' \in \bar{p}} H(x', x) \quad (\bar{p} \in [X], x \in \widehat{X}).
\]

Then \( [H] \) represents the hyper-duality \( \widehat{\psi} \) of Theorem 7.2(ii).

**Proof.** (i): Assume that the Hadamard and primitive idempotents of \( M \) and \( \widehat{M} \) are in standard orders for the hyper-duality represented by \( H \). Fix \( i \in I \), \( x \in [p] \), and \( \widehat{x} \in \widehat{X} \). By Theorem 6.4, \( A_i H = H A_i^\ast \), so \( \sum_{x' \in \widehat{x}} \bar{A}_i H(x, x') = \sum_{x' \in \widehat{x}} H \bar{A}_i(x, x') \). Compute both sides of this equation. On the left side, for
all \( \bar{x}' \in \bar{x} \) 

\[ 'A_i H(x, \bar{x}') = \sum_{w \in X} 'A_i(x, w) H(w, \bar{x}') = \sum_{[w] \in [X]} \sum_{w' \in [w]} 'A_i(x, w') H(w', \bar{x}'). \]

But \( 'A_i(x, w') = 0 \) unless \( x \sim w' \) (i.e. \( [w] = [x] = [p] \)) since \( i \in \mathcal{I} \). Thus \( 'A_i H(x, \bar{x}') = \sum_{w' \in [p]} 'A_i(x, w') H(w', \bar{x}') \). Now \( 'A_i(x, w') = \| 'A_i \| (x, w') \) by Lemma 4.3 as \( x, w' \in [p] \), and \( \sum_{\bar{x}' \in [\bar{x}]} H(w', \bar{x}') = \| H \| (w', [\bar{x}]) \). Thus \( \sum_{\bar{x}' \in [\bar{x}]} 'A_i H(x, \bar{x}') = \| 'A_i \| \| H \| (x, [\bar{x}]) \). On the right side, \( H 'A_i^*(x, \bar{x}') = H(x, \bar{x}) 'A_i^*(\bar{x}', \bar{x}') \) for all \( \bar{x}' \in [\bar{x}] \). Note that \( 'A_i^*(\bar{x}', \bar{x}') = \| 'A_i^* \| ([\bar{x}], [\bar{x}]) \) by Lemma 5.5. Thus \( \sum_{\bar{x}' \in [\bar{x}]} H 'A_i^*(x, \bar{x}') = \sum_{\bar{x}' \in [\bar{x}]} \| H \| \| 'A_i^* \| (x, [\bar{x}]) \). Hence \( \| 'A_i \| \| H \| = \| H \| \| 'A_i^* \| \).

By Theorem 6.4, \( E_i^* H = H^{\bar{p}} \bar{E}_i \), so \( \sum_{\bar{x}' \in [\bar{x}]} E_i^* H(x, \bar{x}') = \sum_{\bar{x}' \in [\bar{x}]} H^{\bar{p}} \bar{E}_i(x, \bar{x}') \).

Compute both sides of this equation. On the left side, for all \( \bar{x}' \in [\bar{x}] \)

\[ H^{\bar{p}} \bar{E}_i(x, \bar{x}') = \sum_{\bar{w} \in \bar{X}} H(x, \bar{w}) \bar{E}_i(\bar{w}, \bar{x}') = \sum_{[\bar{w}] \in [\bar{X}]} \sum_{\bar{w} \in [\bar{w}]} H(x, \bar{w}') \bar{E}_i(\bar{w}', \bar{x}'). \]

By Lemma 5.4, \( \bar{E}_i(\bar{w}', \bar{x}') = \bar{k}_{\bar{I}}^{-1} \| \bar{E}_i \| ([\bar{x}], [\bar{x}]) \), independent of \( \bar{x}' \) and \( \bar{w}' \), so this term factors out of the sum over \( \bar{w}' \in [\bar{w}] \). Thus

\[ H^{\bar{p}} \bar{E}_i(x, \bar{x}') = \sum_{[\bar{w}] \in [\bar{X}]} \bar{k}_{\bar{I}}^{-1} \| H \| ([\bar{x}], [\bar{w}]) \| \bar{E}_i \| ([\bar{w}], [\bar{x}]) \]

\[ = \bar{k}_{\bar{I}}^{-1} \| H \| \| \bar{E}_i \| (x, [\bar{x}]). \]

