

## **Cocycle Invariants of Knots**

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## CHAPTER 1

# Introduction

### 1. What are these web pages about

This article is written for web pages posted at <http://shell.cas.usf.edu/quandle> under the title *Background*.

The purpose of this paper is to give an overview of *quandle cocycle invariants* of knots, including definitions of the invariants. The main goal is to provide the reader with necessary background information quickly, in an effective way.

### 2. Target readers

These pages are intended for advanced undergraduate students who are familiar with basic concepts in knot theory, as well as professionals who want to get a quick overview and current status of this research project. The knowledge we assume are basic group theory and linear algebra, and laymen's knowledge in knot theory, such as Reidemeister moves of knot diagrams. For such basics, we refer the reader to any introductory book on knots, such as [Ad94, Liv93, Mura96].

### 3. Other sources of data used in our calculations

We used the knot table available from Livingston's web site [Liv]. For quandles, we used the table of quandles in Appendix of [CKS00].

### 4. Comments on history

Racks and quandles are fundamental algebraic structures. Quandles have been rediscovered and studied extensively over the past sixty years [Brs88, Deh00, GK03, Joy82, Kau91, Mat82, Taka42]. It appears that a special case of quandles (involutory quandles), called *Kei*, first appeared in literature in [Taka42]. The name *quandle* is due to [Joy82]. It is beyond our scope here to review such a rich history of quandles in details.

In [CJKLS03], quandle cohomology theory was constructed as a modification of rack cohomology theory [FRS95, FRS\*], and quandle cocycle invariants were also defined in a state-sum form. It is our goal here to briefly review this invariant, and post our results in on-going research projects.



## CHAPTER 2

# Knot Diagrams and Reidemeister Moves

It is assumed throughout that a (classical) *knot* is a smooth embedding  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^3$  or  $\mathbb{R}^3$ , where  $\mathbb{S}^n$  denotes the  $n$ -dimensional sphere,  $\mathbb{S}^n = \{ x \in \mathbb{R}^{n+1} \mid |x| = 1 \}$ . If no confusion arises, we also mean the image  $K = f(\mathbb{S}^1)$  by a knot. The equivalence of knots is by smooth ambient isotopies. Another set-up commonly used is in the Piecewise-Linear (PL), locally flat category. In general knot theory is study of embeddings.

Instead of going through detailed set-up, such as detailed definitions involved and basic and fundamental theorems used for smooth manifolds, their embeddings and projections, we take a common practice where we take the results on Reidemeister moves, and take combinatorially represented knot diagrams as our subject of study, and Reidemeister moves as their equivalence. Furthermore, such diagrammatic approaches are explained in details in many books in knot theory, so the purpose of this chapter is to briefly review this combinatorial set-up and establish our conventions.

### 1. Knot Diagrams

Let  $K = f(\mathbb{S}^1) \subset \mathbb{R}^3$  be a knot, where  $f : \mathbb{S}^1 \rightarrow \mathbb{R}^3$ . Let  $P \subset \mathbb{R}^3$  be a plane such that  $K \cap P = \emptyset$ . Let  $p : \mathbb{R}^3 \rightarrow P$  be the orthogonal projection. Then we may assume without loss of generality that the restriction  $p \circ f : \mathbb{S}^1 \rightarrow P$  is a generic immersion. This implies that the image  $(p \circ f)(\mathbb{S}^1)$  is an immersed closed curve only with transverse double intersection points (called *crossings* or *crossing points*), other than embedded points.

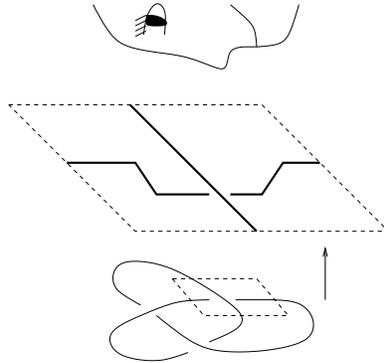


FIGURE 1. Knot diagrams

The situations excluded under this assumption are points of tangency, triple intersection points, and points where a tangent line does not exist, for example. A reason why we may assume such a nice situation is that such projection directions are open and dense among all possible directions.

A *knot diagram* is such a projection image  $(p \circ f)(S^1)$  with under-arc broken to indicate the crossing information. Here, a direction orthogonal to  $P$  is chosen as a height direction, and if a crossing point is formed by  $p(\alpha)$  and  $p(\beta)$  where  $\alpha, \beta$  are arcs in  $K$ , and  $\alpha$  is located in higher position than  $\beta$  is with respect to the fixed height direction, then  $p(\alpha)$  is called an *over-arc* (or *upper-arc*) and  $\beta$  a *under-arc* (or *lower-arc*). The preimages of the crossing points in these arcs are called *over-crossing* and *under-crossing*, respectively. This information as to which arc is upper and which is lower, is the crossing information. It may be convenient to take the height function to be pointing toward the viewer of a knot, see Fig. 1.

To represent knot projections and diagrams symbolically, *Gauss codes* [Gauss1883] has been used. Such codes have been used to produce knot tables [HTW98], and compute knot invariants by computer. There are several conventions for such codes. Here we follow the conventions in [Kau99]. We start with codes for projections. Pick a base point on a projection, and travel along the curve in the given orientation direction. Assign positive integers  $1, \dots, k$  in this order to the crossings that are encountered, where  $k$  is the number of crossings. Then the Gauss code of the projection is a sequence of integers read when the whole circle is traced back to the base point. A typical projection of a trefoil has the code 123123 (Fig. 2 (1)).

To represent over- and under-crossing, if the number  $j$  corresponds to an over- or under-crossing, respectively, then replace  $j$  by  $j_O$  or  $j_U$ , respectively. Thus the right-hand trefoil has the code  $1_O 2_U 3_O 1_U 2_O 3_U$  (Fig. 2 (2)). To represent the sign of the crossing, replace them further by  $j_{C+}$  or  $j_{C-}$  if  $j$  corresponds to a positive or negative crossing, respectively, where  $C = O$  or  $U$  depending on over or under. The convention for the sign of a crossing is depicted in Fig. 2 (3). The signed code for the right-hand trefoil is  $1_{O+} 2_{U+} 3_{O+} 1_{U+} 2_{O+} 3_{U+}$ . When one of the signs is reversed, a diagram no longer is realized on the plane as depicted in Fig. 2 (4) for  $1_{O+} 2_{U+} 3_{O-} 1_{U+} 2_{O+} 3_{U-}$  (this example is taken from [Kau99]). In this figure a non-existent crossing, called a *virtual crossing*, is represented by a circled crossing. Such a generalization of knot diagrams and non-planar Gauss codes up to equivalence by analogues of Reidemeister moves is called *virtual knot theory* [Kau99]. These symbolic representations are useful in applications to DNA.

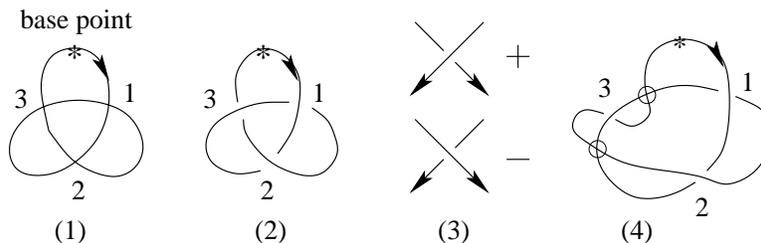


FIGURE 2. Gauss codes

## 2. Reidemeister Moves

We list the Reidemeister moves in Fig. 3 and would like to leave details to other books in knot theory, and recall the most fundamental theorem of Reidemeister we rely on.

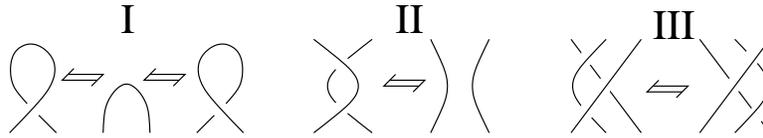


FIGURE 3. reidmoves

**THEOREM 2.1 (Reidemeister).** *Let  $D_1, D_2$  be two diagrams representing the same (equivalent) knot. Then  $D_2$  is obtained from  $D_1$  by a sequence of Reidemeister moves and isotopy of the plane  $\mathbb{R}^2$ .*

**REMARK 2.2.** There are variations of Reidemeister moves in various situations and set-ups. Here we mention three variations.

