Derivation of Squared Eigenfunctions in Integrable Equations

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1. Introduction

What are squared eigenfunctions in integrable equations?

They provide a mapping from variations of the scattering data to variations of the potential.

Example: for the NLS equation, let

$$\rho = b/a, \quad \bar{\rho} = \bar{b}/\bar{a}, \quad \text{(AKNS notations)}$$

Then if

$$\begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} = \int_{-\infty}^{\infty} \left[Z^-(x,\zeta)\delta\rho(\zeta) + Z^+(x,\zeta)\delta\bar{\rho}(\zeta) \right] d\zeta,$$

then $Z^+(x,\zeta), Z^-(x,\zeta)$ are squared eigenfunctions.

1. Introduction

Adjoint squared eigenfunctions provide the opposite mapping: from variations of the potential to variations of the scattering data

$$\delta\rho(\zeta) = \frac{1}{\bar{a}^2(\zeta)} \left\langle \Omega^-(x,\zeta), \begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} \right\rangle,$$

$$\delta\bar{\rho}(\zeta) = \frac{1}{a^2(\zeta)} \left\langle \Omega^+(x,\zeta) \begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} \right\rangle,$$

Here $\Omega^+(x,\zeta)$ and $\Omega^-(x,\zeta)$ are adjoint squared eigenfunctions.

1. Introduction

Why are they called squared eigenfunctions?

Because for some simple equations such as the AKNS and KdV hierarchies, these functions turn out to be “squares" of the Jost functions.

In the more general case, these functions are often quadratic products of Jost functions and adjoint Jost functions.

Importance of squared eigenfunctions:

They are intimately related to many aspects of the integrable theory (such as the recursion operators, self-consistent sources, etc)

More importantly, they are essential for any treatment of perturbed integrable systems (such as the soliton perturbation theory).

1. Introduction

How to derive squared eigenfunctions?

The original method of AKNS (1974) and Kaup (1976):

1. find variations of the scattering data when the potential varies. The coefficients of these are then the adjoint squared eigenfunctions.
2. show that these adjoint squared eigenfunctions are eigenfunctions of an integro-differential operator. (effort)
3. find the adjoint of that operator, which will be the eigenvalue operator for the squared eigenfunctions.
4. by guess and trial, find this adjoint operator's eigenfunctions, which are then the squared eigenfunctions. (effort)
5. calculate the inner products between the squared eigenfunctions and the adjoint squared eigenfunctions explicitly from the asymptotics of the Jost functions.
6. construct and prove the closure relation. (effort)

1. Introduction

This procedure works for simple cases, but can run into difficulties when the situation is less trivial.

For instance, our implementation of this procedure for the Sasa-Satsuma equation encountered serious problems:

Why? Steps 2 and 4 were hard to implement. Because the adjoint squared eigenfunctions turn out to be quite unusual, this makes it difficult to guess the integro-differential operator and eigenfunctions of its adjoint operator.

2. Our method of derivation

In this talk, we propose another method to derive squared eigenfunctions and the adjoint squared eigenfunctions.

Our method:

1. find variations of the scattering data when the potential varies. The coefficients of these are then the adjoint squared eigenfunctions.
2. find variations of the potential when the scattering data varies. The coefficients of these are then the squared eigenfunctions. effort
3. from the above two steps, we can obtain the inner products and closure relation automatically

2. Our method of derivation

Unlike the previous method, we obtain the squared eigenfunctions directly by carrying out Step 2.

Advantages of this new method:

- it is a streamlined method
- no need for guess and trial
- can handle more complicated cases
- closure relation and inner products automatically follow (without any extra effort)

2. Our method of derivation

How do we carry out step 2? by Riemann-Hilbert method.

The RH method provides a way to reconstruct the potential from the scattering data. Thus we can use it to calculate variations of the potential when the scattering data changes.

3. Riemann-Hilbert formulation of Sasa-Satsuma equation

Below we use Sasa-Satsuma equation to demonstrate this method.

