Does the Supersymmetric Integrability Imply the Integrability of Bosonic Sector ?

Z I E M O W I T P O P O W I C Z

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15.07.2009 - 21.07.2009 Shihezi China

Plan :

1.) Integrability.

2.) Classification of 2 component KdV like system.

a.) Generalized symmetries
b.) Recursion opertor
c.) Lax operator

3.) New system of interacted two KdV equations and its Lax operator.

4.) Manin-Radul N=1 Supersymmetric Kadomtsev – Pietvisahvilli hierarchy.

a.) $MRSKP_{3,0}$ as supersymmetric Sawada – Kotera equation

b.) $MRSKP_{5,0}$ the bosonic sector gives us this new system

Korteweg de Vries equation

$$u_t = u_{xxx} + 6uu_x .$$

Many different generalization to 2-component system.

I.) Svinolupov 1991

$$u_t^i = u_{xxx}^i + a_{j,k}^i u^j u_x^k$$

where the constants a satisfy

$$a_{j,k}^n (a_{n,k}^j a_{m,s}^r - a_{m,r}^i a_{n,s}^r) + cyclic(j,k,m) = 0$$

and hence they are the structural constants of Jordan algebra.
It has the infinite number of generalized symmetries

II.) Gűrses and Karasu 1998

$$u_t^i = b_j^i u_{xxx}^i + s_{j,k}^i u^j u_x^k$$

where b, s are constants

Investigated conditions on b,s in order to find the recursion operator of second and fourth order

Decomposable hereditary operator Wen – Xiu Ma, Fordy, Antonowicz

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & a_0\partial \\ a_0\partial & a_1\partial \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} = \begin{pmatrix} 0 & M_0 \\ M_0 & M_1 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} \quad \text{where} \quad M_i = c_1\partial^3 + d_i(\partial u_i + u_i\partial)$$

Definiton of Integrability of a system of equations:

A.) If possesse the recursion operator.

B.) Has infinite number of conserved quantities and generalized symmetries.

C.) Has the Lax representation.

Foursov in 2003 investigated 2-component kdV system

$$u_t = F[u, v]$$
$$v_t = G[u, v]$$

where F and G are polynomial functions of u,v, u_x, v_x

Definition : A system of t-independednt evolution equation

$$u_t = Q_1[u, v]$$
$$v_t = Q_2[u, v]$$

Is said to be a generalized symmetry if their flows commute

$$D_K(Q) - D_Q(K) = 0$$

where $Q = (Q_1, Q_2)$ $K[u, v] = (F[u, v], G[u, v])$ and D_K is a Frechet derivative

Foursov using CA found 5 systems with finite number of generalized symmetries and conserved quantities , 3 of them are known to be integrable

A.) Hirota – Satsuma

$$u_t = u_{xxx} + 6uu_x - 12vv_x$$
$$v_t = -2v_{xxx} - 6uu_x$$

B.) Ito system

$$u_t = u_{xxx} + 3uu_x + 3vv_x$$
$$v_t = u_x v + uv_x$$

C.) Drinfeld – Sokolov

$$u_t = u_{xxx} + 2vu_x + uv_x$$
$$v_t = uu_x$$

Two of them are new

D.)

$$u_t = u_{xxx} + v_{xxx} + 2vu_x + 2uv_x$$
$$v_t = v_{xxx} - 9uu_x + 6vu_x + 3uv_x + 2vv_x$$

This system has been also found by Meshkov.

E.

$$u_t = 4u_{xxx} + 3v_{xxx} + 4uu_x + vu_x + 2uv_x$$
$$v_t = 3u_{xxx} + v_{xxx} - 4uu_x - 2uv_x - 2vv_x$$

Foursov found conserved densities of weight 2,4,8,10,12,14 and generalized symmetries of weight 9,11,13,15 and 19.

We show that the last system is integrable because it has the Lax representation and has the Bi – Hamiltonian structure

There is a possibility to construct 2 component system of interacted KdV equtions with the nonlocal generalized symmetries

$$m_t = m_{xxx} - 3n_{xxx} - 3m_x(4m - 9n) + 3n_x(8m - 15\mathrm{n})$$
$$n_t = -3m_{xxx} + 4n_{xxx} + 12m_x n + 6n_x(m - 4n)$$

Meshkov A.G Teor.Math.Phys. 156 (2008) 351 - 363.

