Quandle Cocycle Invariants and Tangle Embeddings

Linked from: http://shell.cas.usf.edu/quandle

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1 Introduction

1.1 What are these web pages about

This article is written as web pages posted at http://shell.cas.usf.edu/quandle under the title *Application: Tangle Embedding*.

The purpose of this paper is to present our results on using quandle cocycle invariants to study tangle embeddings. This is a group research project, with group members: Khiera Ameur, Mohamed Elhamdadi, Tom Rose, Masahico Saito, and Chad Smudde.

1.2 What are tangles and tangle embeddings?

A *tangle* is a properly embedded arcs in a (3-)ball B, also represented by a pair $T = (B, A)$ of arcs A in B, but often T is simply regarded as the arcs A if no confusion occurs. In this article a tangle will have four end points unless otherwise specified. A tangle T is *embedded* in a link (or a knot) L if there is a ball B in 3-space such that $T = (B, B \cap L)$. Tangles are represented by diagrams in a manner similar to knot diagrams. Usually the end points are located at four corners of a circle at angles $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$, and these end points are labeled by NE, NW, SW, and SE, respectively.

Tangle embeddings have been studied by several authors recently [CL05*, Kre99, KSW00, PSW04*, Rub00]. In this section we prove our main theorem, that uses quandle cocycle invariants for obstructions to tangle embeddings.

In this article we use the table of tangles presented in [KSS03], in particular those with two arcs in a ball. The tangles are parametrized by a pair of numbers in a symbol similar to those representing knots. Those tangles consisting of two arcs are named 5_1, $6_1 - 6_4$, $7_1 - 7_{18}$ (representing that they listed up to, including, 7 crossing tangles). Among these, some that are of our interest are depicted in Fig. 1.

1.3 Target readers and background information

These pages are intended for undergraduate students, as well as professionals who want to get the current status of the project. For background information on quandle cocycle invariants of knots and their applications, we refer the reader to the article posted at http://shell.cas.usf.edu/quandle under the title *Background*.

2 Preliminary

2.1 Quandles, colorings, and cocycle invariants

Again we refer the reader to the article posted at http://shell.cas.usf.edu/quandle under the title *Background*, for the following terms we use: quandles, Alexander quandles, colorings of knot diagrams, coloring of regions of a knot diagram, quandle 2- and 3-cocycles, 2- and 3-cocycle invariants of knots.

Here we review a definition of cocycle invariants in terms of multisets, that will be used in this article.

Let K be a knot diagram on the plane. Let X be a finite quandle and A an abelian group. Let $\phi : X \times X \to A$ be a quandle 2-cocycle, which can be regarded as a function satisfying the 2-cocycle condition

$$\phi(x, y) - \phi(x, z) + \phi(x * y, z) - \phi(x * z, y * z) = 0, \quad \forall x, y, z \in X$$

and $\phi(x, x) = 0, \forall x \in X$. Let $\mathcal{C}$ be a coloring of a given knot diagram K by X.

The *Boltzmann weight* $B(\mathcal{C}, \tau) = B_\phi(\mathcal{C}, \tau)$ at a crossing τ of K is then defined by $B(\mathcal{C}, \tau) = \epsilon(\tau)\phi(x_\tau, y_\tau)$, where $(x_\tau,\ y_\tau)$ is the ordered pair of colors at τ and $\epsilon(\tau)$ is the sign (± 1) of τ. Then the 2-cocycle invariant $\Phi(K) = \Phi_\phi(K)$ in a multiset form is defined by

$$\Phi_\phi(K) = \left\{ \sum_\tau B(\mathcal{C}, \tau) \;\middle|\; \mathcal{C} \in \mathrm{Col_X(K)} \right\},$$

where $\mathrm{Col_X(K)}$ denotes the set of colorings of K by X. (A multiset a collection of elements where a single element can be repeated multiple times, such as $\{0, 0, 1, 1, 1\}$).

Let $\theta : X \times X \times X \to A$ be a quandle 3-cocycle, which can be regarded as a function satisfying

$$\begin{aligned}&\theta(x,z,w) - \theta(x,y,w) + \theta(x,y,z) - \theta(x*y,z,w)\\&\quad +\theta(x*z,y*z,w) - \theta(x*w,y*w,z*w) = 0, \quad \forall x,y,z,w \in X,\end{aligned}$$

and $\theta(x,x,y) = 0 = \theta(x,y,y), \forall x,y \in X$.

Let $\mathcal{C}$ be a coloring of arcs and regions of a given diagram K. Let $(x,y,z)(= (x_\tau, y_\tau, z_\tau))$ be the ordered triple of colors at a crossing τ. Then the weight in this case is defined by $B(\mathcal{C},\tau) = \epsilon(\tau)\phi(x_\tau, y_\tau, z_\tau)$. The (3-)cocycle invariant is defined in a similar way to 2-cocycle invariants by the multiset $\Phi_\theta(K) = \{\sum_\tau B(\mathcal{C},\tau) \mid \mathcal{C} \in \mathrm{Col_X(K)}\}$, where $\mathrm{Col_X(K)}$ denotes the set of colorings with region colors of K by X.

