Applied Mathematics and Computation 256 (2015) 252–256

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Rational solutions to a KdV-like equation

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ARTICLE INFO

Keywords:
Rational solution
Generalized bilinear equation
KdV equation

ABSTRACT

Two classes of rational solutions to a KdV-like nonlinear differential equation are constructed. The basic object is a generalized bilinear differential equation based on a prime number $p = 3$. A conjecture is made that the two presented classes of rational solutions contain all rations solutions to the considered KdV-like equation, which are generated from polynomial solutions to the corresponding generalized bilinear equation.

1. Introduction

In recent years, there has been a growing interest in rational solutions to nonlinear differential equations. One kind of particular rational solutions are rogue wave solutions, which describe significant nonlinear wave phenomena in oceanography [1,2].

Rational solutions to integrable equations have been systematically considered in the literature by using the Wronskian formulation or the Casoratian formulation (see, e.g., [3–5]). Those integrable equations include the KdV, Boussinesq, and Toda equations (see [6–8] for integrable theories). Rational solutions to the non-integrable $(3+1)$-dimensional KP I [10,11] and KP II [9] are also considered by different approaches. In particular, rational solutions to the $(3+1)$-dimensional KP II have been transformed into a problem of finding rational solutions to the good Boussinesq equation [9].

In this paper, we would like to consider a KdV-like nonlinear differential equation induced from a generalized bilinear differential equation of KdV type. From a class of polynomial generating functions, a Maple search tells us two classes of rational solutions to the considered KdV-like equation, along with some special interesting solutions. A conjecture on rational solutions to the considered KdV-like equation is made at the end of the paper.

2. A KdV-like differential equation

Let us consider a generalized bilinear differential equation of KdV type:

$$(D_{3,x}D_{3,t} + D_{3,x}^4)f \cdot f = 2f_{xt}f - 2f_tf_x + 6(f_{xx})^2 = 0. \tag{2.1}$$

This is the same type bilinear equation as the KdV one [12]. The above differential operators are some kind of generalized bilinear differential operators introduced in [13]:

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http://dx.doi.org/10.1016/j.amc.2015.01.027
0096-3003/

$$\begin{aligned} D_{p,x}^m D_{p,t}^n f\cdot f &= \left(\frac{\partial}{\partial x}+\alpha_p\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}+\alpha_p\frac{\partial}{\partial t'}\right)^n f(x,t)f(x',t')|_{x'=x,t'=t} \\ &= \sum_{i=0}^m\sum_{j=0}^n\binom{m}{i}\binom{n}{j}\alpha_p^i\alpha_p^j\frac{\partial^{m-i}}{\partial x^{m-i}}\frac{\partial^i}{\partial x'^{(i)}}\frac{\partial^{n-j}}{\partial t^{n-j}}\frac{\partial^j}{\partial t'^{(j)}}f(x,t)f(x',t')|_{x'=x,t'=t}, \quad m,n\geqslant 0, \end{aligned} \tag{2.2}$$

where

$$\alpha_p^s = (-1)^{r_p(s)}, \quad s = r_p(s) \mod p. \tag{2.3}$$

In particular, we have

$$\alpha_3 = -1, \quad \alpha_3^2 = 1, \quad \alpha_3^3 = 1, \quad \alpha_3^4 = -1, \quad \alpha_3^5 = 1, \quad \alpha_3^6 = 1$$

and thus

$$D_{3,x}D_{3,t}f\cdot f = 2f_{xt}f - 2f_tf_x, \quad D_{3,x}^4 f\cdot f = 6(f_{xx})^2.$$

In the case of $p = 2$, i.e., the Hirota case, we have

$$D_{2,x}D_{2,t}f\cdot f = 2f_{xt}f - 2f_tf_x, \quad D_{2,x}^4 f\cdot f = 2f_{xxxx}f - 8f_{xxx}f_x + 6(f_{xx})^2,$$

which generates the standard bilinear KdV equation [12].