The Lax operator classification of 3 system :

i.) Hirota –i Satsuma

Recursion operator of 4 order

$$L=(\partial^2+u+v)(\partial^2+u-v) = L_{KdV}L_{KdV} \quad \Rightarrow \quad L_t=[L_+^{\frac{3}{4}}, L]$$

i i.) Drinfeld – Sokolov

Recursion operator of 6 order

$$L=(\partial^3+u\partial+\frac{u_x}{2})(\partial^3-v\partial-\frac{v_x}{2}) = L_{KK}L_{KK} \quad \Rightarrow \quad L_t=[L_+^{\frac{1}{2}}, L]$$

i i i.) New equation

Recursion operator of 10 order

$$L=(\partial^3+\frac{2}{3}u\partial+\frac{1}{3}u_x)(\partial^2-\frac{1}{3}v) = L_{KK}L_{KdV} \quad \Rightarrow \quad L_t=[L_+^{\frac{3}{5}}, L]$$

$$\begin{pmatrix} L_{K_{dV}} & L_{KK} \end{pmatrix}$$

$$\begin{pmatrix} L_{KdV} \\ L_{KK} \end{pmatrix} \quad \begin{pmatrix} Hirota-Satsuma & new\,equation \\ new\,equation & Drinfeld-Sokolov \end{pmatrix}$$

Bi - Hamiltonian structure of new equation:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = P \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} = \begin{pmatrix} 3\partial^3 + \partial u + u\,\partial & 0 \\ 0 & 3\partial^3 - 2\partial v - 2\mathrm{v}\,\partial \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix}$$

$$H = \int dx\,(v^2 + 4u^2 + 6uv)\,.$$

The recursion operator has been found using the method of Gurses, Karasu and Sokolov.

$$L_{t_3} = [(L^{\frac{3}{5}})_+, L], \qquad L_{t_{13}} = [(L^2 L^{\frac{3}{5}})_+, L], \qquad (L^2 L^{\frac{3}{5}})_\Delta = (L^2 (L^{\frac{3}{5}})_+ + L^2 (L^{\frac{3}{5}})_\Delta)_\Delta$$

$$(L^2 (L^{\frac{3}{5}})_+)_\Delta = Q = A_9 \partial^9 + \ldots + A_0 \qquad L_{t_{13}} = L^2 L_{t_3} + [Q, L]$$

$$\mathrm{R} = \begin{pmatrix} -\frac{18}{125}\partial^{10} + 268\ terms, & -\frac{11}{375}\partial^{10} + 268\ terms \\ -\frac{11}{375}\partial^{10} + 268\ terms, & -\frac{7}{375}\partial^{10} + 268\ terms \end{pmatrix}$$

and we can obtain the next Hamiltonian structure factorizing R=P Z where

$$\mathrm{Z} \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} -\frac{6}{125}\partial^7 + 74\ terms, & -\frac{11}{375}\partial^7 + 104\ terms \\ -\frac{11}{375}\partial^7 + 104\ terms, & -\frac{7}{375}\partial^7 + 104\ terms \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix}$$

and $$H = \int dx \left(21 u_{10x} u + 26 v_{10x} u + 8 v_{10x} v + 158\ terms\right)$$

The supersymmetric Integrability for N=1 .

The Manin – Radul supersymmetric N=1 KP hierarchy $SKP_{r,m}$

$$L := D^r + \sum_{i=0}^{r-2} v_i^r D^i + \sum_{j=1}^{m} \Phi_j D^{-j}$$

where $D = \partial_\theta + \theta \partial$ and v and Φ are the superbosonic or superfermionic superfields

For even r it is well known hierarchy.

A. For r=3 and m=0 we have

$$\Lambda = D^3 + \Phi \quad \Rightarrow \quad L_{t_k} = 9[\Lambda, \Lambda_+^{\frac{k}{3}}]$$

where $\Phi = \zeta + \theta u$ ξ is a fermionic , u is bosonic function.

$$\Phi_{t,2} = \Phi_x$$

$$\Phi_{\tau,7} = \left((D\Phi)_{xx} + \frac{1}{2}(D\Phi)^2 + 2\Phi\Phi_x \right)_x$$

$$\Phi_{t,10} = \Phi_{5x} + 5\Phi_{xxx}(D\Phi) + 5\Phi_{xx}\Phi_x + 5\Phi_x(D\Phi)^2$$

There are several interesting observations :

A.) t is usual time while τ is an odd time

B.) for k=10 we have the susy N=1 Sawada – Kotera equation and in components it is

$$u_t = u_{5x} + 5u_{xxx}u + 5u_x u_{xx} + 5u^2 u_x - 5\zeta_{xxx}\zeta_x$$
$$\zeta_t = \zeta_{5x} + 5u\zeta_{xxx} + 5u_x\zeta_{xx} + 5u^2\zeta_x$$

and has been recently considered by Tian and Liu.