The Sasa-Satsuma equation is

$$u_t + u_{xxx} + 6|u|^2 u_x + 3u(|u|^2)_x = 0.$$

Its spectral equation of the Lax pair:

$$Y_x = -i\zeta \Lambda Y + QY,$$

where $\Lambda = \mathrm{diag}(1, 1, -1)$,

$$Q = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & u^* \\ -u^* & -u & 0 \end{pmatrix}$$

3. RH formulation of Sasa-Satsuma equation

Introducing new variables

$$J = YE^{-1}, \qquad E = e^{-i\zeta\Lambda x},$$

then

$$J_x = -i\zeta[\Lambda, J] + QJ.$$

Jost solutions $J_{\pm}(x,\zeta)$ are defined by

$$J_{\pm} \to I, \quad \text{as} \quad x \to \pm\infty.$$

$$\Phi = J_- E, \quad \Psi = J_+ E.$$

Scattering matrix S defined by:

$$\Phi = \Psi S, \qquad \det S = 1$$

3. RH formulation of Sasa-Satsuma equation

Analytical properties of Jost functions:

Let

$$\Phi = [\phi_1, \phi_2, \phi_3]$$

$$\Psi = [\psi_1, \psi_2, \psi_3]$$

then

$$P^+ \equiv [\phi_1, \phi_2, \psi_3] e^{i\zeta\Lambda x}$$

is analytic in UHP

3. RH formulation of Sasa-Satsuma equation

To obtain the counterpart of P^+ in LHP, consider the adjoint spectral equation

$$K_x = -i\zeta[\Lambda, K] - KQ,$$

whose solutions are

$$J_-^{-1} = E\Phi^{-1}, \quad J_+^{-1} = E\Psi^{-1}.$$

Let

$$\Phi^{-1} = \begin{bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\phi}_3 \end{bmatrix}, \qquad \Psi^{-1} = \begin{bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{bmatrix},$$

then

$$P^- = e^{-i\zeta\Lambda x} \begin{bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\psi}_3 \end{bmatrix}$$

is analytic in LHP

3. RH formulation of Sasa-Satsuma equation

P^+ and P^- are linearly related on the real axis:

$$P^-(\zeta)P^+(\zeta) = G(\zeta), \qquad \zeta \in \mathbb{R},$$

where

$$G = E \begin{pmatrix} 1 & 0 & \bar{s}_{13} \\ 0 & 1 & \bar{s}_{23} \\ s_{31} & s_{32} & 1 \end{pmatrix} E^{-1}.$$

Here

$$S = (s_{ij}), \quad S^{-1} = (\bar{s}_{ij}).$$

3. RH formulation of Sasa-Satsuma equation

If this RH problem can be solved, then potential Q can be reconstructed by

$$Q(x) = i[\Lambda, P_1(x)],$$

where

$$P^+(x, \zeta) = I + \zeta^{-1} P_1(x) + \mathcal{O}(\zeta^{-2}), \quad \zeta \to \infty.$$

4. Calculation of variations of scattering data

Now we perform Step 1 of our procedure, which is to calculate variations of scattering data when the potential varies.

Starting point:

$$\Phi_x = -i\zeta\Lambda\Phi + Q\Phi.$$

Taking the variation to this equation:

$$\delta\Phi_x = -i\zeta\Lambda\delta\Phi + Q\delta\Phi + \delta Q\Phi.$$

Its solution can be found by the method of variation of parameters as

$$\delta\Phi(\zeta;x) = \Phi(\zeta;x)\int_{-\infty}^{x}\Phi^{-1}(\zeta;y)\delta Q(y)\Phi(\zeta;y)dy.$$

4. Calculation of variations of scattering data

Now take the limit of $x \to +\infty$ to this equation. Since

$$\Phi = \Psi S, \qquad \lim_{x \to +\infty} \Psi = E$$

so

$$\lim_{x \to +\infty} \Phi = ES, \qquad \lim_{x \to +\infty} \delta\Phi = E\delta S.$$

Thus from the last equation of the previous slide, we get

$$\delta S(\zeta) = \int_{-\infty}^{\infty} \Psi^{-1}(\zeta; x)\, \delta Q(x)\, \Phi(\zeta; x) dx, \qquad \zeta \in \mathbb{R}.$$

4. Calculation of variations of scattering data

Defining scattering coefficient

$$\rho_1 \equiv \frac{s_{31}}{s_{33}}, \quad \rho_2 \equiv \frac{s_{32}}{s_{33}}, \quad \bar{\rho}_1 \equiv \frac{\bar{s}_{13}}{\bar{s}_{33}}, \quad \bar{\rho}_2 \equiv \frac{\bar{s}_{23}}{\bar{s}_{33}},$$

then we get

$$\delta\rho_1(\zeta) = \frac{1}{s_{33}^2(\zeta)} \left\langle \Omega^-(\zeta;x), \begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} \right\rangle.$$