2.2 Addition and closure of tangles

The following are standard definitions and notations found in many knot theory books.

The addition $T_1 + T_2$ of two tangles T_1, T_2 is another tangles defined from the original two as depicted in Fig. 2. The closures are two methods of obtaining a link from a tangle by closing the end points, and there are two ways called the *numerator* $N(T)$ and *denominator* $D(T)$ of a tangle T, defined as depicted in Fig. 3.

There is a family of “trivial” or “rational” tangles, some of which are depicted in depicted in Fig. 4. These are obtained from the trivial tangle of two vertical straight arcs by successively twisting end points vertically and horizontally. See, again, [Mura96] or [Ad94], for example, for more details.

3 Realizing tangle embeddings

A straightforward way of identifying a tangle embedded in a knot is to construct a knot from a given tangle. Since we are interested in the knots in the table up to 9 crossings and tangles in the table up to 7 crossings, we try the following procedures: For a given tangle T from the table, add a rational tangle R to obtain $T + R$, then take closures $N(T + R)$ and $D(T + R)$, and see which knots in the table result. We list all such embeddings we find this way.

4 Obstructions to tangle embeddings

We use quandle cocycle invariants as obstructions to embedding tangles in knots. To use the cocycle invariants, we first define cocycle invariants for tangles.

Definition 4.1 Let T be a tangle and X a quandle. A (boundary-monochromatic) *coloring* $\mathcal{C} : \mathcal{A} \to X$ is a map from the set of arcs in a diagram of T to X satisfying the same quandle coloring condition as for knot diagrams at each crossing, such that the (four) boundary points of the tangle diagram receives the same element of X.

For a coloring $\mathcal{C}$ of a tangle diagram T, a region colorings are defined in a similar manner as in the knot case. In this case, we allow region colors to change (not necessarily colored by the same element as the one assigned to the boundary points).

Denote by $\mathrm{Col}_X(T)$ the set of colorings of a diagram of T by X. Denote by $\mathrm{Col}_X(T, s)$ the set of colorings of a diagram of T by X with the color of the leftmost region (between the boundary arcs NW and SW) being $s \in X$. It is seen in a way similar to the knot case that the number of colorings $|\mathrm{Col}_X(T)|$ does not depend on a choice of a diagram of T, and that the set of colorings are in one-to-one correspondence between Reidemeister moves.

The quandle 2- and 3-cocycle invariants are defined for tangles in a manner similar to the knot case, and denoted by $\Phi_\phi(T)$.

Definition 4.2 The inclusion of multisets are denoted by $\subset_m$. Specifically, if an element x is repeated n times in a multiset, call n the multiplicity of x, then $M \subset N$ for multisets M, N means that if $x \in M$, then $x \in N$ and the multiplicity of x in M is less than or equal to the multiplicity of x in N.

Theorem 4.3 *Let T be a tangle and X a quandle. Suppose T embeds in a link L. Then we have the inclusion $\Phi_\phi(T) \subset_m \Phi_\phi(L)$.*

Proof. Suppose a diagram of T embeds in a diagram of L. We continue to use T and L for these diagrams. For a coloring $\mathcal{C}$ of T, let x be the color of the boundary points. Then there is a unique coloring $\mathcal{C}'$ of L such that the restriction of $\mathcal{C}'$ on T is $\mathcal{C}$ and all the arcs of L outside of T receive the color x. Then the contribution of $\sum_{\tau \in T} B(\mathcal{C}, \tau)$ to $\Phi_\phi(T)$ is equal to the contribution $\sum_{\tau \in L} B(\mathcal{C}', \tau)$ to $\Phi_\phi(L)$, and the theorem follows. $\square$

In this project we examine the cocycle invariants of tangles in the table and those of knots in the table that do not satisfy the condition of the above theorem, detecting the tangles that do not embed in knots in the tables.

