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The Schwarz Reflection Principle Revisited

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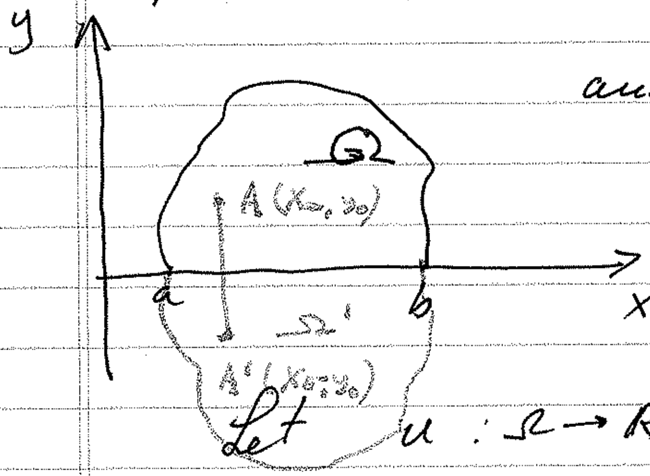
(I) Simplest Form of The SRP in two dimensions

Let Ω be a nice domain in the upper half-plane $\mathbb{R}^2_+ = \{(x, y) : y > 0\}$,

and $[a, b] \subset \partial\Omega$,

$[a, b] \subset \mathbb{R}$ = real line.



Let $u : \Omega \to \mathbb{R}$ be a harmonic function in Ω, ~~vani~~ continuous in $\overline{\Omega}$ and vanishing on $[a, b]$. Then,

u extends as a harmonic function to the domain

Ω' = reflection of Ω about $\mathbb{R} = \{(x,y) : (x,-y) \in \Omega\}$ and the following point-to-point Reflection Law (RL) holds for $A = (x_0, y_0)$, $A' = (x_0, -y_0)$

(RL) $$u(A) + u(A') \equiv 0, \text{ i.e.}$$
$$u(A') = -u(A).$$

There are various proofs of this SPP resting on conformal mappings and analytic functions. A proof that does not use complex analysis essentially reduces to the following:

(i) Define u' in Ω' by (RL), $u'(A') = -u(A)$

(ii) So defined u is harmonic in Ω', continuous in Ω' and vanishes on $[a,b]$, i.e. coincides with u on $[a,b]$

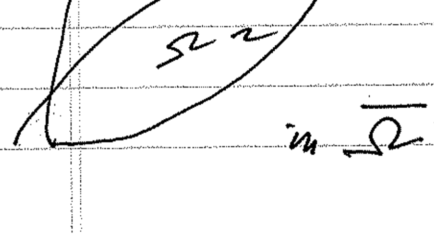
Moreover,

(iii) $\frac{\partial u'}{\partial y} = -\frac{\partial u}{\partial y}$ on $[a,b]$, i.e.

$\frac{\partial u'}{\partial n_{\Omega'}} = \frac{\partial u}{\partial n_{\Omega}}$ on $[a,b]$, where $\vec{n}$ represents outward normal towards $\partial\Omega$ or $\partial\Omega'$ respectively.

Now there is a general fact going back to Cauchy and independentally to S. Kovalevskaya, that claims that u' is a harmonic continuation of u into Ω'.

Fact. Let Ω be a smoothly bounded domain (in R^n, in fact) and Γ is a smooth curve (hypersurface) dividing Ω into two parts Ω_1, Ω_2



Let u be a function harmonic in $\Omega \setminus \Gamma$, continuous in $\overline{\Omega}$ with its first derivatives

Then, by Green's formula, for any $\vec{x}^{\,0} \in \Omega_2(\text{or } \Omega_1)$ we have

$$u(x_0) = \frac{1}{2\pi} \int_{\partial\Omega_2 \setminus \Gamma} \left[u \frac{\partial}{\partial n_y} \log\frac{1}{|x^0-y|} - \log\frac{1}{|x^0-y|}\frac{\partial u}{\partial n_y} \right] dS(y)$$

$$+ \frac{1}{2\pi} \int_{\Gamma} [\quad] \, dS(y) =$$

$$= \frac{1}{2\pi} \int_{\partial\Omega_2 \setminus \Gamma} [\quad] dS_y + \frac{1}{2\pi} \int_{\partial\Omega_1 \setminus \Gamma} [\quad] \, dS(y)$$