C.) The trace formula for Lax operator gives us conserved quantities which however are not reduced to the classical conserved charges of Sawada – Kotera equation.

$$H_1 = \int d\theta\, dx\, \Phi\Phi_x = 2\int dx\, u\zeta_x$$
$$H_2 = \int d\theta\, dx\, 3(D\Phi_{xx})D\Phi_x + 2(D\Phi)^3 = 6\int dx(\zeta_{xxx}u + \zeta_x u^2)$$

D.) The odd time hirerachy is generated by even hamiltonian structure

$$\Phi_{\tau,7} = P\frac{\delta H_1}{\delta\Phi} = (D^5 + 2\partial\Phi + 2\Phi\partial + D\Phi D)\frac{\delta H_1}{\delta\Phi}$$

P is the usual hamiltonian operator of the supersymmetric KdV equation which is connected with the N-1 supersymmetrical Virasoro algebra.

E. Second Hamiltonian Structure of susy Sawada – Kotera equation is odd

$$\Omega = (D\partial^2 + 2\partial\Phi + 2\Phi\partial + D\Phi D)\partial^{-1}(D\partial^2 + 2\partial\Phi + 2\Phi\partial + D\Phi D),$$
$$\Phi_t = \Omega \frac{\delta H_1}{\delta \Phi}.$$

Proof:

A.) We have to check the Jacobi identity

$$<\alpha, P'_{(P\beta)}\gamma> + <\beta, P'_{(P\gamma)}\alpha> + <\gamma, P'_{(P\alpha)}\beta> = 0,$$
$$<\alpha,\beta> = \int dx\, d\theta\, \alpha\beta.$$

where P'_{α} denote the Gatoux derivative along the vector α.

B.) R- matrix approach to Bi-Hamiltonian systems

$$\frac{\partial L}{\partial t} = \Gamma_1 \Delta F = (L(\Delta F))_+ - ((\Delta F)L)_+,$$
$$\frac{\partial L}{\partial t} = \Gamma_2 \Delta F = L((\Delta F)L)_+ - (L(\delta F)_+)L,$$
$$\frac{\partial L}{\partial t} = \Gamma_3 \Delta F = L(L(\Delta F)L)_+ - (L(\Delta F)L)_+ L - L((\Delta F)L)_+ L + L(L(\Delta F))_+ L$$

where gradient is parametrized as

$$\Delta H = \sum_{k>0} \partial^{-k-1}\left(-D\frac{\delta H}{\delta a_k} + \frac{\delta H}{\delta b_k}\right),$$

$$L = \sum_{k>0} (a_k + b_k D)\partial^{k.}$$

We have to use the Dirac reduction, because we need to embed the Lax operator into a larger space.

Ad 1. For Γ_1 it is impossible to carry out such reduction.

Ad 2. For Γ_2 we have 3 dimensional matrix and Dirac reduction gives us P operator which generate odd time flows.

Ad 3. For Γ_3 we have 6 dimensional matrix and Dirac reductio gives us Ω which generates the even time flows.

Second Hamiltonian structure.

Is obtained from the factorization of the recursion operator which was derived by Tian and Liu

$$J = \partial^2 + (D\Phi) - \partial^{-1}(D\Phi)_x + \partial^{-1}\Phi_{xD} + \Phi_x \partial^{-1} D$$

$$J\Phi_t = \frac{\delta H}{\delta \Phi}$$

The SKP hierarchy for r=5 and m=0 . Results

$$L = D^5 + \frac{1}{3}(\partial Z + \partial Z) - \frac{1}{3} D P D$$

where $Z = \varsigma_1 + \theta z_1$, $P = p_1 + \theta \varsigma_2$ are superfermionic and superbosonic supermultiplet.

Computing

$$\frac{\partial L}{\partial t} = 5[L, (L^{\frac{6}{5}})_+]$$

we obtain

$$Z_t = 4 Z_{xxx} + 3 P_{xxx} - 2 Z_x (DZ + DP) + Z(6DZ_x + 2DP_x) + P(3 DZ_x + DP_x) - P_x DP$$

$$P_t = 3 Z_{xxx} + P_{xxx} + 8Z_x DZ - Z(8DZ_x + 6DP_x) + P_x(4DZ + DP) - P_x(4DZ_x + 3DP_x)$$

The bosonic sector in which

$$Z = \theta u, \quad P = \theta v.$$

gives us new 2-component interacted KdV equations.