Motivated by a general Bell polynomial theory [14–16], we adopt a dependent variable transformation

$$u = 2(\ln f)_x \tag{2.4}$$

and then can directly show that the generalized bilinear equation (2.1) is linked to a KdV-like scalar nonlinear differential equation

$$u_t + \frac{3}{2}(u_x)^2 + \frac{3}{2}u^2u_x + \frac{3}{8}u^4 = 0. \tag{2.5}$$

More precisely, by virtue of the transformation (2.4), the following equality holds:

$$\frac{(D_{3,x}D_{3,t} + D_{3,x}^4)f\cdot f}{f^2} = u_t + \frac{3}{2}(u_x)^2 + \frac{3}{2}u^2u_x + \frac{3}{8}u^4 \tag{2.6}$$

and thus, f solves (2.1) if and only if $u = 2(\ln f)_x$ presents a solution to the KdV-like Eq. (2.5).

Resonant solutions in term of exponential functions and trigonometric functions have been considered for generalized bilinear equations [15–17]. In this paper, we would like to discuss their polynomial solutions which generate rational solutions to scalar nonlinear differential equations by focusing on the KdV-like Eq. (2.5).

3. Rational solutions

By symbolic computation with Maple, we look for polynomial solutions, with degrees of x and t being less than 10:

$$f = \sum_{i=0}^9\sum_{j=0}^9 c_{ij}x^it^j, \tag{3.1}$$

where the c_{ij}'s are constants, and find 29 classes of polynomial solutions to the generalized bilinear equation (2.1). The six of those classes of solutions are

$$f = \frac{c_{01}x^3}{36} + c_{20}x^2 + c_{01}t + \frac{12c_{20}^2x}{c_{01}} + c_{00}, \tag{3.2}$$

$$\begin{aligned} f &= c_{32}t^2x^3 + c_{22}t^2x^2 + c_{31}tx^3 + 36c_{32}t^3 + \frac{{c_{22}}^2xt^2}{3c_{32}} + \frac{c_{22}c_{31}x^2t}{c_{32}} + c_{30}x^3 + c_{02}t^2 + \frac{c_{22}^2c_{31}tx}{3c_{32}^2} + \frac{c_{22}c_{30}x^2}{c_{32}} + \frac{c_{02}c_{31}t}{c_{32}} \\ &\quad + 36c_{30}t - \frac{36{c_{31}}^2t}{c_{32}} + \frac{c_{22}^2c_{30}x}{3c_{32}^2} + \frac{c_{02}c_{30}}{c_{32}} - \frac{36c_{30}c_{31}}{c_{32}}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} f &= c_{33}t^3x^3 + c_{23}t^3x^2 + c_{32}t^2x^3 + 36c_{33}t^4 + \frac{c_{23}^2xt^3}{3c_{33}} + \frac{c_{23}c_{32}x^2t^2}{c_{33}} + c_{31}tx^3 + c_{03}t^3 + \frac{c_{23}^2c_{32}t^2x}{3c_{33}^2} + \frac{c_{23}c_{31}x^2t}{c_{33}} + c_{30}x^3 \\ &\quad + \frac{c_{03}c_{32}t^2}{c_{33}} + 36c_{31}t^2 - \frac{36c_{32}^2t^2}{c_{33}} + \frac{c_{23}^2c_{31}tx}{3c_{33}^2} + \frac{c_{23}c_{30}x^2}{c_{33}} + \frac{c_{03}c_{31}t}{c_{33}} + 36c_{30}t - \frac{36c_{31}c_{32}t}{c_{33}} + \frac{c_{23}^2c_{30}x}{3c_{33}^2} + \frac{c_{03}c_{30}}{c_{33}} - \frac{36c_{30}c_{32}}{c_{33}} \end{aligned} \tag{3.4}$$

and

$$f = c_{40}x^4 + \frac{1}{36}c_{01}x^3 + 144c_{40}tx + c_{20}x^2 + c_{01}t + c_{10}x + \frac{c_{01}c_{10} - 12c_{20}^2}{144c_{40}}, \tag{3.5}$$