- The Jones polynomial was first discovered through braid group representations [Jones87]. Two theorems play key roles (see, for example, [Kau91, Kama02]): Alexander's theorem says that any knot or link can be represented as a closed braid, and Markov's theorem says how closed braid forms of the same knot are related (related by conjugations and Markov (de-)stabilizations). Thus the relations of the braid groups and Markov (de-)stabilizations play the role of Reidemeister moves.
- In defining the Jones polynomial from an operator approach [Kau91], a height function is fixed on the plane, and local maxima and minima of knot diagrams plays an important role (a pairing and a copairing are assigned to them). Then moves involving these become necessary to take into considerations. These additional moves, called *Freyd-Yetter moves* and depicted in Fig. 4, are considered also in formulating category formed by knot diagrams [FY89].
- For virtual knots, Reidemeister moves are symbolically interpreted in terms of Gauss codes, then reinterpreted in terms of virtual knot diagrams, see [Kau99], for example. Those moves involving virtual crossings are depicted in Fig. 5.
- Another variation is to assign parenthesis structures on arcs of the diagrams, called non-associative tangles [BN97]. These were used for the study of finite type invariants of knots.



FIGURE 4. Moves with a height function specified

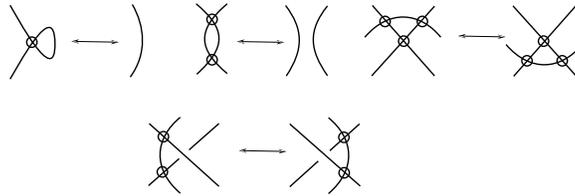


FIGURE 5. Moves for virtual knots

### 3. Knotted Surface Diagrams and Their Moves

Knot diagrams and their moves are defined for higher dimensional knots (codimension 2 smooth embeddings). In particular, for surfaces in 4-space, they are explicitly known, and depicted in Figs. 6, 7, taken from [CKS00]. Specifically, a *knotted surface* is a smooth embedding  $f : F \rightarrow \mathbb{S}^4$  or  $\mathbb{R}^4$  and all definitions are given in completely analogous manner as for the classical case. The set of crossing points, in this case, are described by embedded double point curves (A), isolated triple points (B), and branch points (C) in Fig. 6. The moves analogous to Reidemeister moves, in particular, are obtained by Roseman [Rose98] and called Roseman moves.

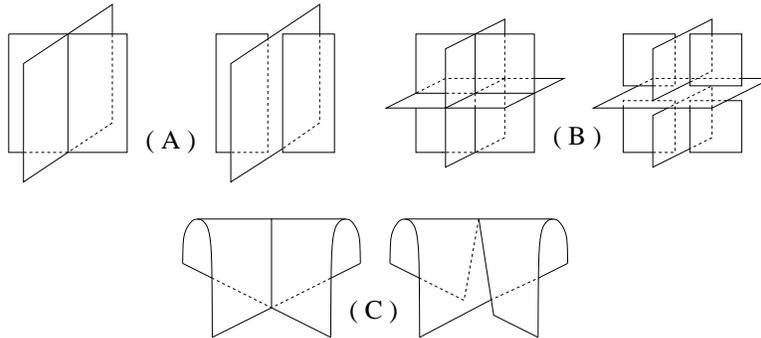


FIGURE 6. Broken surface diagrams near singular points

Referring the reader to [CKS00] for details, we only mention here the relation between knotted surface diagrams and Reidemeister moves for classical knots. The continuous family of projection curves in  $\mathbb{R}^2 \times [0, 1]$  of a knot, where  $[0, 1]$  represents the time parameter during which a Reidemeister move occurs, forms a surface mapped in  $\mathbb{R}^2 \times [0, 1]$ . An example for the type I move is depicted in Fig. 8. As indicated in the figure, the exact moment that the type I move happens corresponds to a branch point of a mapped surface, a generic singularity of smooth maps  $F \rightarrow \mathbb{R}^3$  from surfaces to  $\mathbb{R}^3$ . Similarly, a type II move corresponds to Morse critical points of double point curves of surfaces, and a type III move to an isolated triple point (see also [CrSt98]). Thus a generic singularities and critical points of one dimensional higher case give rise to moves of diagrams. This will be the case for triangulated manifolds as we will see later.

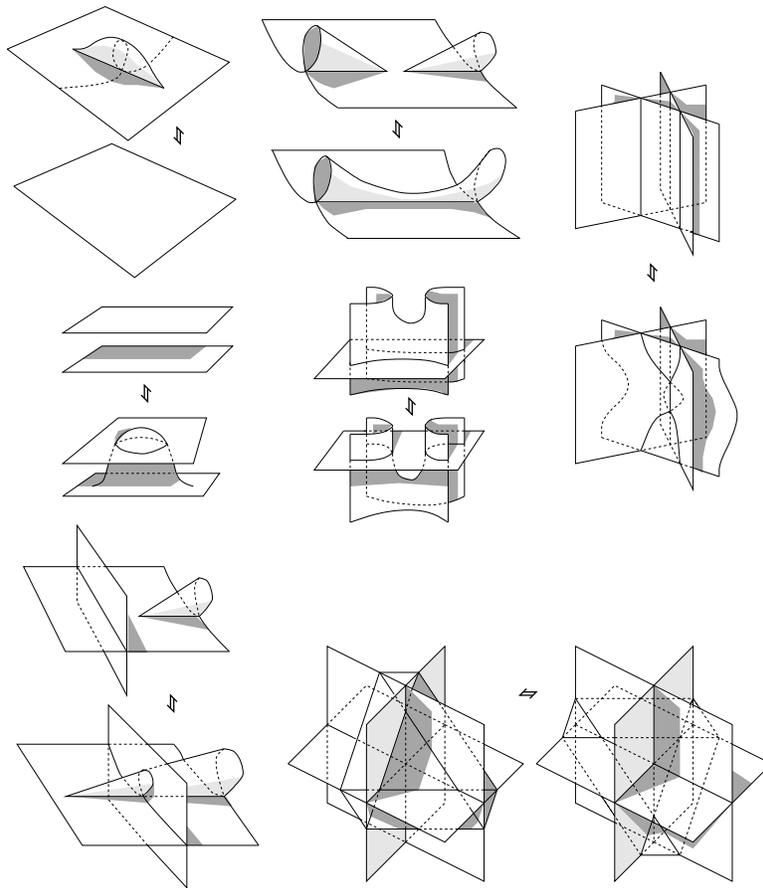


FIGURE 7. Roseman moves

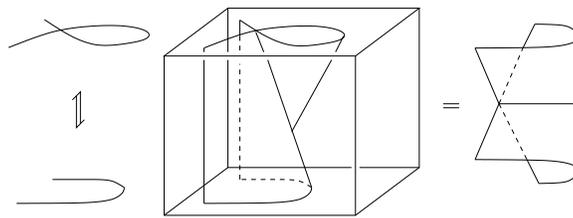


FIGURE 8. Continuous family of curves at type I move



## Quandles and Their Colorings of Knot Diagrams

Racks and quandles are fundamental algebraic structures. Quandles have been rediscovered and studied extensively over the past sixty years [Brs88, Deh00, GK03, Joy82, Kau91, Mat82, Taka42]. It appears that a special case of quandles (involutory quandles), called *Kei*, first appeared in literature in [Taka42], and studied for their algebraic structures in relation to symmetry of geometric objects. The name *quandle* is due to [Joy82].

The famous Fox tri-colorings (and  $n$ -colorings for  $n \leq 3$ , [Fox61]) of knot diagrams, that are identified with homomorphisms from the knot groups (the fundamental groups of the complements of knots) to dihedral groups, are generalized to colorings by quandles [FR92].

In [CJKLS03], quandle cohomology theory was constructed as a modification of rack cohomology theory [FRS95, FRS\*], and quandle cocycle invariants were defined in a state-sum form. It is our goal in this chapter to review colorings of knot diagrams by quandles and quandle cocycle invariants.