Similarly, we get

$$\delta\bar{\rho}_1(\zeta) = \frac{1}{\bar{s}_{33}^2(\zeta)} \left\langle \Omega^+(\zeta;x), \begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} \right\rangle,$$

where

$$\Omega^- = \begin{bmatrix} \bar{\psi}_{31}\varphi_3 - \bar{\psi}_{33}\varphi_2 \\ \bar{\psi}_{32}\varphi_3 - \bar{\psi}_{33}\varphi_1 \end{bmatrix}, \quad \Omega^+ = \begin{bmatrix} \psi_{32}\bar{\varphi}_3 - \psi_{33}\bar{\varphi}_1 \\ \psi_{31}\bar{\varphi}_3 - \psi_{33}\bar{\varphi}_2 \end{bmatrix},$$

and

$$\varphi = \bar{s}_{33}\psi_1 - \bar{s}_{21}\psi_2, \quad \bar{\varphi} = s_{33}\bar{\psi}_1 - s_{12}\bar{\psi}_2.$$

These Ω^- and Ω^+ are adjoint squared eigenfunctions.

4. Calculation of variations of scattering data

Features of these adjoint squared eigenfunctions:

$$\Omega^{-}=\left[\begin{array}{c}\bar{\psi}_{31}\varphi_3-\bar{\psi}_{33}\varphi_2\\ \bar{\psi}_{32}\varphi_3-\bar{\psi}_{33}\varphi_1\end{array}\right],\quad \Omega^{+}=\left[\begin{array}{c}\psi_{32}\bar{\varphi}_3-\psi_{33}\bar{\varphi}_1\\ \psi_{31}\bar{\varphi}_3-\psi_{33}\bar{\varphi}_2\end{array}\right]$$

They are definitely not “squares" of Jost functions.

Rather, they are sums of quadratic products of Jost functions and adjoint Jost functions.

This unusual form of adjoint squared eigenfunctions causes significant difficulties when the previous method for squared eigenfunctions was used.

Example: the recursion operator very complicated, guessing its eigenfunctions very difficult, etc

4. Calculation of variations of scattering data

What caused this unusual form of adjoint squared eigenfunctions?

The symmetries of the potential matrix Q:

$$Q = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & u^* \\ -u^* & -u & 0 \end{pmatrix}$$

This potential has two u and two u^*. Their variations thus can be grouped together, which caused this "sum" feature of adjoint squared eigenfunctions.

4. Calculation of variations of scattering data

Question: why not calculate variations $\delta\rho_2(\zeta)$ and $\delta\bar{\rho}_2(\zeta)$?

Because they are related to $\delta\rho_1(\zeta)$ and $\delta\bar{\rho}_1(\zeta)$ due to symmetry properties of the potential matrix Q.

This makes sense, because the potential matrix Q has only two variables (u, u^*) (treating u and u^* as independent), thus there can only be two independent scattering coefficients $\rho_1(\zeta)$ and $\bar{\rho}_1(\zeta)$.

5. Calculation of variations of the potential

Now, we perform the opposite calculation, i.e. calculate variations of the potential when the scattering data changes.

i.e. we calculate

$$\begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} = \int_{-\infty}^{\infty} [??? \; \delta\rho_1(\zeta) + ??? \; \delta\bar{\rho}_1(\zeta)] \, d\zeta.$$

The coefficients on the RHS will be squared eigenfunctions.

This calculation will be done by the Riemann-Hilbert method.

5. Calculation of variations of the potential

Starting point: first assume s_{33} and $\bar{s}_{33}$ have no zeros.
Define

$$F^+ = P^+ \operatorname{diag}(1, 1, \frac{1}{\bar{s}_{33}}), \qquad F^- = (P^-)^{-1} \operatorname{diag}(1, 1, s_{33}).$$

Then the RH problem for $F^\pm$ is

$$F^+(\zeta) = F^-(\zeta)\tilde{G}(\zeta), \quad \zeta \in \mathbb{R},$$

where

$$\tilde{G} = E \begin{pmatrix} 1 & 0 & \bar{\rho}_1 \\ 0 & 1 & \bar{\rho}_2 \\ \rho_1 & \rho_2 & 1 + \rho_1\bar{\rho}_1 + \rho_2\bar{\rho}_2 \end{pmatrix} E^{-1}.$$

The reason for introducing $F^\pm$: to make the RH problem to contain only scattering coefficients $(\rho_1, \rho_2, \bar{\rho}_1, \bar{\rho}_2)$.