$$\begin{aligned} f = {} & c_{41}tx^4 + c_{31}tx^3 + c_{40}x^4 + 144c_{41}t^2x + c_{21}tx^2 + \frac{c_{31}c_{40}x^3}{c_{41}} + 36c_{31}t^2 + c_{11}tx + \frac{c_{21}c_{40}x^2}{c_{41}} + \frac{c_{11}c_{31}t}{4c_{41}} - \frac{c_{21}^2 t}{12c_{41}} \\ & + \frac{c_{11}c_{40}x}{c_{41}} - \frac{144c_{40}^2x}{c_{41}} + \frac{c_{11}c_{31}c_{40}}{4c_{41}^2} - \frac{{c_{21}}^2c_{40}}{12c_{41}^2} - \frac{36c_{31}c_{40}^2}{c_{41}^2}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} f = {} & c_{42}t^2x^4 + c_{32}t^2x^3 + c_{41}tx^4 + 144c_{42}t^3x + c_{22}t^2x^2 + \frac{c_{32}c_{41}x^3t}{c_{42}} + c_{40}x^4 + 36c_{32}t^3 + c_{12}t^2x + \frac{c_{22}c_{41}x^2t}{c_{42}} + \frac{c_{32}c_{40}x^3}{c_{42}} \\ & + \frac{c_{12}c_{32}t^2}{4c_{42}} - \frac{c_{22}^2t^2}{12c_{42}} + \frac{c_{12}c_{41}tx}{c_{42}} + 144c_{40}tx - \frac{144c_{41}^2tx}{c_{42}} + \frac{c_{22}c_{40}x^2}{c_{42}} + \frac{c_{12}c_{32}c_{41}t}{4c_{42}^2} - \frac{c_{22}^2c_{41}t}{12c_{42}^2} + \frac{36c_{32}c_{40}t}{c_{42}} \\ & - \frac{36c_{32}c_{41}^2t}{c_{42}^2} + \frac{c_{12}c_{40}x}{c_{42}} - \frac{144c_{40}c_{41}x}{c_{42}} + \frac{c_{12}c_{32}c_{40}}{4c_{42}^2} - \frac{c_{22}^2c_{40}}{12c_{42}^2} - \frac{36c_{32}c_{40}c_{41}}{c_{42}^2}, \end{aligned} \tag{3.7}$$

where the involved constants c_{ij}'s are arbitrary provided that the solutions make sense. Taking the concrete forms of the resulting polynomial solutions to (2.1) into consideration, we can verify by Maple that among the solutions generated from (3.1), there are two distinct classes of rational solutions to the KdV-like Eq. (2.5):

$$u = \frac{6(a_1^2x^2 + 2a_1a_2x + a_2^2)}{a_1^2x^3 + 3a_1a_2x^2 + 36a_1^2t + 3a_2^2x + a_3} \tag{3.8}$$

and

$$u = \frac{12a_1(2a_1x^3 + a_2x^2 + 72a_1t + 2a_3x + a_4)}{3a_1^2x^4 + 2a_1a_2x^3 + 432a_1^2tx + 6a_1a_3x^2 + 72a_1a_2t + 6a_1a_4x + a_2a_4 - a_3^2}, \tag{3.9}$$

where $a_i,\ 1 \leqslant i \leqslant 4$, are arbitrary constants. Actually, the polynomial solutions in the first group of (3.2)–(3.4) and the second group of (3.5)–(3.7) generate the rational solutions in (3.8) and (3.9), respectively. Note that in (3.8) and (3.9), the constants were rescaled and renamed.

The first class of solutions in (3.8), on one hand, reduces to

$$u = \frac{6a_2^2}{3a_2^2x + a_3} = \frac{2}{x + c}, \quad c = \text{const.}, \tag{3.10}$$

when $a_1 = 0$, and further

$$u = \frac{2}{x}, \tag{3.11}$$

when $a_1 = a_3 = 0$ or $c = 0$. On the other hand, the first class of solutions in (3.8) reduces to

$$u = \frac{6a_1^2x^2}{a_1^2x^3 + 36a_1^2t + a_3} = \frac{6x^2}{x^3 + 36t + c}, \quad c = \text{const.}, \tag{3.12}$$

when $a_2 = 0$, and further

$$u = \frac{6x^2}{x^3 + 36t}, \tag{3.13}$$

when $a_2 = a_3 = 0$ or $c = 0$. Two pictures of the solution (3.12) with different values of c are given in Fig. 1.

From the second class of solutions in (3.9), we obtain

$$u = \frac{4(2a_1x^3 + 72a_1t + a_4)}{x(a_1x^3 + 144a_1t + 2a_4)} = \frac{8(x^3 + 36t + c)}{x(x^3 + 144t + 4c)}, \quad c = \text{const.}, \tag{3.14}$$

when $a_2 = a_3 = 0$, and further

$$u = \frac{8(x^3 + 36t)}{x(x^3 + 144t)}, \tag{3.15}$$

when $a_2 = a_3 = a_4 = 0$ or $c = 0$. Two pictures of the solution (3.14) with different values of c are given in Fig. 2.