### 1. Racks and Quandles

A *quandle*,  $X$ , is a non-empty set with a binary operation  $X \times X \rightarrow X$  denoted by  $(a, b) \mapsto a * b$  such that

- (I) For any  $a \in X$ ,  $a * a = a$ .
- (II) For any  $a, b \in X$ , there is a unique  $c \in X$  such that  $a = c * b$ .
- (III) For any  $a, b, c \in X$ , we have  $(a * b) * c = (a * c) * (b * c)$ .

A *rack* is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [Brs88, FR92, Joy82, Mat82].

The following are typical examples of quandles. Any set  $X (\neq \emptyset)$  with  $a * b = a$  for any  $a, b \in X$  is a quandle called a *trivial* quandle.

A group  $G$  with  $n$ -fold conjugation as the quandle operation:  $a * b = b^n a b^{-n}$  is a quandle for an integer  $n$ . Any subset of  $G$  that is closed under such conjugation is also a quandle.

Another family is modules over Laurent polynomial ring  $\Lambda_p = \mathbb{Z}_p[t, t^{-1}]$ , with the operation  $a * b = ta + (1-t)b$ , called *Alexander quandles*. Often we use quotient rings  $M = \mathbb{Z}_p[t]/f(t)$ , for a prime  $p$  and modulo a polynomial  $f(t)$  of degree  $d$ . The elements of these rings  $M$  are represented by remainders of polynomials when divided by  $f \bmod p$ , so that  $M$  consists of  $p^d$  elements.

Let  $n > 2$  be a positive integer, and for elements  $i, j \in \mathbb{Z}_n$  (identified with representatives  $\{0, 1, \dots, n-1\}$ ), define  $i * j \equiv 2j - i \pmod{n}$ . Then  $*$  defines a quandle structure  $R_n$  called the *dihedral quandle* on  $\mathbb{Z}_n$ . This is an Alexander quandle  $R_n = \mathbb{Z}_n[t]/(t+1)$ , and also, is the set of reflections of a regular  $n$ -gon (elements of the dihedral group  $D_{2n}$  represented by reflections) with conjugation as the quandle operation.

A map  $f : X \rightarrow Y$  between quandles  $X, Y$  is a quandle *homomorphism* if  $f(a, b) = f(a) * f(b)$  for any  $a, b \in X$ , where we abuse notation and use  $*$  for quandle operations on different quandles unless confusion arises. A quandle homomorphism is an isomorphism if it is surjective and injective. If there is an isomorphism between two quandles, they are called *isomorphic*. An isomorphism between the same quandle  $X$  is an *automorphism*, and they form a group  $\text{Aut}(X)$ , the automorphism group of  $X$ .

Classification of quandles up to isomorphism are discussed in [Gra02b\*] for those with order  $p^2$  for prime  $p$ , and in [Nel03] for Alexander quandles, for example. Another related issue is to realize quandles using familiar structures, and this is discussed in [Joy82] as follows. Consider a group  $G$  and a group automorphism  $s \in \text{Aut}(G)$ . The operation  $x * y = s(xy^{-1})y$  defines a quandle structure  $(G; s)$  on  $G$ , and if  $s(h) = h$  for all  $h$  in a subgroup  $H$ , then  $H \setminus G$  inherits the operation:  $Hx * Hy = Hs(xy^{-1})y$ , forming another quandle  $(H \setminus G; s)$ . In particular, for a fixed  $z \in G$ , the inner-automorphism  $s(x) = z^{-1}xz$  can be used, and if  $z \in H$  and  $H$  satisfies the above condition, then  $(H \setminus G; s)$  has the quandle operation  $Hx * Hy = Hxy^{-1}zy$ . Then it is proved in [Joy82] that any *homogeneous* quandle is, in fact, represented this way [Joy82], where a quandle  $X$  is homogeneous if for any  $x, y \in X$  there is an  $f \in \text{Aut}(X)$  such that  $y = f(x)$ .

## 2. Colorings of Knot Diagrams by Quandles

Let  $X$  be a fixed quandle. Let  $D$  be a given oriented classical knot or link diagram, and let  $\mathcal{R}$  be the set of (over-)arcs. The normals (normal vectors) are given in such a way that the ordered pair (tangent, normal) agrees with the orientation of the plane (the ordered pair of the standard  $x$ - and  $y$ -axes), see Fig. 1.

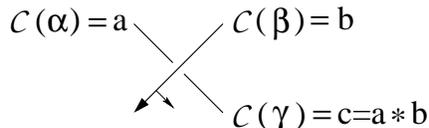


FIGURE 1. Quandle relation at a crossing

A (quandle) *coloring*  $\mathcal{C}$  is a map  $\mathcal{C} : \mathcal{R} \rightarrow X$  such that at every crossing, the relation depicted in Fig. 1 holds. If the normal to the over-arc  $\beta$  points from the arc  $\alpha$  to  $\gamma$ , then it is required that  $\mathcal{C}(\alpha) * \mathcal{C}(\beta) = \mathcal{C}(\gamma)$ . The ordered pair  $(a, b)$  is called the *source colors* or *ordered pair of colors* at the crossing.

Let  $\text{Col}_X(D)$  denote the set of colorings of a knot diagram  $D$  of a knot  $K$  by a quandle  $X$ . It is known [FR92], and can be checked on diagrams, that for any coloring  $\mathcal{C}$  of a diagram  $D$  of a knot  $K$  by a quandle  $X$ , there is a unique coloring  $\mathcal{C}'$  of a diagram  $D'$  of the same knot  $K$  that is obtained from  $D$  by a single Reidemeister move of any type (I, II, or III), such that the colorings coincide outside of a small portion of the diagrams where the move is performed. Thus there is a one-to-one correspondence between the sets of colorings of two diagrams of the same knot. In particular, the cardinality  $|\text{Col}_X(D)|$  is a knot invariant. Note that any knot diagram is colored by a single element of a given quandle. Such a monochromatic coloring is called *trivial*. Hence for any knot  $K$  and any quandle  $X$ ,  $|\text{Col}_X(D)|$  is at least the cardinality  $|X|$  of the quandle.

In fact, the Reidemeister moves I, II, III correspond to the quandle axiom I, II, III, respectively. In particular, a type III move represents the self-distributivity as depicted in Fig. 2.

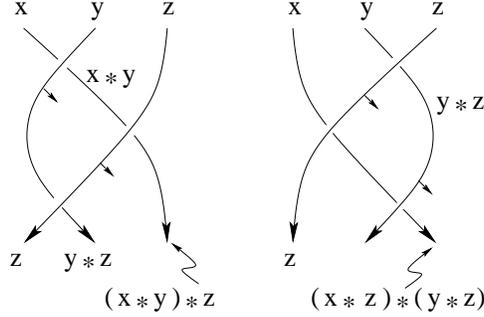


FIGURE 2. Type III move and self-distributivity

A coloring is defined for virtual knot diagrams in a completely similar manner, and the sets of colorings of two virtual diagrams of the same virtual knot are in one-to-one correspondence, as checked by Reidemeister moves for virtual knots.

For a coloring  $\mathcal{C}$  of a knot diagram  $D$ , there is a coloring of regions ([FRS95, FRS\*]) that extend  $\mathcal{C}$  as depicted in Fig. 3. The set  $\mathcal{R}g$  of regions is defined to be the set of connected components of the complement of (underlying) projection of a knot diagram in  $\mathbb{R}^2$  (or  $S^2$ ). A coloring of regions is an assignment of quandle elements to regions, together with a given coloring  $\mathcal{C}$  of arcs,  $\tilde{\mathcal{C}} : \mathcal{R} \cup \mathcal{R}g \rightarrow X$ ,  $\tilde{\mathcal{C}}|_{\mathcal{R}} = \mathcal{C}$ , that satisfies the following requirement: Suppose the two regions  $R_1$  and  $R_2$  are divided by an arc  $\alpha$  such that the normal to  $\alpha$  points from  $R_1$  to  $R_2$ , and let  $x$  be a color of  $R_1$ . Then the color of  $R_2$  is required to be  $x * \mathcal{C}(\alpha)$ . The set of colorings  $\tilde{\mathcal{C}} : \mathcal{R} \cup \mathcal{R}g \rightarrow X$  of arcs and regions is denoted by  $\text{Colr}_X(D)$ .

