Fractional linear multitype branching processes

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A multitype branching process $(Z_n)_{n\geq 0}$ is a Markov chain on $\mathbb{N}^k$ governed by k probability distributions on $\mathbb{N}^k$ called fertility laws with respective generating functions $f_1(s_1,\dots,s_k),\dots,f_k$, such that

$$\mathbb{E}(s_1^{(Z_{n+1})_1}\dots s_k^{(Z_{n+1})_k}|Z_n) = f_1(s)^{(Z_n)_1}\dots f_k(s)^{(Z_n)_k}.$$

For $k=1$ explicit calculations about this process are trivial when the fertility law is given by $f(s)=\frac{as+b}{cs+d}$. The lecture will extend this to $k>1$ for fractional linear transformations of $\mathbb{R}^k$, which can be illustrated for $k=2$ as

$$f_1(s_1,s_2)=\frac{a_{11}s_1+a_{12}s_2+b_1}{c_1s_1+c_2s_2+d},\quad f_2(s_1,s_2)=\frac{a_{21}s_1+a_{22}s_2+b_2}{c_1s_1+c_2s_2+d}.$$

Surprisingly enough, this simple case has not been studied. We denote by ρ the largest eigenvalue of the matrix M of the means of the fertility laws. We shall compute here

1. The extinction probability $\lim_{n\to\infty}\Pr(Z_n=0)$.
2. The distribution of the total progeny $\sum_{n=0}^{\infty}Z_n$ when $\rho\leq 1$.
3. The limit distribution of $\rho^{-n}Z_n$ when $\rho>1$.
4. The limit distribution of $Z_n|Z_n\neq 0$ when $\rho<1$.
5. Stationary measures.