$$= \frac{1}{2\pi} \int_{\partial\Omega} [\quad] \, dS_y$$

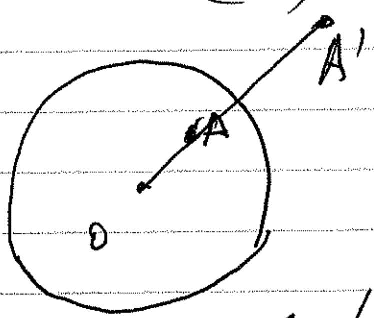
and we are done.

The second equality follows from (iii) and the fact that orientations on Γ viewed as part of $\partial\Omega_1$ or $\partial\Omega_2$ are opposite and since $x^0 \in \Omega_2^{+}$, u and $\log\frac{1}{|x^0-y|}$ are (both) harmonic in Ω_1, so

$$\int [\qquad] \, dS_y = 0.$$

(Note: For $\Omega_{n \neq 2}$ $\log \frac{1}{|x-y|}$ must be replaced by $|x-y|^{2-n}$)

(II) It is well-known that the SRP in (I) extends in two dimensions to more general situations, e.g., when R is replaced by a circle $\gamma : \{|z| = R\}$ then, (RL) holds with A' = inversive image of A wrt γ $= R^2 \frac{x}{|x|^2}$ $(A = \vec{x})$, using complex notation and assuming WLOG $R=1$

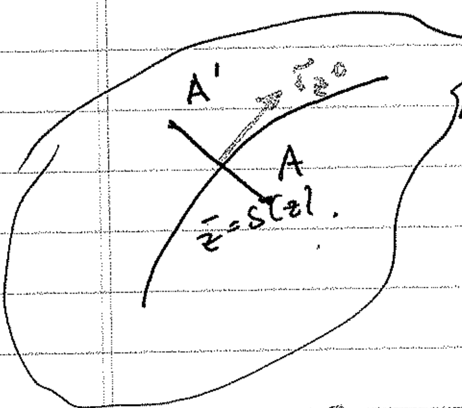


$$A' = \frac{1}{\bar{z}} : \qquad \left(R \neq 1, \quad A' = \frac{R^2}{\bar{z}}\right)$$

Note, that the equation of the real line in complex notation is given by $\bar{z} = S(z) := z$, and that of the circle $\{|z| = R\}$ — by $\bar{z} = S(z) := \frac{R^2}{z}$, $S(z)$ being an analytic function near the curve in question. More generally, if an (analytic!) curve Γ is given by the equation in complex form

$$\Gamma := \{\bar z = S(z)\},\ S \text{ analytic}$$

in a "tubular" neighborhood of Γ.



Then the (RL) holds with $A' = \overline{S(A)}$.

Also, infinitesimally, denoting $R(z) = \overline{S(z)}$, assuming $z - z^0 = i\tau|z - z^0|$, $z^0 \in \Gamma$, z lies on the normal to Γ at z^0, we obtain

$$S'(z^0) = \frac{d\bar z}{ds}\Big|_{z^0} \cdot \frac{ds}{dz}\Big|_{z^0} = \overline{\left(\frac{dz}{ds}\right)^2}\Big|_{z^0} = \bar\tau^2,$$

τ = unit tangent vector at z^0 to Γ,

$$R(z) - z^0 = -i\,\overline{\tau}_{z^0}\,|z - z^0|\,\tau_{z^0}^2 + o(|z - z^0|) =$$

$$= (\bar\tau\tau = 1\ !) = -i\,\tau_{z^0}\,|z - z^0| + o(|z - z^0|)$$

in other words R infinitisemally maps points on the normal to Γ into points symmetric about the tangent line to Γ. Usually, this general form of the reflection principle is obtained from the one for lines via conformal mapping. Since ~~we~~ our goal is to discuss SPP in higher dimensions where conformal maps are essentially non-existent let us arrive at this general RP following E. Study (1907) via ... wave equation. (The paper of Study was apparently well forgotten and "rediscovered" independentally in 1950s by H. Lewy and P. Garabedian.