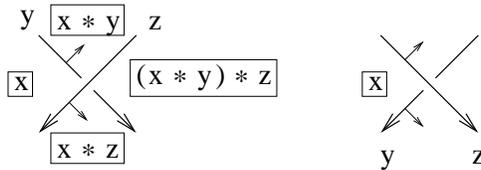


FIGURE 3. Quandle colorings of regions

In Fig. 3, it is seen that a coloring of regions (region colors) are well-defined near a crossing. In the left of the figure, a *positive* crossing is depicted. Near a crossing, there are four regions, one of which is a unique region from which all the normal vectors of the arcs point to other regions. Such a region is called the *source* region at a crossing. There are two ways to go from the source region (leftmost region) to the rightmost region, through upper arcs and lower arcs. Through upper arcs we obtain the color  $(x * y) * z$ , and through the lower arcs  $(x * z) * (y * z)$ , that coincide by the self-distributivity of a quandle. The situation is similar at a negative crossing (the right of Fig. 3, the colors of the regions unspecified are for an

exercise). The triple of elements  $(x, y, z)$  of  $X$  is called the *source colors* or *ordered triple of colors* at the crossing. Specifically,  $x$  is the color of the source region,  $y$  is the color of the under-arc from which the normal of the over-arc points, and  $z$  is the color of the over-arc (the pair  $(y, z)$  is the source colors of arcs).

For any coloring  $\mathcal{C}$  of a knot diagram  $D$  on  $\mathbb{R}^2$  (or  $\mathbb{S}^2$ ) by a quandle  $X$ , and any specific choice of a color  $x_0 \in X$  for a region  $R_0 \in \mathcal{R}g$ , there is a unique region coloring  $\tilde{\mathcal{C}} : \mathcal{R} \cup \mathcal{R}g \rightarrow X$  that extends  $\mathcal{C}$ ,  $\tilde{\mathcal{C}}|_{\mathcal{R}} = \mathcal{C}$ , such that  $\tilde{\mathcal{C}}(R_0) = x_0$ . This is seen as follows. Let  $R \in \mathcal{R}g$  and take a path  $\gamma$  from  $R_0$  to  $R$  that avoids crossings of  $D$  and intersects transversely in finite points with  $D$ . Along  $\gamma$  the element  $\tilde{\mathcal{C}}(R)$  is determined by the coloring rule of regions from the colors of arcs. Take two paths,  $\gamma_1$  and  $\gamma_2$ , from  $R_0$  to  $R$ , that miss crossing points. There is a homotopy from  $\gamma_1$  to  $\gamma_2$ , and it is assumed without loss of generality that during the homotopy the paths experience, themselves and in relation to the knot projection, a sequence of Reidemeister moves. In particular, when a path goes through a crossing of the knot projection, it corresponds to a type III move, and well-definedness under such a move is checked in Fig. 3. Therefore the color of regions are uniquely determined by a color of a single region. Note that we used the fact that the plane is simply connected, i.e., any two paths are homotopic. In general, we cannot define region colors for knots on a surface, or virtual knots.

### 3. Colorings of Knotted Surface Diagrams by Quandles

A coloring of knotted surface diagrams by a quandle is defined (cf. [CKS00]) in an analogous manner as for (classical) knots and links. We specify a given orientation of a knotted surface diagram by orientation normal vectors to broken sheets. A *coloring* for a knotted surface diagram  $D$  by a quandle  $X$  is an assignment of an element (called a color) of a quandle  $X$  to each broken sheet such that  $p * q = r$  holds at every double point, where  $p$  (or  $r$ , respectively) is the color of the under-sheet behind (or in front of) the over-sheet with the color  $q$ , where the normal of the over-sheet points from the under-sheet behind it to the front. The pair  $(p, q) \in X \times X$  is called the *source colors* or *ordered pair of colors of the double point curve*. The coloring rule is depicted in the left of Fig. 4. In the right of Fig. 4, the situation of a coloring near a triple point is depicted. Note that a coloring of the “bottom” sheet that is divided into four broken sheet looks exactly like a coloring of a classical knot diagram with region colors. In particular, well-definedness of a coloring at a triple point requires the self-distributivity. The triple  $(p, q, r) \in X \times X \times X$  of colors at a triple point is called the *source colors* or *ordered triple of colors* at a triple point.

In arguments similar to the classical case, it is seen by checking the Roseman moves that there is a one-to-one correspondence between the set of colorings  $\text{Col}_X(D)$  of a diagram of a surface  $K$  and that of a diagram  $D'$ ,  $\text{Col}_X(D')$ , obtained from  $D$  by a Roseman move, and therefore, for any  $D$  and  $D'$  representing the same knotted surface. In particular, again, the number of colorings is a knotted surface invariant. The region colors are defined in a completely similar manner as well.

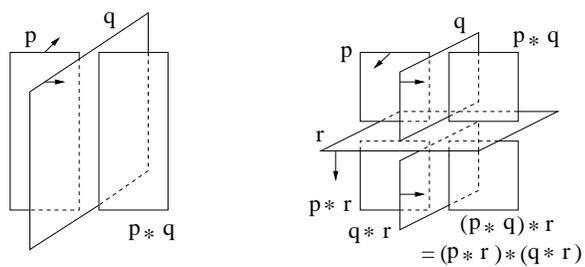


FIGURE 4. Colors at double curves and 3-cocycle at a triple point



## Quandle Cocycle Invariants of Knots

In this chapter, we define quandle cocycle invariants for classical knots and knotted surfaces. Details of algebraic aspects of homology and cohomology of quandles will be discussed in a later chapter, and here we limit discussions of cocycles to functions used as weights, and present useful polynomial cocycles.

### 1. Quandle 2-Cocycle Invariants of Classical Knots

The cocycle invariant for classical knots [CJKLS03] using quandle 2-cocycles was defined as follows. Let  $X$  be a finite quandle and  $A$  an abelian group.

A function  $\phi : X \times X \rightarrow A$  is called a *quandle 2-cocycle* if it satisfies the *2-cocycle condition*

$$\phi(x, y) - \phi(x, z) + \phi(x * y, z) - \phi(x * z, y * z) = 0,$$

for all  $x, y, z \in X$  and  $\phi(x, x) = 0$  for all  $x \in X$ . Let  $\mathcal{C}$  be a coloring of a given diagram  $D$  of a knot  $K$  by  $X$ .

The (*Boltzmann*) *weight*  $B(\mathcal{C}, \tau)$  at a crossing  $\tau$  of  $K$  is then defined by  $B(\mathcal{C}, \tau) = \phi(x_\tau, y_\tau)^{\epsilon(\tau)}$ , where  $(x_\tau, y_\tau)$  is the ordered pair of colors (source colors) at  $\tau$  and  $\epsilon(\tau)$  is the sign ( $\pm 1$  for positive and negative crossings, respectively) of  $\tau$ . Here  $B(\mathcal{C}, \tau)$  is an element of  $A$  written multiplicatively. In Fig. 1 of Chapter 2, if the under-arc of the crossing  $\tau$  depicted is oriented downwards from  $\alpha$  to  $\gamma$ , then  $\epsilon(\tau) = 1$ , and  $B(\mathcal{C}, \tau) = \phi(a, b)$  with the indicated coloring. If the orientation of the under-arc is opposite, then  $B(\mathcal{C}, \tau) = \phi(a, b)^{-1}$ .

DEFINITION 4.1 ([CJKLS03]). The formal sum (called a state-sum) in the group ring  $\mathbb{Z}[A]$

$$\Phi_\phi(K) = \sum_{\mathcal{C} \in \text{Col}_X(D)} \prod_{\tau} B(\mathcal{C}, \tau)$$

is called the quandle 2-cocycle invariant.

The value (an element of  $\mathbb{Z}[A]$ ) of the invariant is written as  $\sum_{g \in G} a_g g$ , so that if  $A = \mathbb{Z}_n = \{u^k \mid 0 \leq k < n\}$ , the value is written as a polynomial  $\sum_{k=0}^{n-1} a_k u^k$ . The fact that  $\Phi_\phi(K)$  is a knot invariant is proved easily by checking Reidemeister moves [CJKLS03]. For example, in Fig. 2 of Chapter 2, the product of weights that appear in the LHS is  $\phi(x, y)\phi(x * y, z)\phi(y, z)$  from the top to bottom crossings, and for the RHS it is  $\phi(y, z)\phi(x, z)\phi(x * z, y * z)$ , and the equality obtained is exactly the 2-cocycle condition in multiplicative notation, after canceling  $\phi(y, z)$ .