Note that this is independent of \( \bar{x}' \in [\bar{x}] \), and that \( |\bar{x}| = |\bar{x}| \) by Lemma 5.3. Thus \( \sum_{\bar{x}' \in [\bar{x}]} H^{\bar{p}} \bar{E}_i(x, \bar{x}') = \bar{k}_{\bar{I}} \bar{k}_{\bar{I}}^{-1} \| H \| \| \bar{E}_i \| (x, [\bar{x}]) = \| H \| \| \bar{E}_i \| (x, [\bar{x}]). \)

On the right side compute \( \sum_{\bar{x}' \in [\bar{x}]} E_i^* H(x, \bar{x}') = \sum_{\bar{x}' \in [\bar{x}]} E_i^*(x, x) H(x, \bar{x}') = E_i^*(x, x) \sum_{\bar{x}' \in [\bar{x}]} H(x, \bar{x}') = \| E_i^* \| (x, x) \| H \| (x, [\bar{x}]) = \| E_i^* \| \| H \| (x, [\bar{x}]). \) Hence \( \| E_i^* \| \| H \| = \| H \| \| \bar{E}_i \|. \)

By Theorems 6.4 and 7.1, \( \| M \| \) to \( \| \bar{M} \| \) are formally dual and the orderings Hadamard and primitive idempotents induced by \( \| A_i \| \| H \| = \| H \| \| A_i \| \) and \( \| E_i \| \| H \| = \| H \| \| E_i \| \) for all \( i \in \mathcal{I} \) are part of a standard ordering for some formal duality. Thus Lemma 6.5 implies that \( \| E_i \| \| H \| = \| H \| \| \bar{E}_i \| \) and \( \| A_i \| \| H \| = \| H \| \| \bar{A}_i \| \) for some orderings of \( \{ \| E_i \| \}_{i \in [\bar{D}]} \) and \( \{ \| \bar{A}_i \| \}_{i \in [\bar{D}]} \).

Note that \( H \) is invertible, and write \( L_H \) to denote the (invertible) linear transformation defined by left multiplication by \( H \) from the column vector space
with entries indexed by $\tilde{X}$ to the column vector space with entries indexed by $X$. Since $H\tilde{E}_i = E^*_i H$ ($i \in I$), $L_H$ maps the span of the column spaces of all $\tilde{E}_i$ ($i \in I)$ onto that of all $E^*_i$ ($i \in I$). The column spaces of all $\tilde{E}_i$ ($i \in I$) are contained in the span of all column vectors with entries indexed by $\tilde{X}$ which are constant on each class of $\tilde{X}$ by Lemma 5.1(i). Equality holds since each of these vector spaces has dimension $\tilde{m}_I$. The span of the column spaces of all $E^*_i$ ($i \in I$) are precisely those column vectors with entries indexed by $X$ with all entries not indexed by elements of $[p]$ equal to zero.

Consider the invertible linear transformation $\theta$ from the column vector space with entries indexed by $\tilde{X}$ to the column vectors space with entries indexed by $\tilde{X}$ which are constant on each class of $\tilde{X}$ defined by $\theta(\tilde{v})(\tilde{x}) = [\tilde{v}](\tilde{x})$ for all $\tilde{x} \in \tilde{X}$. Also consider the invertible linear transformation $\eta$ from the column vector space with entries indexed by $X$ with all entries not indexed by elements of $\tilde{p}$ equal to zero. $\eta$ maps the column vector space with entries indexed by $[p]$ to the column vector space with entries indexed by $[p]$. Observe that the result of multiplication by $[H]$ on column vector space with entries indexed by $[\tilde{X}]$ is just the composition $\eta L_H \theta$ of invertible linear transformations. Thus $[H]$ is invertible.