References

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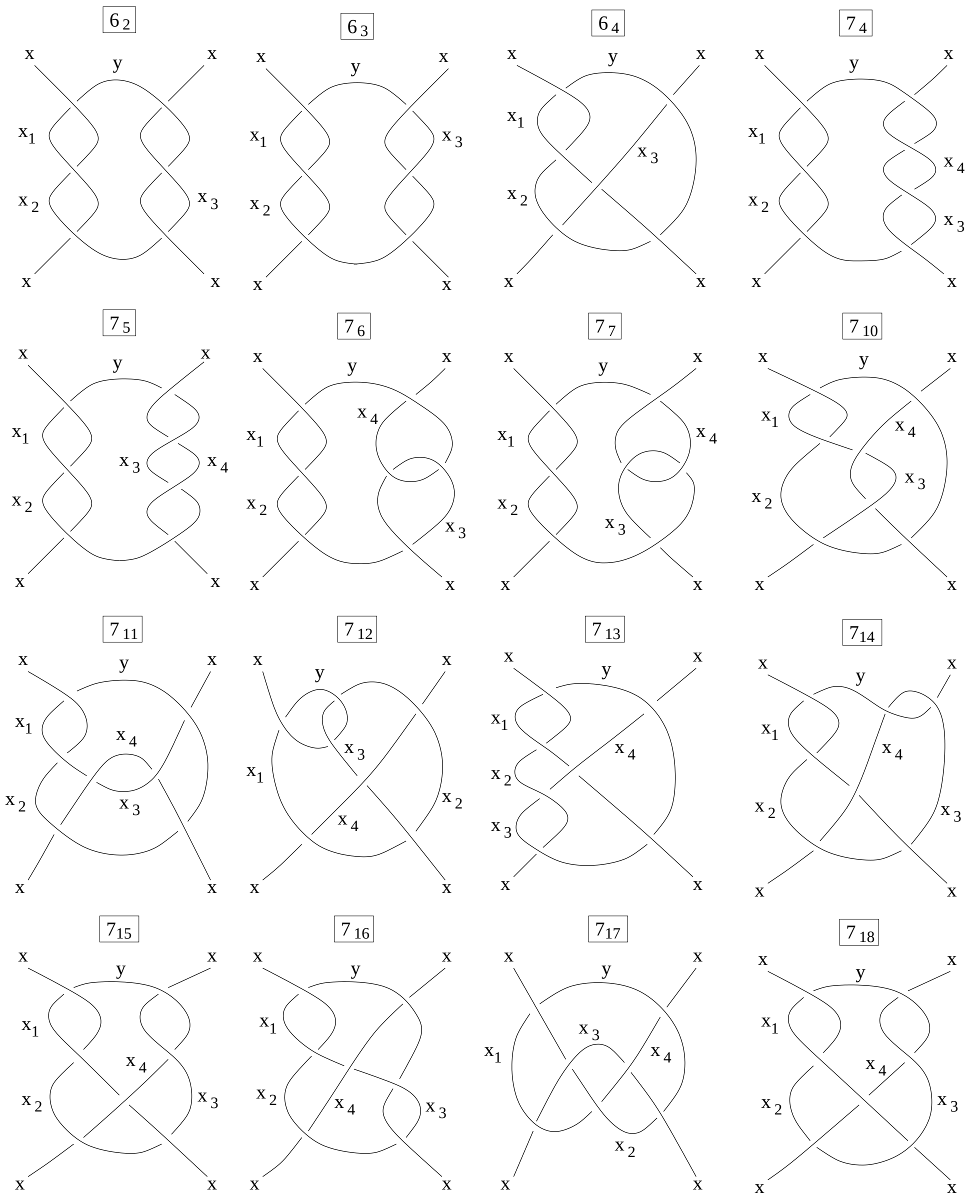

Figure 1: Some tangles

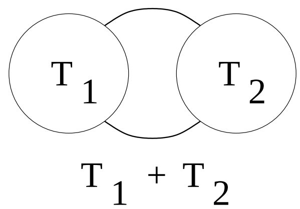

Figure 2: Addition of tangles

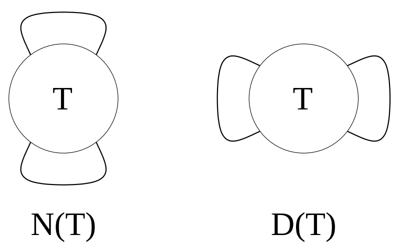

Figure 3: Closures (numerator $N(T)$, denominator $D(T)$) of tangles

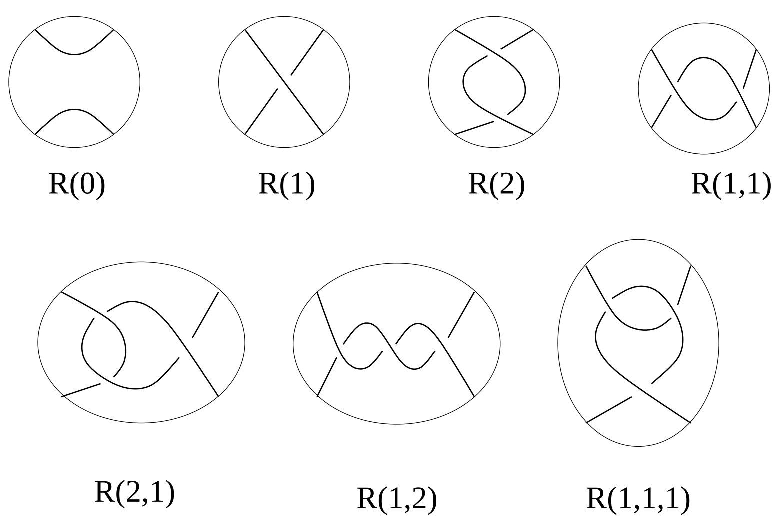

Figure 4: Some rational tangles