The cocycle invariant can be also written as a family (multi-set, a set with repetition allowed) of weight sums [Lop03]

$$\left\{ \sum_{\tau} B_A(\mathcal{C}, \tau) \mid \mathcal{C} \in \text{Col}_X(K) \right\}$$

where now the values of  $B$  in  $A$  are denoted by additive notation,  $B_A(\mathcal{C}, \tau) = \epsilon(\tau)\phi(x_\tau, y_\tau)$ . Then this is well-defined for any quandle of any cardinality. The sum of the value of the weight for a specific coloring is also called the contribution of the coloring to the invariant.

The 2-cocycle invariant is well-defined for virtual knots, as checked in a similar manner.

EXAMPLE 4.2. Consider a trefoil  $K$  represented by the closure of the braid  $(\sigma_1)^3$ , where we denote the standard braid generator by  $\sigma_i$ , which is represented by a positive crossing placed between  $i$ th and  $(i+1)$ st strings.

Let  $X = A = \mathbb{Z}_2[t]/(t^2 + t + 1)$ , an Alexander quandle as  $X$  and an abelian group as  $A$ , and put the source colors  $(1, 0)$ , for example, at the top crossing. This color vector goes down to the next crossing, giving  $(0, t)$ ,  $(t, t(1+t))=(t, 1)$ , and comes back to  $(1, t^2 + t + 1) = (1, 0)$ , by the coloring rule, and defines a non-trivial coloring  $\mathcal{C}$ . In fact any pair extends to a coloring, and  $|\text{Col}_X(K)| = 4^2$ .

Consider the function  $\phi : X \times X \rightarrow A$  defined by  $\phi(x, y) = (x - y)^2 y$ . Then this satisfies the 2-cocycle condition by calculation (we will further discuss this method of construction of 2-cocycles later).

The contribution is computed by

$$\sum_{\tau} B_A(\mathcal{C}, \tau) = 0 + (-t)^2 t + (t - 1)^2 = t + 1 \in A,$$

and after we compute it for all colorings, we obtain, as a multiset,

$$\Phi_{\phi}(K) = \{\sqcup_4(0), \sqcup_{12}(t + 1)\},$$

where  $\sqcup_n x$  denotes  $n$  copies of  $x$  for a positive integer  $n$ .

We make the following notational convention. In the above example, if we denote additive generators 1 and  $t$  multiplicatively by  $h$  and  $k$ , then the elements of  $X$  in multiplicative notation are written as  $\{1, h, k, hk\}$ . Hence the invariant in the state-sum form is written as  $\Phi_{\phi}(K) = 4 + 12hk$ . In *Maple* calculations, it is convenient and easier to see outputs if we use the symbols  $u (= u^1)$  and  $u^t$  for  $h$  and  $k$ . Then the additive notations remain on superscripts and the addition is also retained in exponents ( $hk = u \cdot u^t = u^{(t+1)}$ ). With this convention the invariant value is written as  $4 + 12u^{(t+1)}$ . We use this notation if there is no confusion.

## 2. Quandle 3-Cocycle Invariants of Classical Knots

Let  $X$  be a finite quandle and  $A$  an abelian group. A function  $\theta : X \times X \times X \rightarrow A$  is called a *quandle 3-cocycle* if it satisfies the *3-cocycle condition*

$$\begin{aligned} &\theta(x, z, w) - \theta(x, y, w) + \theta(x, y, z) - \theta(x * y, z, w) \\ &+ \theta(x * z, y * z, w) - \theta(x * w, y * w, z * w) = 0, \end{aligned}$$

for all  $x, y, z, w \in X$ , and  $\theta(x, x, y) = 0 = \theta(x, y, y)$  for all  $x, y \in X$ .

Let  $\mathcal{C} \in \text{Col}_X(D)$  be a coloring of arcs and regions of a given diagram  $D$  of a classical knot  $K$ . Let  $(x, y, z) (= (x_\tau, y_\tau, z_\tau))$  be the ordered triple of colors at a crossing  $\tau$ , see Fig. 3 of Chapter 2. Let  $\theta$  be a 3-cocycle. Then the weight in this case is defined by  $B(\mathcal{C}, \tau) = \theta(x_\tau, y_\tau, z_\tau)^{\epsilon(\tau)}$  where  $\epsilon(\tau)$  is  $\pm 1$  for positive and negative crossing, respectively.

Then the (3-)cocycle invariant is defined in a similar way to 2-cocycle invariants by  $\Phi_{\theta}(K) = \{\sum_{\tau} B(\mathcal{C}, \tau) \mid \mathcal{C} \in \text{Col}_X(D)\}$ . The multiset version is defined similarly.

EXAMPLE 4.3. Let  $D$  be the closure of  $\sigma_1^3$ , a diagram of a trefoil. Take  $X = R_3$  for a quandle and let  $\theta(x, y, z) = (x-y)(y^2+yz+z^2)z \pmod{3}$  for  $x, y, z \in R_3 = \mathbb{Z}_3$ . (This formula came from  $\theta(x, y, z) = (x-y)[(2z^p - y^p) - (2z-y)^p]/p \pmod{p}$  for  $p = 3$  that will be discussed later.) If the source region of all three crossings is colored by 0, the top left and right arcs are colored by (1, 2), respectively, then the contribution is

$$\theta(0, 1, 2) + \theta(0, 2, 0) + \theta(0, 0, 1) = (-1)(1 + 2 + 1)(2) + 0 + 0 = 1 \quad \text{in } \mathbb{Z}_3,$$

and by computing all colors, we obtain  $\Phi_\theta(K) = 9 + 18u$ .

### 3. Quandle 3-Cocycle Invariants of Knotted Surfaces

For a fixed finite quandle  $X$  and a 3-cocycle  $\theta$ , we define a knotted surface invariant as follows: Let  $K$  be a knotted surface diagram and let  $\mathcal{C} : \mathcal{R} \rightarrow X$  be a coloring of  $K$ , where  $\mathcal{R}$  is the set of sheets of  $K$ . The Boltzman weight at a triple point  $\tau$  is defined by  $B(\tau, \mathcal{C}) = \theta(x_\tau, y_\tau, z_\tau)^{\epsilon(\tau)}$ , where  $\epsilon(\tau)$  is the sign of the triple point  $\tau$ . (A triple point is positive if and only if the normal vectors of the top, middle, and bottom sheets, in this order, agree with the (right-hand) orientation of  $\mathbb{R}^3$  [CrSt98]. In the right of Fig. 4 of Chapter 2, the triple point  $\tau$  is positive.) The colors  $x_\tau, y_\tau, z_\tau$  are the colors of the bottom, middle, and top sheets, respectively, around the source region of  $\tau$ . The *source region* is the region from which normals of top, middle and bottom sheets point. The cocycle invariant is defined by

$$\Phi(K) = \sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C})$$

as before. It was shown in [CJKLS03] that  $\Phi(K)$  is an invariant for knotted surfaces, called the (*quandle*) *cocycle invariant*. The multiset form is similarly defined.

Since it requires some preliminary expositions to explain examples of how to compute this invariant for surfaces, we refer the reader to other publications, such as [AS03\*, SatShi01b\*, SatShi04], for example.

### 4. Polynomial Quandle Cocycles

To actually compute the quandle cocycle invariants from the definition, we need to have an explicit cocycles. In this section, we present quandle cocycles of Alexander quandles written by polynomials, called *polynomial (quandle) cocycles*, that can be used for such explicit calculations of the invariant. Such cocycles were first constructed in [Mochi03], and studied in [Ame06, Mochi05]. They have been extensively used in applications.

The following formulas are found in [Ame06].

- $f(x, y) = (x - y)^{p^n}$  is a 2-cocycle for any Alexander quandle mod  $p$ .
- $f(x, y) = (x - y)^{p^{m_1}} y^{p^{m_2}}$  is a 2-cocycle for an Alexander quandle  $\mathbb{Z}_p[t, t^{-1}]/g(t)$  if  $g(t)$  divides  $(t^{(p^{m_1} + p^{m_2})} - 1)$ .
- $f(x, y, z) = (x - y)^{p^{m_1}} (y - z)^{p^{m_2}}$  is a 3-cocycle for any Alexander quandle mod  $p$ .
- $f(x, y, z) = (x - y)^{p^{m_1}} (y - z)^{p^{m_2}} z^{p^{m_3}}$  is a 3-cocycle for an Alexander quandle  $\mathbb{Z}_p[t, t^{-1}]/g(t)$  if  $g(t)$  divides  $(t^{(p^{m_1} + p^{m_2} + p^{m_3})} - 1)$ .