(ii): Similar to (i). By Theorem 6.4, $E_i H = H \tilde{E}^*_i$, so $\sum_{x' \in [x]} E_i H(x', \tilde{x}) = \sum_{x' \in [x]} H \tilde{E}^*_i(x', \tilde{x})$. Computing both sides of this equation with simplifications similar to those used in (i) gives $[E_i] [H] = [H] [\tilde{E}^*_i]$ and $[A_i] H(x', \tilde{x}) = \sum_{x' \in [x]} H \tilde{A}^*_i(x', \tilde{x})$. Computing both sides of this equation gives $[A^*_i] [H] = [H] [\tilde{A}^*_i]$. As in (i), Lemma 6.5 implies that $[A_i] [H] = [H] [A^*_i]$ and $[E^*_i] [H] = [H] [\tilde{E}_i]$ for some orderings of $\{[A_i]\}_{i \in [I]}$ and $\{[E^*_i]\}_{i \in [\tilde{I}]}$. The proof that $[H]$ is invertible is similar to the proof that $[\tilde{H}]$ is invertible.  

Both imprimitivity and formal duality can be developed at the level of $C$-algebras [2] as they depend only on the parameters of a Bose-Mesner algebra. We expect that most of our results can generalized to a similar level in the abstract Terwilliger algebra of [14] which is defined using a dual pair of $C$-algebras.

8 Self-duality and imprimitivity

We briefly consider the cases of formal and hyper-self duality. This case proceeds as expected, so we omit proofs. We will consider a formally self-dual Bose-Mesner algebra $M$ which is imprimitive with respect to $(I, D)$. We shall write $[M]_I$ and $[M]_D$ to denote the block and quotient Bose-Mesner alge-
bras of $\mathcal{M}$ with respect to $(\mathcal{I}, \mathcal{D})$. It turns out that $\mathcal{M}$ is also imprimitive with respect to $(\mathcal{D}, \mathcal{I})$, so there are block and quotient Bose-Mesner algebras $\|\mathcal{M}\|_\mathcal{D}$ and $\|\mathcal{M}\|_\mathcal{I}$ of $\mathcal{M}$ with respect to $(\mathcal{D}, \mathcal{I})$. For simplicity we shall only consider one subconstituent algebra—in principle we could use two distinct base points.

Theorem 8.1 Let $\mathcal{M}$ denote a Bose-Mesner algebra on $X$, and assume that $\Psi: \mathcal{M} \to \mathcal{M}$ is a formal duality of $\mathcal{M}$. Suppose $\mathcal{M}$ is imprimitive with respect to $(\mathcal{I}, \mathcal{D})$.

(i) $\mathcal{M}$ is imprimitive with respect to $(\mathcal{D}, \mathcal{I})$.

(ii) Let $\|\mathcal{M}\|_\mathcal{I}$ denote any block Bose-Mesner algebra of $\mathcal{M}$ with respect to $(\mathcal{I}, \mathcal{D})$, and let $\|\mathcal{M}\|_\mathcal{I}$ denote the quotient Bose-Mesner algebra of $\mathcal{M}$ with respect to $(\mathcal{D}, \mathcal{I})$. Then the linear map $\{\Psi\}: \|\mathcal{M}\|_\mathcal{I} \to \|\mathcal{M}\|_\mathcal{I}$ satisfying $\{\Psi\}(\|A_i\|_\mathcal{I}) = n^i\|E_i\|_\mathcal{I}$ ($i \in \mathcal{I}$) is a formal duality, where $\{A_i\}_{i \in \mathcal{I}}$ and $\{E_i\}_{i \in \mathcal{I}}$ are in a standard order for $\Psi$.

(iii) Let $\|\mathcal{M}\|_\mathcal{D}$ denote the quotient Bose-Mesner algebra of $\mathcal{M}$ with respect to $(\mathcal{I}, \mathcal{D})$, and let $\|\mathcal{M}\|_\mathcal{D}$ denote any block Bose-Mesner algebra of $\mathcal{M}$ with respect to $(\mathcal{D}, \mathcal{I})$. Then the linear map $\{\Psi\}: \|\mathcal{M}\|_\mathcal{D} \to \|\mathcal{M}\|_\mathcal{D}$ satisfying $\{\Psi\}(\|E_i\|_\mathcal{D}) = \|A_i\|_\mathcal{D}$ ($i \in \mathcal{D}$) is a formal duality, where $\{A_i\}_{i \in \mathcal{D}}$ and $\{E_i\}_{i \in \mathcal{D}}$ are in a standard order for $\Psi$.