5. Calculation of variations of the potential

Next take the variation to this $F^{\pm}$ RH problem and get:

$$\delta F^{+} = \delta F^{-}\tilde{G} + F^{-}\delta\tilde{G}, \qquad \zeta \in \mathbb{R}.$$

Rewrite it as

$$\delta F^{+}(F^{+})^{-1} = \delta F^{-}(F^{-})^{-1} + F^{-}\delta\tilde{G}(F^{+})^{-1}, \qquad \zeta \in \mathbb{R},$$

which defines another Riemann-Hilbert problem for $\delta F F^{-1}$.

This RH problem for $\delta F \cdot F^{-1}$ can be solved, and the solution is

$$\delta F \cdot F^{-1}(\zeta; x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Pi(\zeta; x)}{\zeta - \zeta} d\zeta,$$

where

$$\begin{aligned} \Pi(\zeta; x) &\equiv F^{-}(\zeta; x)\,\delta\tilde{G}(\zeta; x)\,(F^{+})^{-1}(\zeta; x) \\ &= \Phi \begin{pmatrix} 0 & 0 & \delta\bar{\rho}_1 \\ 0 & 0 & \delta\bar{\rho}_2 \\ \delta\rho_1 & \delta\rho_2 & 0 \end{pmatrix} \Phi^{-1}. \end{aligned}$$

5. Calculation of variations of the potential

Now we take the $\zeta \to \infty$ limit to the above equation. From the previous RH formulation, we see that

$$F^+(\zeta; x) \to I + \frac{1}{2i\zeta} \left[\begin{array}{ccc} * & * & u(x) \\ * & * & u^*(x) \\ u^*(x) & u(x) & * \end{array} \right], \qquad \zeta \to \infty,$$

hence

$$\delta F^+(\zeta; x) \to \frac{1}{2i\zeta} \left[\begin{array}{ccc} * & * & \delta u(x) \\ * & * & \delta u^*(x) \\ \delta u^*(x) & \delta u(x) & * \end{array} \right], \qquad \zeta \to \infty.$$

In addition, it is easy to see

$$\int_{-\infty}^{\infty} \frac{\Pi(\zeta; x)}{\zeta - \zeta} d\zeta \longrightarrow -\frac{1}{\zeta} \int_{-\infty}^{\infty} \Pi(\zeta; x) d\zeta, \qquad \zeta \to \infty.$$

5. Calculation of variations of the potential

Inserting these asymptotics into the previous $\delta F \cdot F^{-1}$ solution, we get

$$\delta u(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \Pi_{13}(\zeta; x) d\zeta,$$

$$\delta u^*(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \Pi_{31}(\zeta; x) d\zeta,$$

i.e.

$$\begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \left[Z^-(\zeta; x)\delta\rho_1(\zeta) + Z^+(\zeta; x)\delta\bar{\rho}_1(\zeta)\right] d\zeta,$$

where

$$Z^- = \begin{bmatrix} \phi_{31}\bar{\phi}_{13} + \phi_{33}\bar{\phi}_{12} \\ \phi_{32}\bar{\phi}_{13} + \phi_{33}\bar{\phi}_{11} \end{bmatrix}, \quad Z^+ = \begin{bmatrix} \phi_{11}\bar{\phi}_{33} + \phi_{13}\bar{\phi}_{32} \\ \phi_{12}\bar{\phi}_{33} + \phi_{13}\bar{\phi}_{31} \end{bmatrix}.$$

These Z^- and Z^+ are then the squared eigenfunctions.