Consider the ~~the~~ one (space) dimensional wave equation

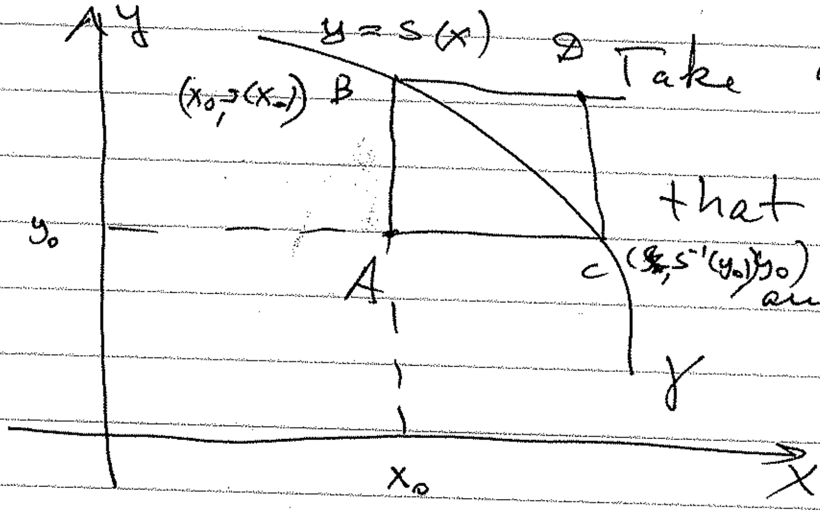
$\frac{\partial^2 u}{\partial w^2} - \frac{\partial^2 u}{\partial t^2} = 0$, which can be rewritten in coordinates $x = w - t$, $y = w + t$ as

$$\frac{\partial^2 u}{\partial x \partial y} = 0. \qquad (1)$$

(1) has an easily obtained general solution

$$u = f(x) + g(y) \qquad (2)$$

Now, suppose $\gamma = \{y = s(x)\}$ is a smooth curve in $\mathbb{R}^2$, that does not have horizontal tangents, so $s(x)$ is monotone



Take any solution u of (1) that vanishes on γ and any point A off γ. Construct the point D as shown "symmetric" to $A(x_0, y_0)$ wrt γ

$$D = (s^{-1}(y_0), s(x_0)).$$

(RL) $\quad u(A) + u(D) = 0, \quad \frac{\partial^2 u}{\partial x \partial y} = 0, \quad u\big|_\gamma = 0.$

In other words, the "signal" at A that "does not penetrate" through the wall γ can be received at D "behind" the wall.

<u>Proof</u> $\quad u(A) = f(A) + g(A) = f(B) + g(C) =$

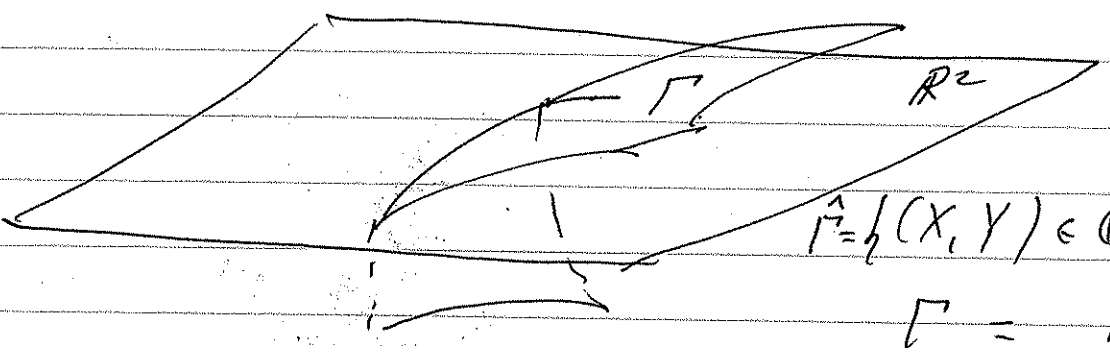
$= -g(B) - f(C) = -g(D) - f(D) =$

$= -u(D)$ #

Now how does it all relate to our situation?

(III) Change of variables in $\mathbb{C}^2$.