Theorem 8.2 Let $\mathcal{M}$ denote a Bose-Mesner algebra on $X$. Fix $p \in X$, and let $\mathcal{T}$ denote the subconstituent algebra of $\mathcal{M}$ with respect to $p$. Suppose that $\psi: \mathcal{T} \to \mathcal{T}$ is a hyper-self-duality. Further suppose that $\mathcal{M}$ is imprimitive with respect to $(\mathcal{I}, \mathcal{D})$ (so $\mathcal{M}$ is imprimitive with respect to $(\mathcal{D}, \mathcal{I})$).

(i) Let $\|\mathcal{T}\|_\mathcal{I}$ denote the subconstituent algebra with respect to $p$ of the $\|p\|_\mathcal{I}$-block Bose-Mesner algebra $\|\mathcal{M}\|_\mathcal{I}$ of $\mathcal{M}$ with respect to $(\mathcal{I}, \mathcal{D})$. Let $\|\mathcal{T}\|_\mathcal{I}$ denote the subconstituent algebra with respect to $\|p\|_\mathcal{I}$ of the quotient Bose-Mesner algebra $\|\mathcal{M}\|_\mathcal{I}$ of $\mathcal{M}$ with respect to $(\mathcal{D}, \mathcal{I})$. Then there is a hyper-duality $\{\psi\}: \|\mathcal{T}\|_\mathcal{I} \to \|\mathcal{T}\|_\mathcal{I}$ satisfying $\{\psi\}(\|A_i\|_\mathcal{I}) = \|A_i^*\|_\mathcal{I}$, $\{\psi\}(\|E_i\|_\mathcal{I}) = \|E_i\|_\mathcal{I}$ ($i \in \mathcal{I}$), where $\{A_i\}_{i \in \mathcal{I}}$ and $\{E_i\}_{i \in \mathcal{I}}$ are in a standard order for $\psi$.

(ii) Let $\|\mathcal{T}\|_\mathcal{D}$ denote the subconstituent algebra with respect to $\|p\|_\mathcal{D}$ of the quotient Bose-Mesner algebra $\|\mathcal{M}\|_\mathcal{D}$ of $\mathcal{M}$ with respect to $(\mathcal{I}, \mathcal{D})$. Let $\|\mathcal{T}\|_\mathcal{D}$ denote the subconstituent algebra with respect to $p$ of the $\|p\|_\mathcal{D}$-block Bose-Mesner algebra $\|\mathcal{M}\|_\mathcal{D}$ of $\mathcal{M}$ with respect to $(\mathcal{D}, \mathcal{I})$. Then there is a hyper-duality $\{\psi\}: \|\mathcal{T}\|_\mathcal{D} \to \|\mathcal{T}\|_\mathcal{D}$ satisfying $\{\psi\}(\|E_i\|_\mathcal{D}) = \|E_i\|_\mathcal{D}$, $\{\psi\}(\|A_i\|_\mathcal{D}) = \|A_i\|_\mathcal{D}$ ($i \in \mathcal{D}$), where $\{E_i\}_{i \in \mathcal{D}}$ and $\{A_i\}_{i \in \mathcal{D}}$ are in a standard order for $\psi$.

Theorem 8.3 With reference to Theorem 8.2, suppose that the matrix $H \in \mathcal{M}_X$ represents the hyper-self-duality $\psi: \mathcal{T} \to \mathcal{T}$. 