More general formula for  $n$ -cocycles are given in [Ame06]. We cite (a slightly simplified version of) her result:

PROPOSITION 4.4 ([Ame06]). Consider an Alexander quandle  $X = \mathbb{Z}_p[t, t^{-1}]/h(t) = A$ . Let  $a_i = p^{m_i}$ , for  $i = 1, \dots, n-1$ , where  $p$  is a prime and  $m_i$  are non-negative integers. For a positive integer  $n$ , let  $f : X^n \rightarrow A$  be defined by

$$f(x_1, x_2, \dots, x_n) = (x_1 - x_2)^{a_1} (x_2 - x_3)^{a_2} \cdots (x_{n-1} - x_n)^{a_{n-1}} x_n^{a_n}.$$

Then  $f$  is an  $n$ -cocycle ( $\in Z_{\mathbb{Q}}^n(X; A)$ ), (1) if  $a_n = 0$ , or (2)  $a_n = p^{m_n}$  (for a positive integer  $m_n$ ) and  $g(t)$  divides  $1 - t^a$ , where  $a = a_1 + a_2 + \cdots + a_{n-1} + a_n$ .

More specific polynomials and Alexander quandles considered are as follows. Non-triviality of quandle cocycle invariants for some of the cocycles below are obtained in [Ame06, StSm].

- 2-cocycles:

- $\mathbb{Z}_2$  coefficients:

- \*  $f(x, y) = (x - y)^{(2^m)} y$ .

- Alexander quandle  $\mathbb{Z}_2[t, t^{-1}]/(t^2 + t + 1)$  has a non-trivial 2-cocycle  $f(x, y) = (x - y)^2 y$ . This 4 element quandle is well-known, and for example, isomorphic to the quandle consisting of 120 degree rotations of a regular tetrahedron. It is known to have 2-dimensional cohomology group  $\mathbb{Z}_2$  with  $\mathbb{Z}_2$  coefficient [CJKLS03]. The invariant values for the coefficient  $\mathbb{Z}_2[t, t^{-1}]/(t^2 + t + 1)$  are all of the form  $k[4 + 12u^{(t+1)}]$ , so we conjecture that it is always the case. It is also an interesting problem to characterize this invariant.

- \*  $f(x, y) = (x - y)^{(2^2)} y$ .

- The quandle must be mod  $g(t)$  where  $g(t)$  divides  $t^5 - 1$ , but  $t^5 - 1$  is factored into prime polynomials  $(t+1)(t^4 + t^3 + t^2 + t + 1)$  mod 2, so we set  $g(t) = t^4 + t^3 + t^2 + t + 1$ . The Alexander quandle we use in this case, thus, is  $\mathbb{Z}_2[t, t^{-1}]/(t^4 + t^3 + t^2 + t + 1)$  with 2-cocycle:  $f(x, y) = (x - y)^{(2^2)} y$ , which gives non-trivial invariants.

- \*  $f(x, y) = (x - y)^{(2^3)} y$ .

- We factor  $t^9 - 1$  mod 2 to  $(t+1)(t^2 + t + 1)(t^6 + t^3 + 1)$ , so we try Alexander quandle  $\mathbb{Z}_2[t, t^{-1}]/(t^6 + t^3 + 1)$ , which gives non-trivial invariants.

- $\mathbb{Z}_3$  coefficients:

- \*  $f(x, y) = (x - y)^3 y$ .

- We have  $t^4 - 1 = (t+1)(t+2)(t^2 + 1)$  mod 3, so we try the quandle  $\mathbb{Z}_3[t, t^{-1}]/(t^2 + 1)$  which gives non-trivial invariants.

- 3-cocycles:

- $\mathbb{Z}_2$  coefficients:  $f(x, y, z) = (x - y)(y - z)^2$ . The cocycle is non-trivial for the quandle:  $\mathbb{Z}_2[t, t^{-1}]/(t^2 + t + 1)$ .

- $\mathbb{Z}_3$  coefficients:  $f(x, y, z) = (x - y)(y - z)^3$ .

- The cocycle is non-trivial for the quandles:  $\mathbb{Z}_3[t, t^{-1}]/(t^2 + 1)$ ,  $\mathbb{Z}_3[t, t^{-1}]/(t^2 - t + 1)$ .

- $\mathbb{Z}_5$  coefficients:  $f(x, y, z) = (x - y)(y - z)^5$ . The cocycle is non-trivial for the quandle:  $\mathbb{Z}_5[t, t^{-1}]/(t^2 - t + 1)$ .

- $\mathbb{Z}_7$  coefficients:  $f(x, y, z) = (x - y)(y - z)^7$ . The cocycle is non-trivial for the quandle:  $\mathbb{Z}_7[t, t^{-1}]/(t^2 - t + 1)$ .

There is another type of 3-cocycles Mochizuki constructed specifically for dihedral quandles  $R_p$  for prime  $p$ . It is given by the formula

$$\theta(x, y, z) = (x - y)[(2z^p - y^p) - (2z - y)^p]/p \pmod p,$$

where the numerator computed in  $\mathbb{Z}$  is divisible by  $p$ , and then after dividing it by  $p$ , the value is taken as an integer modulo  $p$ .

### 5. Quandle Cocycle Invariants with Actions on Coefficients

In [CES02], a cocycle invariant using twisted quandle cohomology theory was defined. Let  $X$  be a quandle, and  $A$  be an Alexander quandle with a variable  $t$  (typically  $A = \mathbb{Z}_p[t, t^{-1}]/h(t)$  for a prime  $p$  and a polynomial  $h(t)$ ). A function  $\phi : X \times X \rightarrow A$  is called a *twisted quandle 2-cocycle* if it satisfies the *twisted 2-cocycle condition*

$$t[\phi(y, z) - \phi(x, z) + \phi(x, y)] - [\phi(y, z) - \phi(x * y, z) + \phi(x * z, y * z)] = 0,$$

for all  $x, y, z \in X$ , and  $\phi(x, x) = 0, \forall x \in X$ .

Let  $K$  be an oriented knot diagram with orientation normal vectors. The underlying projection of the diagram divides the 3-space into regions. For a crossing  $\tau$ , let  $R$  be the source region (i.e., a unique region among four regions adjacent to  $\tau$  such that all normal vectors of arcs near  $\tau$  point from  $R$  to other regions). Take an oriented arc  $\ell$  from the region at infinity to a given region  $H$  such that  $\ell$  intersects the arcs (missing crossing points) of the diagram transversely in finitely many points. The *Alexander numbering*  $\mathcal{L}(H)$  of a region  $H$  is the number of such intersections counted with signs. If the orientation of  $\ell$  agrees with the normal vector of the arc where  $\ell$  intersects, then the intersection is positive (counted as 1), otherwise negative (counted as  $-1$ ). The *Alexander numbering*  $\mathcal{L}(\tau)$  of a crossing  $\tau$  is defined to be  $\mathcal{L}(R)$  where  $R$  is the source region of  $\tau$ .

For a coloring  $\mathcal{C}$  of a diagram  $D$  of a classical knot  $K$  by  $X$ , the *twisted (Boltzmann) weight* at  $\tau$  is defined by  $B_T(\tau, \mathcal{C}) = [\phi(x_\tau, y_\tau)^{\epsilon(\tau)}]^{t^{-\mathcal{L}(\tau)}}$ , where  $\phi$  is a twisted 2-cocycle, and  $\epsilon(\tau)$  is the sign of the crossing  $\tau$ . This is in multiplicative notation, so that the action of  $t$  on  $a \in A$  is written by  $a^t$ . The colors  $x_\tau, y_\tau$  assigned near  $\tau$  are chosen in the same manner as in the ordinary 2-cocycle invariant (source colors, or ordered pair of colors at  $\tau$ ) and  $\mathcal{L}(\tau)$  is the Alexander numbering of  $\tau$ . The *twisted quandle 2-cocycle invariant* is the state-sum

$$\Phi_T(K) = \sum_{\mathcal{C}} \prod_{\tau} B_T(\tau, \mathcal{C}).$$

The value of the weight  $B_T(\tau, \mathcal{C})$  is in the coefficient group  $A$  written multiplicatively, and the value of the state-sum is again in the group ring  $\mathbf{Z}[A]$ . It was proved [CES02] by checking Reidemeister moves that  $\Phi_T(K)$  is an invariant of knots.