Analytic curve $\Gamma := \{(x,y) : \varphi(x,y) = 0\}$,
φ real valued in $\mathbb{R}^2$
where φ is real-analytic function and $\nabla\varphi \neq (0,0)$ on Γ, i.e. φ is an analytic function in $\mathbb{C}^2 = \{(X,Y) : X, Y \in \mathbb{C}\} \supset \mathbb{R}^2 = \{(X,Y) : X, Y \in \mathbb{R}\}$



$\hat{\Gamma} = \{(X,Y) \in \mathbb{C}^2 : \varphi = 0\}$.

$\Gamma = \hat{\Gamma} \cap \mathbb{R}^2$

Change Variables in $\mathbb{C}^2$ $z = X + iY$, $w = X - iY$

Since $\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) \neq (0,0)$ on Γ,

$X = \frac{z+w}{2}$, $Y = \frac{z-w}{2i}$

$$\frac{\partial \varphi}{\partial w} = \frac{1}{2}\frac{\partial \varphi}{\partial x} + \frac{1}{2}\frac{\partial \varphi}{\partial y}\left(-\frac{1}{i}\right) =$$

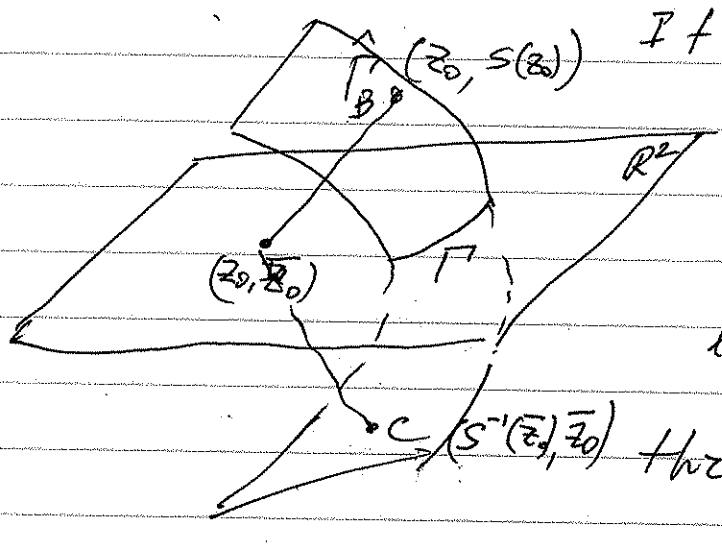
$$= \frac{1}{2}\left(\frac{\partial \varphi}{\partial x} + i\frac{\partial \varphi}{\partial y}\right) \neq 0 \text{ near } \Gamma \subset \mathbb{R}^2.$$

Hence, we can resolve the equation wrt w and write the equation of $\hat{\Gamma}$ as

$\hat{\Gamma} := \{(z, w) : w = S(z)\}$ where

S is an analytic function. On Γ,

$w = \bar{z}$, $\left|\frac{d\bar{z}}{dz}\right| = |S'| = 1 \neq 0$, hence



If we start with point A $(z_0, \bar{z}_0)$ near Γ in $\mathbb{R}^2$ draw a pair of complex lines $z = z_0$, $w = \bar{z}_0$ through A, they will intersect Γ at two points $B(z_0, S(z_0))$ and $C(S^{-1}(\bar{z}_0), \bar{z}_0)$. Now, pass through B, C two complex lines of "opposite kind" $w = S(z_0)$ through B and $z = S^{-1}(\bar{z}^0)$ through C. These ~~two~~ new lines will intersect at point $D(\underbrace{S^{-1}(\bar{z}_0)}_{z'}, \underbrace{S(z_0)}_{w'})$

Note that $D \in \mathbb{R}^2$: Indeed,

$S(\overline{S(z)}) = z$, $S(\overline{S(\bar{z})}) = \bar{z}$, So

$S^{-1}(\bar{z}) = \overline{S(z)}$ near Γ, hence

$D(z', w')$: $w' = \overline{z'}$ is in $\mathbb{R}^2$.

Moreover $R(D) = z_0$ since

$\overline{S(S^{-1}(\bar{z}_0))} = \bar{\bar{z}}_0 = z_0$.

Finally, it remains to notice that in z, w coordinates the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ becomes $4 \frac{\partial^2}{\partial z \partial w}$ and Study's proof