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Let $\lfloor \lfloor H \rceil \rceil \in M_{\lfloor \lfloor p \rceil \rceil} I$, $\lceil \lceil X \rceil \rceil I$ have $(x, \lceil \lceil y \rceil \rceil I)$-entry
$$\lfloor \lfloor H \rceil \rceil (x, \lceil \lceil y \rceil \rceil I) = \sum_{y' \in \lceil \lceil y \rceil \rceil I} H(x, y')(x \in [\lfloor \lfloor p \rceil \rceil], \lceil \lceil y \rceil \rceil I \in [\lceil \lceil X \rceil \rceil])$$.

Then $\lfloor \lfloor H \rceil \rceil$ represents the hyper-duality $\lceil \lceil \psi \rceil \rceil$ of Theorem 8.2(i).

(ii) Let $\lceil \lceil H \rceil \rceil \in M_{\lceil \lceil X \rceil \rceil D}, \lfloor \lfloor p \rceil \rceil D$ have $(\lceil \lceil x \rceil \rceil D, y)$-entry
$$\lceil \lceil H \rceil \rceil(\lceil \lceil x \rceil \rceil D, y) = \sum_{x' \in \lceil \lceil x \rceil \rceil D} H(x', y) \quad (\lceil \lceil x \rceil \rceil D \in [\lceil \lceil X \rceil \rceil D, y \in [\lfloor \lfloor p \rceil \rceil D].$$

Then $\lceil \lceil H \rceil \rceil$ represents the hyper-duality $\lceil \lceil \psi \rceil \rceil$ of Theorem 8.2(ii).

9 Example

The reader may find it illustrative to fill in the details of the following examples associated with the Hamming cubes. The reader is referred to [3] for background material. For example, we shall freely use facts concerning P- and Q-polynomial Bose-Mesner algebras.

Let $X$ denote the set of all $d$-tuples of zeros and ones. The Hamming cube $Q_d$ is the distance-regular graph on $X$ with two vectors adjacent if they differ in exactly one coordinate. Consider the subconstituent algebra of the Hamming cube with respect to $\vec{0}$—the reference [15] describes this subconstituent algebra. The adjacency matrix $A$, Q-polynomial primitive idempotent $E$, dual adjacency matrix $A^* = \rho(2^d E)$, and Q-polynomial dual idempotent $E^* = \rho(A)$ are given by

$$A(\vec{u}, \vec{v}) = \begin{cases} 1 & \text{if } \vec{u}, \vec{v} \text{ differ in exactly 2 entries,} \\ 0 & \text{otherwise,} \end{cases}$$
$$E(\vec{u}, \vec{v}) = 2^{-d}(d - 2)j \text{ whenever } \vec{u}, \vec{v} \text{ differ in exactly } j \text{ entries,}$$
$$A^*(\vec{u}, \vec{v}) = d - 2j \text{ whenever } \vec{u} \text{ has exactly } j \text{ nonzero entries,}$$
$$E^*(\vec{u}, \vec{u}) = \begin{cases} 1 & \text{if } \vec{u} \text{ has exactly one nonzero entry,} \\ 0 & \text{otherwise} \end{cases}$$

for all $\vec{u}, \vec{v} \in X$. Let $H$ denote the matrix with $(\vec{u}, \vec{v})$-entry $H(\vec{u}, \vec{v}) = (-1)^{\vec{u} \cdot \vec{v}}$ for all $\vec{u}, \vec{v} \in X$. Then $H$ represents a hyper-self-duality of $Q_d$:

$$^i A H = HA^*, \quad A^* H = HA, \quad EH = HE^*, \quad E^* H = H^i E \quad (0 \leq i \leq d).$$

In particular, the Bose-Mesner algebra of the Hamming cube is representably hyper-self-dual. The Hamming cube is imprimitive (both bipartite and antipodal). The associated block and quotient Bose-Mesner algebras are respectively those of the halved cube $\frac{1}{2}Q_d$ and the folded cube $Q_d^*$. 
Let \( \lfloor X \rfloor = \{ \bar{u} \in X \mid \bar{u} \) has an even number of nonzero entries\}. The halved cube \( \frac{1}{2} Q_d \) is the distance-regular graph with vertex set \( \lfloor X \rfloor \) and two vectors adjacent if they differ in exactly two coordinates. Consider the subconstituent algebra of \( \frac{1}{2} Q_d \) with respect to \( \bar{0} \). The adjacency matrix \( \lfloor A \rfloor \), primitive idempotent \( \lfloor E \rfloor \), dual adjacency matrix \( \lfloor A^* \rfloor = \rho(2^{d-1}\lfloor E \rfloor) \), and Q-polynomial dual idempotent \( \lfloor E^* \rfloor = \rho(\lfloor A \rfloor) \) are given by