5. Calculation of variations of the potential

$$Z^- = \left[\begin{array}{c} \phi_{31}\bar{\phi}_{13} + \phi_{33}\bar{\phi}_{12} \\ \phi_{32}\bar{\phi}_{13} + \phi_{33}\bar{\phi}_{11} \end{array} \right], \quad Z^+ = \left[\begin{array}{c} \phi_{11}\bar{\phi}_{33} + \phi_{13}\bar{\phi}_{32} \\ \phi_{12}\bar{\phi}_{33} + \phi_{13}\bar{\phi}_{31} \end{array} \right].$$

Features of these squared eigenfunctions:

They are sums of quadratic products of Jost functions and adjoint Jost functions as well, like the ASE $\Omega^\pm$.

What caused this “sum" feature in squared eigenfunctions $Z^\pm$?

Reason:

Originally, calculations give

$$\left[\begin{array}{c} \delta u(x) \\ \delta u^*(x) \end{array} \right] = \int_{-\infty}^{\infty} \left[**\delta\rho_1(\zeta) + **\delta\rho_2(\zeta) + **\delta\bar{\rho}_1(\zeta) + **\delta\bar{\rho}_2(\zeta) \right] d\zeta.$$

5. Calculation of variations of the potential

$$\begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} = \int_{-\infty}^{\infty} \left[{**}\delta\rho_1(\zeta) + {**}\delta\rho_2(\zeta) + {**}\delta\bar{\rho}_1(\zeta) + {**}\delta\bar{\rho}_2(\zeta) \right] d\zeta$$

However, due to symmetries of the potential Q,

$\rho_2(\zeta)$ and $\bar{\rho}_2(\zeta)$ are related to $\rho_1(\zeta)$ and $\bar{\rho}_1(\zeta)$,

thus $\delta\rho_2(\zeta)$ and $\delta\bar{\rho}_2(\zeta)$ terms above are combined together with the $\delta\rho_1(\zeta)$ and $\delta\bar{\rho}_1(\zeta)$ terms

which cause squared eigenfunctions $Z^{\pm}$ to become sums.

Thus, **symmetries of the potential Q determine the forms of squared eigenfunctions.**

6. Inner products and closure relations

One significant advantage of the present method is that we can now obtain the inner products and closure relations of squared eigenfunctions without doing any calculations.

Our starting points: the two variation relations

$$\begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \left[Z^-(\zeta;x)\delta\rho_1(\zeta) + Z^+(\zeta;x)\delta\bar{\rho}_1(\zeta) \right] d\zeta,$$

and

$$\delta\rho_1(\zeta) = \frac{1}{s_{33}^2(\zeta)} \left\langle \Omega^-(\zeta;x), \begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} \right\rangle .$$

$$\delta\bar{\rho}_1(\zeta) = \frac{1}{\bar{s}_{33}^2(\zeta)} \left\langle \Omega^+(\zeta;x), \begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} \right\rangle .$$

6. Inner products and closure relations

To obtain the inner products of squared eigenfunctions, we insert the former into latter and get

$$\begin{bmatrix} \delta\rho(\zeta) \\ \delta\bar{\rho}(\zeta) \end{bmatrix} = \int_{-\infty}^{\infty} B(\zeta,\zeta') \begin{bmatrix} \delta\rho(\zeta') \\ \delta\bar{\rho}(\zeta') \end{bmatrix} d\zeta',$$

where

$$B(\zeta,\zeta') = \begin{pmatrix} -\frac{1}{\pi s_{33}^2(\zeta)} \left\langle \Omega^-(\zeta;x), Z^-(\zeta';x) \right\rangle, & -\frac{1}{\pi s_{33}^2(\zeta)} \left\langle \Omega^-(\zeta;x), Z^+(\zeta';x) \right\rangle \\ -\frac{1}{\pi \bar{s}_{33}^2(\zeta)} \left\langle \Omega^+(\zeta;x), Z^-(\zeta';x) \right\rangle, & -\frac{1}{\pi \bar{s}_{33}^2(\zeta)} \left\langle \Omega^+(\zeta;x), Z^+(\zeta';x) \right\rangle \end{pmatrix}$$

Since $\delta\rho(\zeta)$ and $\delta\bar{\rho}(\zeta)$ are arbitrary, we have

$$B(\zeta,\zeta') = \delta(\zeta-\zeta')\, I.$$

6. Inner products and closure relations

Thus we get the inner products

$$\left\langle \Omega^-(\zeta;x), Z^-(\zeta';x)\right\rangle = -\pi s_{33}^2(\zeta)\,\delta(\zeta-\zeta'),$$

$$\left\langle \Omega^+(\zeta;x), Z^+(\zeta';x)\right\rangle = -\pi \bar{s}_{33}^2(\zeta)\,\delta(\zeta-\zeta'),$$

$$\left\langle \Omega^-(\zeta;x), Z^+(\zeta';x)\right\rangle = \left\langle \Omega^+(\zeta;x), Z^-(\zeta';x)\right\rangle = 0,$$

for any $\zeta, \zeta' \in \mathbb{R}$.