## Applications of Cocycle Invariants

In this chapter we give an overview of applications of quandle cocycle invariants.

### 1. Extensions of Colorings

Let  $X$  be a quandle, and for a given abelian coefficient group  $A$ , take a quandle 2-cocycle  $\phi$ . Let  $E = A \times X$  and define a binary operation by  $(a_1, x_1) * (a_2, x_2) = (a_1 + \phi(x_1, x_2), x_1 * x_2)$ . It was shown in [CKS03] that  $(E, *)$  defines a quandle, called an *abelian extension*, and is denoted by  $E = E(X, A, \phi)$ . There is a natural quandle homomorphism  $p : E = A \times X \rightarrow X$  defined by the projection to the second factor. (This is in parallel to central extensions of groups, see Chapter IV of [Brw82].) Examples are found in [CENS01].

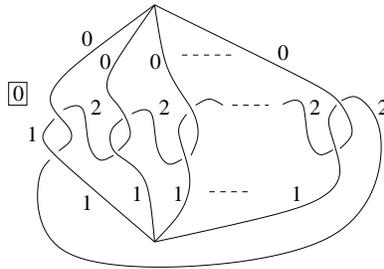
In [CENS01], the interpretation of the cocycle invariant was given as an obstruction to extending a quandle coloring to another coloring by a larger quandle, which is an abelian extension of the original quandle. Specifically, let  $E = E(X, A, \phi)$  be the abelian extension defined above,  $D$  be a diagram of a knot  $K$ , and  $\mathcal{C} \in \text{Col}_X(D)$ . A coloring  $\mathcal{C}' \in \text{Col}_E(D)$  is an *extension* of  $\mathcal{C}$  (or  $\mathcal{C}$  *extends to*  $\mathcal{C}'$ ) if  $p(\mathcal{C}'(\alpha)) = \mathcal{C}(\alpha)$  holds for any arc  $\alpha$  of  $D$ , where  $p : E \rightarrow X$  is the projection defined above. It was proved in [CENS01] that the contribution  $\prod_{\tau} B(\mathcal{C}, \tau)$  is trivial (the identity element of  $A$ ) if and only if there is an extension  $\mathcal{C}'$  of  $\mathcal{C}$ . In particular, every coloring  $\mathcal{C} \in \text{Col}_X(D)$  extends to some  $\mathcal{C}' \in \text{Col}_E(D)$  if and only if the cocycle invariant  $\Phi_{\phi}(K)$  is trivial.

The proof is easily seen by walking along a knot diagram starting from a base point, and trying to extend a given coloring as one goes through each under-arc. If the source colors at a crossing  $\tau$  is  $(x_{\tau}, y_{\tau})$ , and the color  $x_{\tau}$  is extended to  $(x_{\tau}, a_{\tau})$  inductively, then after going under the over-arc colored by  $y_{\tau}$ , the color of the other under-arc is extended to  $(a + \epsilon(\tau)\phi(x_{\tau}, y_{\tau}), x_{\tau} * y_{\tau})$ , whose change in the first factor is exactly the weight at this crossing. Thus after going through the diagram once, the discrepancy in the first factor is the contribution of this coloring.

### 2. Chirality of knots and graphs

A knot or a spatial graph  $K$  is called *achiral* or *amphichaeral* if it is equivalent to its mirror image  $K^*$ . Otherwise  $K$  is called *chiral*. It is well known, for example, that a trefoil  $(3_1)$  is chiral and the figure-eight knot  $(4_1)$  is achiral.

In [RkSn00\*], a new proof of the chirality of trefoil was given using quandle homology theory. Their result can be expressed by the difference in values of the 3-cocycle invariant with the dihedral quandle  $R_3$ . Specifically, in Example 4.3, the value of the invariant for a trefoil  $K$  is  $\Phi_{\theta}(K) = 9 + 18u$ , and one computes  $\Phi_{\theta}(K^*) = 9 + 18u^2$ , so that  $K$  and  $K^*$  are not equivalent.

FIGURE 1. Infinitely many Suzuki's  $\theta_n$ -curves are chiral

In [Sat04\*], for infinitely many spatial graphs called Suzuki's  $\theta_n$ -curves, it was proved that they are chiral. The 3-cocycle invariants of dihedral quandles  $R_p$  with the Mochizuki's cocycle were used to prove this fact. In Fig. 1, a special family of Suzuki's  $\theta_n$ -curves are depicted, with a coloring by  $R_3$ . Suzuki's  $\theta_n$ -curves have the property that any proper subgraph is trivially embedded, but they are non-trivial (the embeddings are not planar although there are planar embeddings), so that any method that uses proper subgraphs can not be applied effectively.

### 3. Colored chirality of knots

For a given knot diagram  $K$  with a (Fox)  $n$ -coloring  $\mathcal{C}$  (a coloring by  $R_n$ ), its mirror  $K^*$  has the  $n$ -coloring  $\mathcal{C}^*$  (called the *mirror coloring*) that is the mirror image of  $\mathcal{C}$ . Specifically, let  $\mathcal{A}$  be the set of (over-)arcs of a diagram  $K$ , then the mirror image  $K^*$  has the set of arcs  $\mathcal{A}^*$  that is in a natural bijection with  $\mathcal{A}$ , such that each arc  $a \in \mathcal{A}$  has its mirror  $a^* \in \mathcal{A}^*$ , and vice-versa (for any arc  $\alpha \in \mathcal{A}^*$ , there is a unique arc  $a \in \mathcal{A}$  such that  $\alpha = a^*$ ). Then  $\mathcal{C}^* : \mathcal{A}^* \rightarrow \mathbb{Z}_n$  is defined by  $\mathcal{C}^*(a^*) = \mathcal{C}(a)$ .

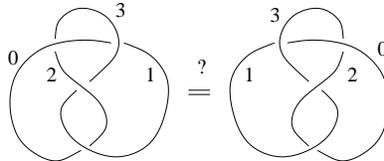


FIGURE 2. Is figure-eight amphicheiral with 5-colorings?

In Fig. 2 a 5-colored figure-eight knot diagram and its mirror with its mirror coloring are depicted. It is, then, natural to ask if an  $n$ -colored knot diagram (a pair  $(K, \mathcal{C})$  of a diagram  $K$  and a coloring  $\mathcal{C}$ ) is equivalent to its mirror with its mirror coloring  $(K^*, \mathcal{C}^*)$ . If this is the case, we call the colored diagram  $(K, \mathcal{C})$  *amphicheiral (or achiral) with (respect to) the  $n$ -coloring  $\mathcal{C}$* . Otherwise we say a diagram is *chiral with (respect to) the  $n$ -coloring*.

The answer to the above question for the colored figure-eight knot is NO, and it is seen from the contributions of the 3-cocycle invariant with the dihedral quandle  $R_5$ , as they are distinct between the left and right of Fig. 2. Many other achiral knots have colorings with respect to which they are chiral. On the other hand, there are many colorings of many knots that are achiral with the colorings.

#### 4. Minimal number of type III Reidemeister moves

The idea of quandle cocycle invariants was used to determine the minimal number of Reidemeister moves needed to move one diagram to another diagram of the same knot for some examples in [CESS05\*].

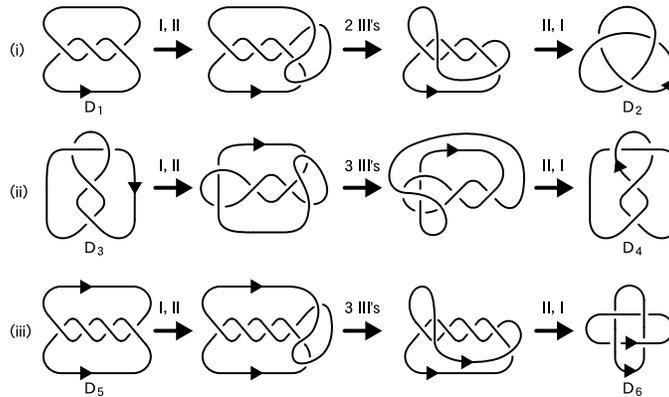


FIGURE 3. Minimal number of type III moves

In Fig. 3 (from [CESS05\*]), a series of Reidemeister moves are sketched for well-known diagrams of trefoil, figure-eight, and the (2, 4)-torus link, from top to bottom. From these figures, it is seen that these different diagrams are related by 2, 3 and 3 type III moves, respectively. It was proved that at least these numbers of type III moves are actually needed.