An easy class of solutions to the generalized bilinear equation (2.1) is given by

$$f = (a_1x + a_2)t^m, \tag{3.16}$$

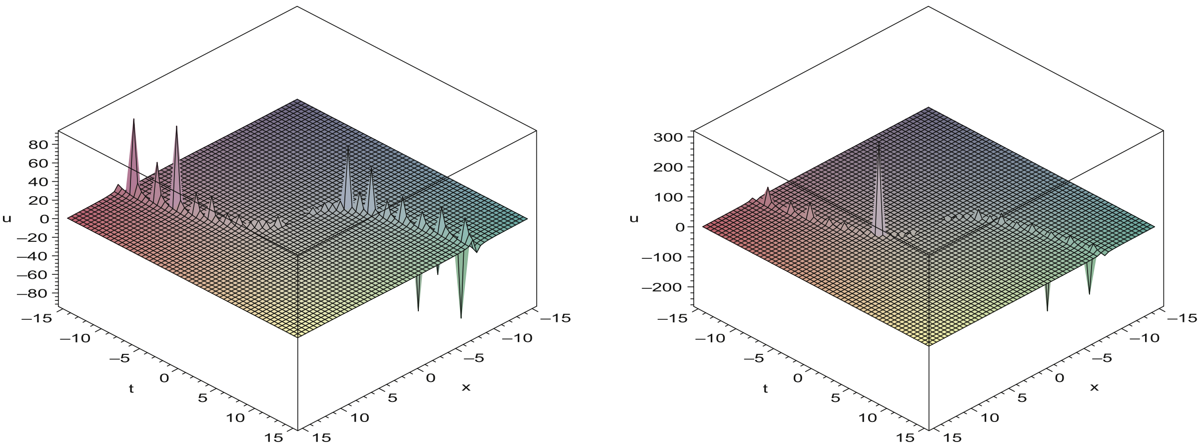

Fig. 1. Pictures of the solution (3.12) with $c = 0$ (left) and $c = 2$ (right).

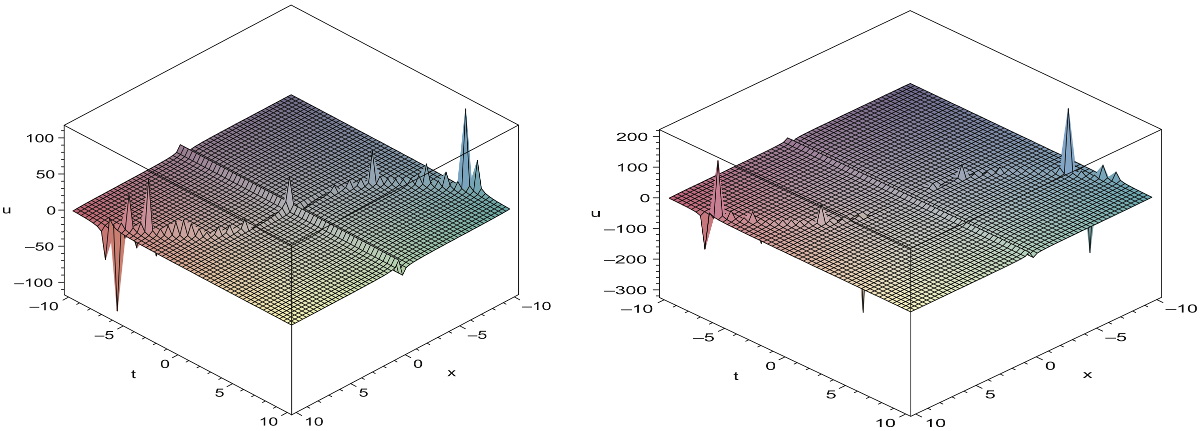

Fig. 2. Pictures of the solution (3.14) with $c = 0$ (left) and $c = 2$ (right).

where $a_1,\ a_2$ are two arbitrary constants and m is an arbitrary nonnegative integer. However, this leads to the solution presented in (3.10).

4. Concluding remarks

We considered a generalized bilinear equation which yields a KdV-like nonlinear differential equation, and constructed two classes of rational solutions to the resulting KdV-like equation. The basic starting point is a kind of generalized bilinear differential operators introduced in [13].