\[
\lfloor A \rfloor(\bar{u}, \bar{v}) = \begin{cases} 1 & \text{if } \bar{u}, \bar{v} \text{ differ in exactly 2 entries,} \\ 0 & \text{otherwise,} \end{cases}
\]

\[
\lfloor E \rfloor(\bar{u}, \bar{v}) = 2^{1-d}(d-2j) \text{ if } \bar{u}, \bar{v} \text{ differ in exactly } j \text{ entries,}
\]

\[
\lfloor A^* \rfloor(\bar{u}, \bar{v}) = d - 2j \text{ if } \bar{u} \text{ has exactly } j \text{ nonzero entries,}
\]

\[
\lfloor E^* \rfloor(\bar{u}, \bar{v}) = \begin{cases} 1 & \text{if } \bar{u} \text{ has exactly two nonzero entries,} \\ 0 & \text{otherwise} \end{cases}
\]

for all \( \bar{u}, \bar{v} \in \lfloor X \rfloor \).

Let \( \lceil X \rceil = \{ \bar{u} \in X \mid \text{the first coordinate of } \bar{u} \text{ is zero}\} \), a set of representatives of antipodal pairs of vertices in \( Q_d \). The folded cube \( Q_d \) is the distance-regular graph with vertex set \( \lceil X \rceil \) and two vectors adjacent if they differ in either 1 or \( d - 1 \) coordinates. Consider the subconstituent algebra of \( Q_d \) with respect to \( \bar{0} \). The adjacency matrix \( \lceil A \rceil \), Q-polynomial primitive idempotent \( \lceil E \rceil \), dual adjacency matrix \( \lceil A^* \rceil = \rho(2^{d-1}\lceil E \rceil) \), and Q-polynomial dual idempotent \( \lceil E^* \rceil = \rho(\lceil A \rceil) \) are given by

\[
\lceil A \rceil(\bar{u}, \bar{v}) = \begin{cases} 1 & \text{if } \bar{u}, \bar{v} \text{ differ in exactly 1 or } d - 1 \text{ or entries,} \\ 0 & \text{otherwise,} \end{cases}
\]

\[
\lceil E \rceil(\bar{u}, \bar{v}) = 2^{1-d}((d - 2 \min(j, d - j))^2 - d)/2 
\]

\[
\text{if } \bar{u}, \bar{v} \text{ differ in exactly } j \text{ or } d - j \text{ entries,}
\]

\[
\lceil A^* \rceil(\bar{u}, \bar{v}) = ((d - 2 \min(j, d - j))^2 - d)/2 
\]

\[
\text{if } \bar{u} \text{ has exactly } j \text{ or } d - j \text{ nonzero entries,}
\]

\[
\lceil E^* \rceil(\bar{u}, \bar{v}) = \begin{cases} 1 & \text{if } \bar{u} \text{ has exactly 1 or } d - 1 \text{ nonzero entries,} \\ 0 & \text{otherwise} \end{cases}
\]

for all \( \bar{u}, \bar{v} \in \lceil X \rceil \).