#### 5. Tangle embeddings

The number of Fox colorings, as well as branched coverings and quantum invariants, were used as obstructions to tangle embeddings (see, for example, [Kre99, KSW00, PSW04\*, Rub00]). Quandle cocycle invariants can be used as obstructions as well [AERSS].

For a tangle  $T$ , the cocycle invariant  $\Phi_\phi(T)$  is defined by the formula similar to the case of knots, where, for the purpose of applications to tangle embeddings, we require that the colors on the boundary points of  $T$  are monochromatic (have the same color). Such a coloring is called *boundary monochromatic*.

Suppose a tangle  $T$  embeds in a knot (or link) diagram  $D$ . For any boundary monochromatic coloring  $\mathcal{C}$  of  $T$ , there is a unique coloring  $\tilde{\mathcal{C}}$  of  $D$  such that the restriction on  $T$  is the given coloring of  $T$  ( $\tilde{\mathcal{C}}|_T = \mathcal{C}$ ) and that all the other arcs (the arcs of  $D$  outside of  $T$ ) receive the same color as that of the boundary color of  $T$ . Then the contribution of  $\tilde{\mathcal{C}}$  to  $\Phi_\phi(K)$  agrees with the contribution of  $\mathcal{C}$  to  $\Phi_\phi(T)$ , since the contribution from the outside of the tangle is trivial, being monochromatic. Hence we obtain the condition  $\Phi_\phi(T) \subset_m \Phi_\phi(K)$ , where the multi-subset  $M \subset_m N$  is defined as follows. If an element  $x$  is repeated  $n$  times in a multiset, call  $n$  the multiplicity of  $x$ , then  $M \subset_m N$  for multisets  $M, N$  means that if  $x \in M$ , then  $x \in N$  and the multiplicity of  $x$  in  $M$  is less than or equal to the multiplicity of  $x$  in  $N$ .

In [AERSS], tables of tangles [KSS03] and knots [Liv] were examined. For those in the tables, the cocycle invariants were computed, and compared to obtain information on which tangles do not embed in which knots in the tables. For one case it was possible to completely determine, using the cocycle invariants, in which knots up to 9 crossings a tangle embeds.

## 6. Non-invertibility of knotted surfaces

A (classical or higher dimensional) knot is called *invertible* (or *reversible*) if it is equivalent to its orientation reversed counterpart, with the orientation of the ambient space fixed. A knot is called *non-invertible* (or *irreversible*) if it is not invertible.

The cocycle invariant provided a diagrammatic method (by a state-sum) of detecting non-invertibility of knotted surfaces. In [CJKLS03], it was proved that the 2-twist spun trefoil  $K$  and its orientation-reversed counterpart  $-K$  have distinct cocycle invariants ( $6 + 12u$  and  $6 + 12u^2$ ) with a cocycle of the dihedral quandle  $R_3$ , and therefore,  $K$  is non-invertible. This result was the first application of the cocycle invariant, and was extended to infinite families of twist spun knots in [AS03\*, CEGS05, Iwa04].

The higher genus surfaces (called stabilized surfaces) obtained from the surface  $K$ , whose non-invertibility is detected by the cocycle invariant, by adding an arbitrary number of trivial 1-handles are also non-invertible, since such handle additions do not alter the cocycle invariant. This property that the conclusion is applied right away to stabilized surfaces is characteristic and an advantage of the cocycle invariant.

## 7. Minimal number of triple points on knotted surface projections

One of the fundamental problems in classical knot theory is to determine the (minimal) crossing number of a given knot. An analogue of the crossing number for knotted surfaces is the minimal number of triple points on projections, called the *triple point number*. The quandle cocycle invariants have been used to obtain lower bounds for the triple point number.

In [SatShi04], the triple point number of the 2-twist spun trefoil was determined to be 4 using cocycle invariants. It was the first time that the triple point number was determined for a specific knot (earlier, only inequalities were known). See [Hata04, SatShi01b\*] for further results on triple point numbers.

## 8. Minimal number of broken sheets in knotted surface diagrams

For a classical knot, the number of crossings and the number of arcs coincide. For knotted surfaces, then, another analogue of the crossing number is the minimal number of broken sheets that are needed to form a diagram of a given knotted surface.

In [SaiSat03\*], a quandle  $X$  defined by a 2-cocycle by extension was used to show that the minimal number of sheets for the untwisted spun trefoil is four. The quandle  $X$  is an extension of  $\mathbb{Z}_2[t]/(t^2 + t + 1)$  by a non-trivial 2-cocycle with coefficient group  $\mathbb{Z}_2$  (hence as a set  $X = \mathbb{Z}_2 \times \mathbb{Z}_2[t]/(t^2 + t + 1)$ ). More generally, it was proved that if a surface diagram is colored non-trivially by  $X$ , then any broken surface diagram of the surface has at least four sheets.

An example, on the other hand, of a diagram of untwisted spun trefoil with exactly four sheets is easily constructed – the diagram obtained from the standard diagram of a trefoil tangle with two end points and three crossings, by spinning it around an axis that goes through the end points of the tangle, has four sheets, since the original tangle diagram has four arcs. It is also easy to color this diagram by  $X$  non-trivially, as the original tangle has such a coloring.

### 9. Ribbon concordance

Let  $F_0$  and  $F_1$  be knotted surfaces of the same genus. We say that  $F_1$  is *ribbon concordant* to  $F_0$  if there is a concordance  $C$  (a properly embedded orientable submanifold diffeomorphic to  $F_0 \times I$ ) in  $\mathbb{R}^4 \times [0, 1]$  between  $F_1 \subset \mathbb{R}^4 \times \{1\}$  and  $F_0 \subset \mathbb{R}^4 \times \{0\}$  such that the restriction to  $C$  of the projection  $\mathbb{R}^4 \times [0, 1] \rightarrow [0, 1]$  is a Morse function with critical points of index 0 and 1 only. We write  $F_1 \geq F_0$ .

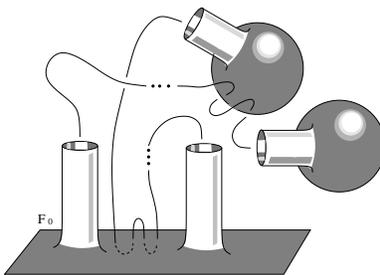


FIGURE 4. Ribbon concordance

Note that if  $F_1 \geq F_0$ , then there is a set of  $n$  1-handles on a split union of  $F_0$  and  $n$  trivial sphere-knots, for some  $n \geq 0$ , such that  $F_1$  is obtained by surgeries (1-handle additions) along these handles (Fig. 4). Ribbon concordance was first defined in [Gor81]. It is defined in general for knots in any dimension. In [CSS03\*], quandle cocycle invariants were used as obstructions to ribbon concordance for surfaces, and explicit examples of surfaces that are not related by ribbon concordance were given.

This obstruction is described as follows. For two multi-sets  $A'$  and  $A''$  of  $A$ , we use the notation  $A' \stackrel{m}{\subset} A''$  if for any  $a \in A'$  it holds that  $a \in A''$ . In other words,  $A' \stackrel{m}{\subset} A''$  if and only if  $\tilde{A}' \subset \tilde{A}''$  where  $\tilde{A}'$  and  $\tilde{A}''$  are the subsets of  $A$  obtained from  $A'$  and  $A''$  by eliminating the multiplicity of elements, respectively.

If  $F_1 \geq F_0$ , then with respect to a natural diagram of  $F_1$  described as above, that is obtained from  $F_0$  by adding small spheres and this tubes, any coloring of such a diagram of  $F_1$  restricts to a coloring of a diagram of  $F_0$ , and since adding tubes do not introduce triple points, we obtain the condition  $\Phi_\theta(F_1) \stackrel{m}{\subset} \Phi_\theta(F_0)$ .



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