We remark that it is worth checking if there exists a kind of Wronskian solutions and multiple soliton type solutions to the KdV-like nonlinear equation (2.5). We also conjecture that the two classes of rational solutions in (3.8) and (3.9) would contain all rational solutions to the KdV-like nonlinear equation (2.5), generated from polynomial solutions to the generalized bilinear equation (2.1) under the link (2.4).

There is, additionally, a kind of generalized tri-linear differential equations [18]. Their rational solutions, both singular and non-singular, or even rogue wave solutions will be a very interesting topic. Particularly, higher-order rogue wave solutions should be connected with generalized Wronskian solutions [19,20] and generalized Darboux transformations [21,22].

Acknowledgments

The work was supported in part by NNSFC under the Grants 11371326, 11271008 and 61072147, NSF under the Grant DMS-1301675, Natural Science Foundation of Shanghai (Grant No. 11ZR1414100), Zhejiang Innovation Project of China (Grant No. T200905), and the First-class Discipline of Universities in Shanghai and the Shanghai Univ. Leading Academic Discipline Project (No. A.13-0101-12-004).

References

[1] P. Müller, C. Garrett, A. Osborne, Rogue waves, Oceanography 18 (2005) 66–75.
[2] C. Kharif, E. Pelinovsky, A. Slunyaev, Rogue Waves in the Ocean, Advances in Geophysical and Environmental Mechanics and Mathematics, Springer-Verlag, Berlin, 2009.
[3] W.X. Ma, Y. You, Solving the Korteweg–de Vries equation by its bilinear form: Wronskian solutions, Trans. Am. Math. Soc. 357 (2005) 1753–1778.
[4] W.X. Ma, C.X. Li, J.S. He, A second Wronskian formulation of the Boussinesq equation, Nonlinear Anal. TMA 70 (2009) 4245–4258.
[5] W.X. Ma, Y. You, Rational solutions of the Toda lattice equation in Casoratian form, Chaos Solitons Fract. 22 (2004) 395–406.
[6] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.
[7] S. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons – The Inverse Scattering Method, Consultants Bureau/A Division of Plenum Publishing Corporation, New York, 1984.
[8] D.Y. Chen, Introduction to Solitons, Science Press, Beijing, 2006.
[9] W.X. Ma, Comment on the 3 + 1 dimensional Kadomtsev–Petviashvili equations, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 2663–2666.
[10] M. Khalfallah, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 1169–1175.
[11] D.I. Sinelshchikov, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 3235–3236.
[12] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.
[13] W.X. Ma, Generalized bilinear differential equations, Stud. Nonlinear Sci. 2 (2011) 140–144.
[14] C. Gilson, F. Lambert, J. Nimmo, R. Willox, Proc. R. Soc. Lond. A 452 (1996) 223–234.
[15] W.X. Ma, Bilinear equations, Bell polynomials and linear superposition principle, J. Phys.: Conf. Ser. 411 (2013) 012021.
[16] W.X. Ma, Bilinear equations and resonant solutions characterized by Bell polynomials, Rep. Math. Phys. 72 (2013) 41–56.
[17] H.C. Zheng, W.X. Ma, X. Gu, Hirota bilinear equations with linear subspaces of hyperbolic and trigonometric function solutions, Appl. Math. Comput. 220 (2013) 226–234.
[18] W.X. Ma, Trilinear equations, Bell polynomials, and resonant solutions, Front. Math. China 8 (2013) 1139–1156.
[19] W.X. Ma, Wronskians, Generalized Wronskians and solutions to the Korteweg–de Vries equation, Chaos Solitons Fract. 19 (2004) 163–170.
[20] Y. Zhang, L.G. Jin, Y.L. Kang, Generalized Wronskian solutions for the (3 + 1)-dimensional Jimbo–Miwa equation, Appl. Math. Comput. 219 (2012) 2601–2610.
[21] B.L. Guo, L.M. Ling, Q.P. Liu, Nonlinear Schrödinger equation: generalized Darboux transformation and rogue wave solutions, Phys. Rev. E 85 (2012) 026607.
[22] N. Vishnu Priya, M. Senthilvelan, Generalized Darboux transformation and Nth order rogue wave solution of a general coupled nonlinear Schrödinger equations, Commun. Nonlinear Sci. Numer. Simul. 20 (2015) 401–420.