6. Inner products and closure relations

To get the closure relation, we do the opposite, i.e. we insert the latter into the former and get

$$\begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} = \int_{-\infty}^{\infty} A(x,x') \begin{bmatrix} \delta u(x') \\ \delta u^*(x') \end{bmatrix} dx',$$

where

$$A(x,x') = -\frac{1}{\pi}\int_{-\infty}^{\infty} \left[\frac{1}{s_{33}^2(\zeta)} Z^-(\zeta;x)\Omega^{-T}(\zeta;x') \right. \\ \left. + \frac{1}{\bar{s}_{33}^2(\zeta)} Z^+(\zeta;x)\Omega^{+T}(\zeta;x') \right] d\zeta.$$

Since $(\delta u, \delta u^*)$ are arbitrary, $A(x,x')$ must be the delta function $\delta(x-x')I$. Thus we get the closure relation

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} \left[\frac{1}{s_{33}^2(\zeta)} Z^-(\zeta;x)\Omega^{-T}(\zeta;x') \right. \\ \left. + \frac{1}{\bar{s}_{33}^2(\zeta)} Z^+(\zeta;x)\Omega^{+T}(\zeta;x') \right] d\zeta = \delta(x-x')I.$$

6. Inner products and closure relations

Extension of closure relation with discrete contributions

all one needs to do is to add the residues of continuous terms:

$$
\begin{aligned}
&-\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{s_{33}^2(\xi)}Z^-(\xi;x)\Omega^{-T}(\xi;x') + \frac{1}{\bar{s}_{33}^2(\xi)}Z^+(\xi;x)\Omega^{+T}(\xi;x')\right]d\xi \\
&-\sum_{j=1}^{N}\frac{2i}{s_{33}'^2(\bar{\zeta}_j)}\left[Z^-(\bar{\zeta}_j;x)\Theta^{-T}(\bar{\zeta}_j;x') + \dot{Z}^-(\bar{\zeta}_j;x)\Omega^{-T}(\bar{\zeta}_j;x')\right] \\
&+\sum_{j=1}^{N}\frac{2i}{\bar{s}_{33}'^2(\zeta_j)}\left[Z^+(\zeta_j;x)\Theta^{+T}(\zeta_j;x') + \dot{Z}^+(\zeta_j;x)\Omega^{+T}(\zeta_j;x')\right] = \delta(x-x')I,
\end{aligned}
$$

where

$$
\Theta^-(\bar{\zeta}_j;x) = \dot{\Omega}^-(\bar{\zeta}_j;x) - \frac{s_{33}''(\bar{\zeta}_j)}{s_{33}'(\bar{\zeta}_j)}\Omega^-(\bar{\zeta}_j;x),
$$

$$
\Theta^+(\zeta_j;x) = \dot{\Omega}^+(\zeta_j;x) - \frac{\bar{s}_{33}''(\zeta_j)}{\bar{s}_{33}'(\zeta_j)}\Omega^+(\zeta_j;x).
$$

7. Squared eigenfunctions and the linearization operator

Squared eigenfunctions are intimately related to the linearization operator of the integrable equation.

Using a simple and general argument (Yang and Kaup 2008), we can show that

- squared eigenfunctions satisfy the linearized equation for any integrable equation,
- adjoint squared eigenfunctions satisfy the adjoint linearized equation for any integrable equation

These properties are the key for direct soliton perturbation theories (where one considers how a soliton evolves when an integrable equation is perturbed).

8. Summary

- we presented a new method for calculating squared eigenfunctions of an integrable equation
- the key part of this method is based on the Riemann-Hilbert problem
- this method can treat cases where squared eigenfunctions are less-trivial
- in this method, inner products and closure relation come out automatically
- we can show that squared eigenfunctions satisfy the linearized equation of any integrable equation