Define \( \lceil H \rceil \in \mathbb{M}_{\lceil X \rceil, \lceil X \rceil} \) to have entries \( \lceil H \rceil(\bar{u}, \bar{v}) = (-1)^{\bar{u} \cdot \bar{v}} \) (\( \bar{u} \in \lceil X \rceil, \bar{v} \in \lceil X \rceil \)). Then as predicted by Lemma 8.3, \( \lceil H \rceil \) represents a hyper-duality from \( \frac{1}{2} Q_d \) to \( Q_d \):

\[
\lceil A \rceil \lceil H \rceil = \lceil H \rceil \lceil A^* \rceil, \quad \lceil E \rceil \lceil H \rceil = \lceil H \rceil \lceil E^* \rceil, \\
\lceil E^* \rceil \lceil H \rceil = \lceil H \rceil \lceil E \rceil, \quad \lceil A^* \rceil \lceil H \rceil = \lceil H \rceil \lceil A \rceil.
\]

If \( d \neq 6 \), the folded \( d \)-cube is characterized by its parameters. There are exactly
two other distance-regular graphs with the same parameters as the folded 6-cube (see [3, p. 264]), i.e. whose Bose-Mesner algebra is isomorphic to that of the folded 6-cube. In fact, their subconstituent algebras (with respect to every base point) are isomorphic to that of the folded 6-cube (the tools of [8] are helpful in verifying this). Thus each is representably hyper-dual to the halved 6-cube.

Suppose \( d \) is even. Then the halved cube \( \frac{1}{2}Q_d \) is antipodal and the folded cube \( Q_d \) is bipartite, i.e. both are imprimitive. The folded halved cube is isomorphic to the halved folded cube, and each is formally self-dual. It can be checked directly that the halved folded cube is representably hyper-self-dual. Let \( Y = \{ \vec{u} \in X \mid \text{the first entry of } \vec{u} \text{ is zero, } \vec{u} \text{ has an even number of nonzero entries} \} \). The halved folded cube \( \frac{1}{2}Q_d \) has vertex set \( Y \) with two vectors adjacent if they differ in 2 or \( d - 2 \) places. Consider the subconstituent algebra with respect to the zero vector. The adjacency matrix \( B \), Q-polynomial primitive idempotent \( F \), dual adjacency matrix \( B^* = \rho(2^{d-2}F) \), and Q-polynomial dual idempotent \( F^* = \rho(B) \) are given by

\[
B(\vec{u}, \vec{v}) = \begin{cases} 
1 & \text{if } \vec{u}, \vec{v} \text{ differ in exactly 2 or } d - 2 \text{ or entries,} \\
0 & \text{otherwise,}
\end{cases}
\]

\[
F(\vec{u}, \vec{v}) = 2^{2-d}(2(d/2 - \min(j, d - j))^2 - d/2)
\]

\[
B^*(\vec{u}, \vec{u}) = (2(d/2 - \min(j, d - j))^2 - d/2)
\]

\[
F^*(\vec{u}, \vec{u}) = \begin{cases} 
1 & \text{if } \vec{u} \text{ has exactly 2 or } d - 2 \text{ nonzero entries,} \\
0 & \text{otherwise}
\end{cases}
\]

for all \( \vec{u}, \vec{v} \in Y \). Let \( K \) denote the matrix with \((\vec{u}, \vec{v})\)-entry \( K(\vec{u}, \vec{v}) = (-1)^{\vec{u} \cdot \vec{v}} \) for all \( \vec{u}, \vec{v} \in Y \). Then \( K \) represents a hyper-self-duality of \( \frac{1}{2}Q_d \):

\[
BK = KB^*, \quad FK =KF^*, \quad F^*K =KF, \quad B^*K = KB.
\]

Although \( K \) is not in the subconstituent algebra of \( \frac{1}{2}Q_d \), it can be shown using module theoretic arguments that this hyper-self-duality can be represented by some matrix \( K' \) in the subconstituent algebra, i.e. \( \frac{1}{2}Q_d \) is strongly hyper-self-dual.

For small \( d \), there are other distance-regular graphs with the same parameters as the folded halved cube, so they are both formally self-dual and formally dual to the folded half-cube. It remains to check if all of them are hyper-self dual (those with diameter 2 are by [7]) and if any are hyper-dual to the folded halved cube. A folded halved cube with diameter at least 5 is the unique distance-regular graph with its parameters [19].
The Hamming cubes are 2-homogeneous bipartite distance-regular graphs. All of these graphs are bipartite, antipodal, and hyper-self-dual [9], so the results of this paper may be applied to these examples as well.

References


